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Research Article

Noniterative Design of 2-Channel FIR Orthogonal Filters

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This paper addresses the problem of obtaining an explicit expression of all real FIR paraunitary filters. In this work, we present a general parameterization of 2-channel FIR orthogonal filters. Unlike other approaches which make use of a lattice structure, we show that our technique designs any orthogonal filter *directly*, with no need of iteration procedures. Moreover, in order to design an L-tap 2-channel paraunitary filterbank, it suffices to choose $L/2$ independent parameters, and introduce them in a simple expression which provides the filter coefficients directly. Some examples illustrate how this new approach can be used for designing filters with certain desired properties. Further conditions can be eventually imposed on the parameters so as to design filters for specific applications.

1. INTRODUCTION

Filterbanks are widely used in all signal processing areas. In the 2-channel case, the filterbank decomposes any signal into its lowpass and highpass components; this is achieved by convolution with a lowpass filter $\mathbf{h}$ and a highpass filter $\mathbf{g}$. When using finite impulse response (FIR) filters, the implementation is even more direct.

In particular, for signal compression applications, orthogonal subband transforms are desired; hence, paraunitary filterbanks are required. Thus, some design techniques have been addressed in the literature. Moreover, the appearance of the wavelet theory gave a new insight into the filter bank theory, and also provided new methods for the design of real FIR paraunitary filters.

Despite the wide number of publications, we have focused on the most well-known results, which are contained in [1–5]. We can merge these main approaches into three groups.

(a) *Methods based on spectral factorizations.* Commonly used in wavelet theory [1, 3], they are based on the following characterization: a real L-tap filter $\mathbf{h} = (h_1, h_2, \dots, h_L)$ is paraunitary if and only if its transfer function $H(z) = \sum_{n=1}^{L} h_n z^{1-n}$ verifies

$$\left|H(z)\right|^2 + \left|H(-z)\right|^2 = 2, \quad |z| = 1. \tag{1}$$

Hence, it suffices to find the power spectral $P(z) = |H(z)|^2$ which satisfies $P(z) + P(-z) = 2$, and then factors it as $P(z) = |H(z)|^2 = H(z)H(z^{-1})$ so as to get the real filter coefficients. Thus, it is necessary to compute roots of a $2L-1$ degree polynomial P; but the main drawback is that the corresponding iterative algorithms generally become numerically unstable for long filters.

(b) *Lattice filters design.* Instead of computing a polynomial, this approach designs the polyphase matrix associated with the FIR 2-channel cell given by filters $\mathbf{h}, \mathbf{g}$. The polyphase matrix is defined as

$$H_p(z) = \begin{pmatrix} H_{\text{even}}(z) & H_{\text{odd}}(z) \\ G_{\text{even}}(z) & G_{\text{odd}}(z) \end{pmatrix} \tag{2}$$

and the filterbank is paraunitary if and only if $H_p(z)$ is unitary for every $|z| = 1$. Thus, it suffices to build unitary matrices of this kind. The paradigm of these methods is Vaidyanathan's algorithm [2, 6, 7]. Basically, any paraunitary real L-tap FIR causal filter is obtained through iteration, because its polyphase matrix can be factorized as

$$H_p(z) = \left(\prod_{j=1}^{L/2-1} \left(I + (z^{-1} - 1)\mathbf{v}_j \mathbf{v}_j^t \right) \right) Q, \tag{3}$$

where Q is unitary of order 2, and $\mathbf{v}_j$ are unitary column vectors of $\mathbb{R}^2$. Thus, in order to design a paraunitary filter of length L, we need a total amount of $L/2$ parameters in $[-1, 1]$, and $L/2$ signs. This algorithm behaves well numerically, but it is difficult to apply when imposing extra desired properties upon the filter. Besides, as we will see later on, this representation is redundant, in the sense that it could eventually give rise to filters of smaller length $L-2$.

(c) *Lifting scheme* [5, 8]. this is apparently another approach for either orthogonal or biorthogonal two-channel filter banks. The key idea is to build filters of length L with desirable properties by *lifting* filters of length smaller than L. But, for the orthogonal case [5], it turns out to be another iterative algorithm, equivalent to the lattice factorization already mentioned. Thus, we will consider it as a particular example of the *lattice filters design* approach.

We summarize that all these well-known approaches present some disadvantages.

On the other hand, in our recent work [9] we have proposed the first parameterization for paraunitary filters of length L by means of only $L/2$ independent parameters and 1 sign; besides, the filter coefficients are obtained explicitly, using neither an iteration process nor a root finding procedure. In this paper, we improve our design technique, by obtaining a simpler expression; we also make for the first time a rigorous proof of its validity for lowpass filters. Finally, as one of the main contributions, we present a novel explicit expression of the power spectral response $P(z)$ of paraunitary filters. It constitutes a new tool for the design of filters. For each specific application, it may be used to design the paraunitary filter to satisfy the desired properties.

The paper is organized as follows: in Section 2 we derive the explicit expression of all real 2-channel FIR orthogonal filterbanks. In Section 3 we study the particular case of lowpass paraunitary filterbanks, and illustrative examples are shown. In Section 4 we obtain the general explicit expression of the halfband power spectral response of a paraunitary filter, by means of the free parameters; conclusions are finally discussed in Section 5.

Let us now introduce the following notation, necessary to follow the development of our work: for any set of real numbers $(a_1, \ldots, a_m)$, let us denote the Toeplitz low-triangular matrix of order m which contains these numbers in its first column,

$$T(a_1, \ldots, a_m) = \begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ a_2 & a_1 & 0 & \ddots & \vdots \\ a_3 & a_2 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ a_m & \cdots & a_3 & a_2 & a_1 \end{pmatrix}. \qquad (4)$$

Throughout this paper, only real matrices and vectors are considered. Matrices are denoted by capital letters, and vectors by boldface lowercase letters. The superscript t denotes transposition.

Finally, P will denote the exchange matrix, say, the one that produces a reversal. Recall that any Toeplitz matrix T verifies $PTP = T^t$. In effect, by reversing the order of its rows and columns we obtain its transpose.

2. NEW EXPRESSION OF ORTHOGONAL FILTERS

Throughout this paper, we will say that an L-tap filter $\mathbf{h}$ is *orthogonal* if it is orthogonal to its even shifts, that is, if it satisfies

$$\forall k = 1, \ldots, \frac{L}{2} - 1, \quad 0 = \sum_{n=1}^{L-2k} h_n h_{n+2k}. \qquad (5)$$

If we additionally impose the norm 1 condition ($\sum_{n=1}^{L} h_n^2 = 1$), then $\mathbf{h}$ will be called paraunitary.

The orthogonality condition implies that L is even; then, (5) can be rewritten, for any $k = 1, \ldots, L/2 - 1$, as

$$\sum_{n \,\text{odd}} h_n h_{n+2k} = -\sum_{n \,\text{even}} h_n h_{n+2k}. \qquad (6)$$

For instance, if $k = L/2 - 1$, we have that $h_1 h_{L-1} = -h_2 h_L$; as $h_1 \cdot h_L \neq 0$, there must be a real parameter a_1 such that

$$\frac{h_{L-1}}{h_L} = -\frac{h_2}{h_1} = a_1. \qquad (7)$$

Hence,

$$h_2 = -a_1 h_1, \qquad h_{L-1} = a_1 h_L. \qquad (8)$$

In other words, h_2, h_{L-1} can be derived from h_L, h_1, respectively.

Now the key question arises: can we always write the even components of the filter by means of the odd ones, and vice versa? In [9], we have proved the next result, which guarantees that the answer is *yes*. Its demonstration is also included here.

Theorem 1. *$\mathbf{h} = (h_1, h_2, \ldots, h_L)$ is an orthogonal real filter if and only if there exists a unique set of real numbers $a_1, \ldots, a_{L/2-1}$ such that, for any $k = 1, \ldots, L/2 - 1$,*

$$h_{2k} = -\sum_{j=1}^{k} h_{2k+1-2j} a_j, \qquad h_{L+1-2k} = \sum_{j=1}^{k} h_{L-2k+2j} a_j. \qquad (9)$$

Or, in an equivalent matricial way,

$$\begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{L-2} \end{pmatrix} = -T(a_1, \ldots, a_{L/2-1}) \begin{pmatrix} h_1 \\ h_3 \\ \vdots \\ h_{L-3} \end{pmatrix}, \qquad (10)$$

$$\begin{pmatrix} h_3 \\ \vdots \\ h_{L-3} \\ h_{L-1} \end{pmatrix} = \left(T(a_1, \ldots, a_{L/2-1})\right)^t \begin{pmatrix} h_4 \\ \vdots \\ h_{L-2} \\ h_L \end{pmatrix}. \qquad (11)$$

Proof. Equation (5) may be easily rewritten matricially as

$$T(h_1, h_3, \dots, h_{L-3}) \begin{pmatrix} h_{L-1} \\ h_{L-3} \\ \vdots \\ h_3 \end{pmatrix} = -T(h_L, h_{L-2}, \dots, h_4) \begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{L-2} \end{pmatrix}, \tag{12}$$

where we have used our notation for lower triangular Toeplitz matrices. As $h_1 \cdot h_L \neq 0$, both matrices are nonsingular; besides, their inverses are also lower triangular Toeplitz matrices; finally, such matrices always commute, so we can state that

$$\left(T(h_L, h_{L-2}, \dots, h_4)\right)^{-1} \begin{pmatrix} h_{L-1} \\ h_{L-3} \\ \vdots \\ h_3 \end{pmatrix} = -\left(T(h_1, h_3, \dots, h_{L-3})\right)^{-1} \begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{L-2} \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{L/2-1} \end{pmatrix}; \tag{13}$$

in other words, we define $(a_1, \dots, a_{L/2-1})^t$ as any of these two vectors. For instance, the first coefficient a_1 is the one that verifies $a_1 = h_{L-1}/h_L = -h_2/h_1$. Thus, we simultaneously have

$$\begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{L-2} \end{pmatrix} = -T(h_1, h_3, \dots, h_{L-3}) \begin{pmatrix} a_1 \\ \vdots \\ a_{L/2-1} \end{pmatrix}, \quad \begin{pmatrix} h_{L-1} \\ h_{L-3} \\ \vdots \\ h_3 \end{pmatrix} = T(h_L, h_{L-2}, \dots, h_4) \begin{pmatrix} a_1 \\ \vdots \\ a_{L/2-1} \end{pmatrix}. \tag{14}$$

Note also that the set of parameters $(a_j)_{j=1}^{L/2-1}$ which satisfies any of these conditions is unique. Besides, these equations are equivalent to

$$\begin{pmatrix} h_2 \\ h_4 \\ \vdots \\ h_{L-2} \end{pmatrix} = -T(a_1, \dots, a_{L/2-1}) \begin{pmatrix} h_1 \\ h_3 \\ \vdots \\ h_{L-3} \end{pmatrix}, \quad \begin{pmatrix} h_{L-1} \\ h_{L-3} \\ \vdots \\ h_3 \end{pmatrix} = T(a_1, \dots, a_{L/2-1}) \begin{pmatrix} h_L \\ h_{L-2} \\ \vdots \\ h_4 \end{pmatrix}. \tag{15}$$

The former identity yields (10) directly. On the other hand, by reversing the rows of the second identity we obtain (11). Just recall that $T(a_1, \dots, a_{L/2-1})$ is a Toeplitz matrix, so the exchange matrix P satisfies

$$PT(a_1, \dots, a_{L/2-1}) = T(a_1, \dots, a_{L/2-1})^t P, \tag{16}$$

which concludes the proof. □

For example, for $k = 2$, Theorem 1 implies that $h_{L-3} = a_1 h_{L-2} + a_2 h_L$ and $-h_4 = a_1 h_3 + a_2 h_1$. So we have shown that it is possible to express every odd coefficient h_{2k+1} by means of its following even coefficients of the filter, and every even coefficient h_{2k} by means of its former odd ones.

2.1. New simplified expression of orthogonal filters

Once we have demonstrated the existence of the vector of parameters $\mathbf{a} = (a_j)_{j=1}^{L/2-1}$, then we define

(i) a Toeplitz low-triangular matrix of order $L/2 - 1$:

$$A := T(0, a_1, \dots, a_{L/2-2}); \tag{17}$$

(ii) and two vectors of length $L/2 - 1$:

$$\mathbf{b} = -\left(I + AA^t\right)^{-1} \mathbf{a}, \qquad \mathbf{c} = A^t \mathbf{b} = -A^t \left(I + AA^t\right)^{-1} \mathbf{a}, \tag{18}$$

which are well defined because $I + AA^t$ is always a positive definite matrix. Note also that $b_1 = -a_1$ and $c_{L/2-1} = 0$ because of the null diagonal of A.

For the sake of simplicity, from now on we will denote

$$\mathbf{h}_{\text{even}} = (h_2, h_4, h_6, \dots, h_{L-2})^t, \qquad \mathbf{h}_{\text{odd}} = (h_3, h_5, \dots, h_{L-3}, h_{L-1})^t, \tag{19}$$

which contain the even and odd indexed coefficients of $\mathbf{h}$ *except the first and the last ones*, h_1, h_L.

Now we are able to finally express all the components of the filter by means of h_1, h_L, and the $L/2 - 1$ parameters. This is one of the main results of this paper, which constitutes a new characterization and design method of all orthogonal filters, even simpler than the one obtained in [9].

Theorem 2. $\mathbf{h} = (h_1, h_2, \dots, h_L)$ *is an orthogonal filter if and only if there exist* $L/2 - 1$ *real numbers* $a_1, \dots, a_{L/2-1}$ *such that*

$$\mathbf{h}_{\text{even}} = h_1 \mathbf{b} + h_L P \mathbf{c}, \qquad \mathbf{h}_{\text{odd}} = h_1 \mathbf{c} - h_L P \mathbf{b}. \tag{20}$$

Proof. By making use of the matrix A and the vectors $\mathbf{h}_{\text{even}}$ and $\mathbf{h}_{\text{odd}}$ introduced above, (10) and (11) can be, respectively, rewritten as

$$-\mathbf{h}_{\text{even}} = h_1 \mathbf{a} + A \mathbf{h}_{\text{odd}}, \qquad \mathbf{h}_{\text{odd}} = h_L P \mathbf{a} + A^t \mathbf{h}_{\text{even}} \tag{21}$$

so we have that

$$\mathbf{h}_{\text{even}} + A \mathbf{h}_{\text{odd}} = -h_1 \mathbf{a}, \qquad -A^t \mathbf{h}_{\text{even}} + \mathbf{h}_{\text{odd}} = h_L P \mathbf{a}. \tag{22}$$

It just suffices to solve this linear system with unknowns $\mathbf{h}_{\text{even}}$, $\mathbf{h}_{\text{odd}}$. By elementary Gaussian elimination operations, it is equivalent to the system

$$\begin{aligned}(I+AA^t)\mathbf{h}_{\text{even}} &= -(h_1 I + h_L AP)\mathbf{a},\\ (I+A^tA)\mathbf{h}_{\text{odd}} &= (h_L P - h_1 A^t)\mathbf{a},\end{aligned} \tag{23}$$

from which we can obtain both vectors independently, because $I+AA^t$ and $I+A^tA$ are nonsingular. Moreover, we can exploit the fact that A is Toeplitz: $A^t = PAP$, $A = PA^tP$, and $A^tA = PAA^tP$ so $(I+A^tA) = P(I+AA^t)P$ and

$$(I+A^tA)^{-1}P = P(I+AA^t)^{-1}; \tag{24}$$

besides, it is easy to show that

$$\begin{aligned}(I+AA^t)^{-1}A &= A(I+A^tA)^{-1},\\ (I+A^tA)^{-1}A^t &= A^t(I+AA^t)^{-1}.\end{aligned} \tag{25}$$

Finally, we make use of all these expressions and the definition of $\mathbf{b}$ and $\mathbf{c}$ given in expressions (18) in order to obtain (20):

$$\begin{aligned}\mathbf{h}_{\text{even}} &= -(I+AA^t)^{-1}(h_1 I + h_L AP)\mathbf{a} = h_1\mathbf{b} + h_L P\mathbf{c},\\ \mathbf{h}_{\text{odd}} &= (I+A^tA)^{-1}(h_L P - h_1 A^t)\mathbf{a} = -h_L P\mathbf{b} + h_1\mathbf{c}.\end{aligned} \tag{26}$$

□

We have derived that, by choosing $L/2-1$ arbitrary parameters and 2 arbitrary nonzero numbers h_1, h_L, we are able to parameterize the whole set of orthogonal filters $\mathbf{h}$ of length L. In other words, these filters are characterized by means of just $L/2+1$ parameters. And this representation is *unique*: different sets of parameters always yield different filters, so there is no redundancy in this parameterization.

All the coefficients of the filter are of the following form (first: odd coefficients, last: even coefficients):

$$\begin{pmatrix} h_1 \\ \mathbf{h}_{\text{odd}} \\ \mathbf{h}_{\text{even}} \\ h_L \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \mathbf{c} & -P\mathbf{b} \\ \mathbf{b} & P\mathbf{c} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_L \end{pmatrix}. \tag{27}$$

Thus, any orthogonal filter is a linear combination of these two columns, which are indeed orthogonal filters of length $L-2$. They are orthogonal columns; moreover, it can easily be seen that they are conjugate quadrature filters. In effect, the odd components of the first filter correspond to the even components of the second one, reversed; and the even components of the first filter are the opposite of the odd components of the second one, reversed. This property will be exploited in the next section.

Let us remark that this property confirms the underlying idea of lattice factorization [6] and lifting scheme [5]. L-tap paraunitary filters can be built by means of paraunitary filters of smaller length $(L-2)$. In this sense, our design approach generalizes those existing techniques.

To finish this section, let us notice that we can also write

$$(\mathbf{h}_{\text{odd}} \;\; P\mathbf{h}_{\text{even}}) = (\mathbf{c} \;\; -P\mathbf{b}) \begin{pmatrix} h_1 & h_L \\ h_L & -h_1 \end{pmatrix}. \tag{28}$$

Remark 1. From this expression, we also deduce that any pair of conjugate quadrature mirror filters is associated to the *same set of independent parameters* $a_1, \dots, a_{L/2-1}$; the only difference is the value of the first and last coefficients. If we choose h_1, h_L for the filter $\mathbf{h}$, then we just have to set $g_1 = h_L$, $g_L = -h_1$ for its CQM filter $\mathbf{g}$.

2.2. New expression of paraunitary filters

Next, we impose the constraint that the vector $\mathbf{h}$ has norm 1; regarding (27), let us note that the norm of each column is equal to

$$1 + \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 = 1 - \mathbf{b}^t\mathbf{a} \geq 1, \tag{29}$$

where we have used that

$$\|\mathbf{b}\|^2 + \|A^t\mathbf{b}\|^2 = \mathbf{b}^t(I+AA^t)\mathbf{b} = -\mathbf{b}^t\mathbf{a} \geq 0. \tag{30}$$

Due to the orthogonality of the two columns of this expression (27), the norm of $\mathbf{h}$ is very easy to compute:

$$1 = \|\mathbf{h}\|^2 = (1 - \mathbf{b}^t\mathbf{a})(h_1^2 + h_L^2). \tag{31}$$

As the quantity $1 \leq 1 - \mathbf{b}^t\mathbf{a} < \infty$ and only depends on the election of the parameters, it just suffices to choose h_1, h_L in the circle of radius

$$0 < \frac{1}{\sqrt{1 - \mathbf{b}^t\mathbf{a}}} \leq 1. \tag{32}$$

Corollary 1. *$\mathbf{h} = (h_1, h_2, \dots, h_L)$ is a paraunitary filter if and only if there exist $L/2-1$ real numbers $a_1, \dots, a_{L/2-1}$ verifying* (20), *and*

$$h_1^2 + h_L^2 = (1 - \mathbf{b}^t\mathbf{a})^{-1}. \tag{33}$$

This means that h_L (up to its sign) is expressed by means of h_1. In other words, it is deduced that the set of paraunitary filters of length L is determined by $L/2$ parameters, and 1 sign.

For instance, if all the parameters are chosen null, then vectors $\mathbf{b}$ and $\mathbf{c}$ are null, and the filter obtained is of the type $\mathbf{h} = (h_1, 0, \dots, 0, h_L)$ which is orthogonal, and unitary whenever $h_1^2 + h_L^2 = 1/(1 + 0^2 + 0^2 + 0^2) = 1$.

3. DESIGN OF ORTHOGONAL FILTERS WITH DESIRABLE PROPERTIES

3.1. Design of lowpass orthogonal filters

Lowpass filters must satisfy $H(1) = s \neq 0$. Equivalently, let us now impose $s = H(1) = \sum h_n = \mathbf{u}^t\mathbf{h}$ where $\mathbf{u}$ is the vector whose components are all equal to 1. Again, by using (27), the sum of the coefficients of $\mathbf{h}$ is a linear combination of the sum of each one of the two columns:

$$s = h_1(1 + \mathbf{u}^t(\mathbf{b} + \mathbf{c})) + h_L(1 + \mathbf{u}^t(\mathbf{c} - \mathbf{b})) \tag{34}$$

so we get the equation of a straight line. Note that the normal vector is always nonzero, because the sum of both columns cannot vanish simultaneously. The reason is that they are conjugate quadrature mirror filters. Hence, there are always infinite choices for h_1, h_L in that line.

For example, by choosing all parameters null, the equation of the line is $s = h_1 + h_L$ so the associated orthogonal filter is $\mathbf{h} = (h_1, 0, \ldots, 0, s - h_1)$.

3.2. Design of lowpass paraunitary filters

It is well known that lowpass paraunitary filters must satisfy the DC leakage condition. As $H(-1) = 0$, introducing it into (1) we obtain that $H(1) = \sqrt{2}$. Now we impose both conditions over the orthogonal filter $\mathbf{h}$: norm 1 and sum $\sqrt{2}$.

The equations that h_1, h_L must verify are

$$\begin{aligned} h_1^2 + h_L^2 &= \left(1 - \mathbf{b}^t\mathbf{a}\right)^{-1}, \\ \sqrt{2} &= h_1\left(1 + \mathbf{u}^t(\mathbf{b} + \mathbf{c})\right) + h_L\left(1 + \mathbf{u}^t(-\mathbf{b} + \mathbf{c})\right). \end{aligned} \tag{35}$$

In other words, (h_1, h_L) lies in the intersection between a circle and a line in $\mathbb{R}^2$. *May this intersection be null*? This question was open in our previous work [9] but now we demonstrate that the answer is *no*. The reason is that, for any lowpass filter of first and last coefficients h_1, h_L, the corresponding conjugate highpass filter of the same length is the one whose first and last coefficients are $\pm h_L$, $\mp h_1$. This means that the line which is orthogonal to the previous one and contains the origin will surely intersect such circle in two points: $\pm(h_L, -h_1)$. So there is only one highpass filter (up to the sign); hence, there is only one lowpass orthogonal filter which satisfies both conditions above. So this justifies that such intersection is not null, moreover, it contains only one point.

For example, if all the parameters are chosen to be null, then all these vectors are null, and this condition is clearly satisfied, giving rise to the paraunitary lowpass filters $\pm\sqrt{2}/2(1, 0, 0, \ldots, 0, 0, 1)$; for $L = 2$, we obtain the Haar filter.

3.3. Example: 4-tap lowpass paraunitary filters

As a very simple example, let us consider paraunitary filters of length 4; they must be of the following form: $\mathbf{h} = (h_1, -ah_1, ah_4, h_4)$, they must have norm 1, and satisfy the DC condition:

$$\begin{aligned} h_1^2 + h_4^2 &= \left(1 + a^2\right)^{-1}, \\ \sqrt{2} &= h_1(1 - a) + h_4(1 + a). \end{aligned} \tag{36}$$

But such conditions are *always possible for all* a_1, since the line and the circle intersect in only one point,

$$h_1 = \frac{(1 - a)}{(1 + a^2)\sqrt{2}}, \qquad h_4 = \frac{(1 + a)}{(1 + a^2)\sqrt{2}} \tag{37}$$

obtaining the unique expression for the filter

$$\mathbf{h} = \frac{1}{(1 + a^2)\sqrt{2}}\left(1 - a, a^2 - a, a^2 + a, 1 + a\right). \tag{38}$$

Let us compare it to the other approaches. The spectral method would have required a greater amount of operations; as for the lattice filters, we only would need $4/2 - 1 = 1$ unitary vector, and a unitary matrix of order 2. It is easy to see that the unitary matrix for lowpass filters is always equal to

$$Q = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{39}$$

Next, by choosing an arbitrary unitary vector $\mathbf{v} = (c, d)^t/\sqrt{c^2 + d^2}$ and the unitary matrix Q, the paraunitary lowpass filters computed via the lattice method are

$$\mathbf{h} = \frac{1}{(c^2 + d^2)\sqrt{2}}\left(d^2 - cd, d^2 + cd, c^2 + cd, c^2 - cd\right) \tag{40}$$

so they are of length 4, except when $c = 0$ (and we have Haar filter), or $d = 0$ or $c = d$ (shifted versions of Haar filter). This is an example that the lattice design may provide filters of smaller length.

On the other hand, note that its components are $\mathbf{h} = (h_1, -ah_1, ah_4, h_4)$ with (in case the length is exactly 4)

$$a = \frac{c + d}{c - d} = \frac{1 + d/c}{1 - d/c} \tag{41}$$

and the ratio d/c is the very important direction of vector $\mathbf{v}$, whereas $a \in \mathbb{R}$ is a free parameter which can take all possible real values. This means that our expression (38) is simpler than (40), and yields the same set of paraunitary filters.

3.4. Example: 4-tap lowpass paraunitary filter with maximum attenuation

The attenuation of the lowpass filter may be measured as

$$\int_0^{\pi/2} \left|H(w)\right|^2 dw = \frac{\pi}{2} + 2\sum_{n=0}^{L/2-1} r(2n + 1)\frac{(-1)^n}{(2n + 1)}, \tag{42}$$

where $r(n)$ denotes the autocorrelation coefficients of the filter.

Let us impose now the maximum attenuation to our 4-tap designed filters. In this case we should maximize $r(1) - r(3)/3$. To this aim, we compute such autocorrelation coefficients of (38):

$$r(1) = \frac{3a^2 + a^4}{\left(1 + a^2\right)^2 2}, \qquad r(3) = h_1h_4 = \frac{1 - a^2}{\left(1 + a^2\right)^2 2}. \tag{43}$$

Next, it suffices to maximize

$$r(1) - \frac{r(3)}{3} = \frac{10a^2 + 3a^4 - 1}{6\left(1 + a^2\right)^2}. \tag{44}$$

We obtain that the maximum is achieved for $a = \pm\sqrt{3}$. For $a = \sqrt{3}$, we have

$$\mathbf{h} = \frac{1}{4\sqrt{2}}\left(1 - \sqrt{3}, 3 - \sqrt{3}, 3 + \sqrt{3}, 1 + \sqrt{3}\right), \tag{45}$$

whereas for $a = -\sqrt{3}$, we obtain

$$\mathbf{h} = \frac{1}{4\sqrt{2}}\left(1 + \sqrt{3}, 3 + \sqrt{3}, 3 - \sqrt{3}, 1 - \sqrt{3}\right) \tag{46}$$

which correspond to the 4-tap Daubechies filters (minimum/maximum phase), which are the optimal ones, with attenuation $(\pi/2) + (7/6)$.

Let us remark that our technique confirms the results obtained by means of other approaches, although in a more direct way. Nevertheless, working with longer filters will involve maximizing a functional which depends on more variables, and the expressions will be more complicated.

4. NEW EXPRESSION OF THE POWER SPECTRAL RESPONSE

As another final contribution, we will find the explicit expression of the halfband polynomial $P(z) = |H(z)|^2$ associated to a paraunitary filter of length L. Our final aim would be to design the polynomial P instead of the filter itself. To this end, we first must find the desired expression of P by means of the $L/2 - 1$ independent parameters $(a_1, \dots, a_{L/2-1})$, apart from h_1, h_L which verify the 1-norm condition (33).

We will use the simple expression (27) already obtained. Let us denote $H_1(z)$ the transfer function of the filter given by the first column. On one hand, its even coefficients constitute vector $\mathbf{b}$, while its odd coefficients are $(1, \mathbf{c})$. Note that it is a filter of length $L - 2$ because the last component of $\mathbf{c}$ is zero. So we can write $H_1(z) = C(z^2) + z^{-1}B(z^2)$, where B, C are the respective transfer functions associated to the filters $\mathbf{b}$, and $(1, \mathbf{c})$, both of length $L/2 - 1$.

Moreover, this first column constitutes an orthogonal filter; in effect, the filter $|H_1(z)|^2$ is halfband:

$$\begin{aligned} d = 2(1 - \mathbf{b}^t\mathbf{a}) &= \left|H_1(z)\right|^2 + \left|H_1(-z)\right|^2 \\ &= 2\left|C(z^2)\right|^2 + 2\left|B(z^2)\right|^2. \end{aligned} \tag{47}$$

On the other hand, the second column is a shifted version of its CQF filter, so we easily deduce that

$$H(z) = h_1 H_1(z) + h_L z^{1-L} H_1(-z^{-1}). \tag{48}$$

Let us finally compute the power spectral response, also by making use of (33):

$$\begin{aligned} P(z) &= \left|H(z)\right|^2 \\ &= \left|h_1 H_1(z) + h_L z^{1-L} H_1(-z^{-1})\right|^2 \\ &= (h_1^2 + h_L^2)\frac{d}{2} + 2h_1 h_L \operatorname{Re}\left(z^{L-1} H_1(-z) H_1(z)\right) \\ &\quad + 2(h_1^2 - h_L^2)\operatorname{Re}\left(zC(z^2)B(z^{-2})\right) \\ &= 1 + 2h_1 h_L \operatorname{Re}\left(z^{L-1}\left(C(z^2)^2 - z^{-2}B(z^2)^2\right)\right) \\ &\quad + 2(h_1^2 - h_L^2)\operatorname{Re}\left(zC(z^2)B(z^{-2})\right), \end{aligned} \tag{49}$$

where Re stands for real part of the complex number.

We summarize that the coefficients of P, which are the autocorrelation coefficients $r(n)$ of the filter, can be easily obtained by means of the coefficients of C^2 and B^2 (resp., the autoconvolution of $(1, \mathbf{c})$, and the autoconvolution of $\mathbf{b}$) and the coefficients of $C(z^2)B(z^{-2})$ (say, the correlation between $(1, \mathbf{c})$ and $\mathbf{b}$). Recall that all these vectors are computed directly from the free parameters $\mathbf{a}$. Finally, h_L is simply obtained from h_1 by means of (33), up to a sign.

5. CONCLUSIONS

We have presented a novel characterization of real paraunitary FIR filterbanks. This provides a new method for the direct design of this type of filters. Its main advantage is that it does not need any iteration process. It just suffices to choose arbitrary values of some parameters, and substitutes them into a closed-form expression. We have also obtained the general expression of lowpass paraunitary filters. Moreover, the proposed technique helps us to design filters with desired properties in a very simple and direct way, even more than the existing techniques, as has been illustrated with 4-tap filters. For paraunitary filters of arbitrary length, we have also obtained a simple explicit expression of its power spectral response. This yields a new powerful tool for designing paraunitary filters which satisfy extra conditions, as it is usually requested in specific applications.

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