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Research Article

Online Estimation of Time-Varying Volatility Using a Continuous-Discrete LMS Algorithm

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The following paper addresses a problem of inference in financial engineering, namely, online time-varying volatility estimation. The proposed method is based on an adaptive predictor for the stock price, built from an implicit integration formula. An estimate for the current volatility value which minimizes the mean square prediction error is calculated recursively using an LMS algorithm. The method is then validated on several synthetic examples as well as on real data. Throughout the illustration, the proposed method is compared with both UKF and offline volatility estimation.

1. INTRODUCTION

In 1973 Black, Scholes and Merton [1, 2] reasoned that under certain idealized market assumptions the prices of stocks and the derivatives on these stocks are coupled. One of the crucial assumptions is that the traded asset price S follows

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \tag{1}$$

where B_t is a Brownian motion. μ and σ are called, respectively, drift and volatility of the stock; both are deterministic constants. Nevertheless, it turns out that the assumption of constant volatility does not hold in practice.

Traders in the market are supposed to assess returns which have different horizon times in order to predict volatility. Researchers in empirical finance have, therefore, developed an increasing interest in the possibility of uncovering the complex volatility dynamics that exist both within and across different financial markets. Even to the most casual observer of markets, it should be clear that volatility is a random variable. Stochastic volatility models provide a framework for such modeling, especially when dealing with high frequency data. Shephard and Andersen trace the origins of the subject in [3] and attributes it to five sets of people. Back in 1995, the ARCH/GARCH models were a hot topic in econometrics research, and their discoverer, Robert Engle, published a collection of papers on the topic. Now, ten years later, the ARCH/GARCH models are still widely used but their limitations are motivating research into alternative models, specifically, stochastic volatility models (usually abbreviated as SV models). In modern finance, stochastic volatility models represent the latest research which tries to understand financial volatility in continuous time. The resulting process is the nonnegative spot volatility which is assumed to have càdlàg sample paths. The preference given to SV models necessarily follows from the theoretical development of stochastic calculus, which is closely related to continuous time Markov processes. SV models are expected to allow for more comprehensive empirical investigation of the fundamental determinants of certain phenomena:

(a) options with different strikes and maturities have different implied volatilities;

(b) the empirical distributions of stock returns are leptokurtic.

SV models, consequently, allow for safer measures of risk, for pricing accurately and for hedging options.

We refer to Shephard (2005) [4] in order to have a thorough account of the topic of stochastic volatility. All the following studies, for instance, Hull and White (1987) [5], E. M. Stein and J. C. Stein (1991) [6], Heston (1993) [7], Scott (1997), support only offline processing. They aim to calibrate a given model for the volatility dynamics, on the observed sample path of the asset price. The main feature of the method proposed in this paper is an online estimation of volatility: the object to be estimated is one particular trajectory of the volatility process. We use the trajectory of the stock price process, as and when its observation proceeds. Jazwinski in [8] studied the problem of online estimation within continuous time models. In the context of a nonlinear model identification, the use of nonlinear filters such as the unscented Kalman's filter [9, 10] is required.

It is proven, however, in [10, 11] that traditional UKF is ill-suited for the problem of time-varying volatility estimation. Actually, the UKF never updates prior beliefs, and consequently, it is not able to track volatility fluctuations. We do, however, implement UKF as literature provides no online estimation methods for volatility. Furthermore, we have recourse to an offline estimation method. It is based on an SV model: a continuous time model of volatility dynamics in the form of a stochastic differential equation. Its driving process is Lévy rather than Brownian. The method has been the subject of a recent paper [12]. The model frame is built by a "shaping filter" technique [13], using prior information on the covariance function of the squared volatility process.

2. THE PROPOSED METHOD

To estimate the latent instantaneous volatility σ_t of the stock price S_t, the stochastic differential equation for the log-price $y_t = \log S_t$ is considered. Applying Itô's formula to (1) yields

$$dy_t = \left(\mu - \frac{\sigma_t^2}{2}\right)dt + \sigma_t dB_t. \tag{2}$$

This SDE may be expressed as

$$dy_t = F(t)dt. \tag{3}$$

The basic idea of the proposed method is to build a predictor from (3) for the observation y_t at $t = t_{i+1}$. Consequently, (3) is to be discretized at observation instants; this leads us to the question of numerical stability of discretization schemes. It is well known that implicit schemes, such as

$$y_{i+1} = G(y_{i-1}, y_i, F_i, F_{i+1}, \dots) \quad (F_i = F(t_i)), \tag{4}$$

guarantee numerical stability better [14]. Generally, implicit formulae use constant time steps. However, since observations here are made according to arbitrary sampling (i.e., discretization instants are not necessarily equally spaced), only the so-called order-1 and order-2 Adams Moulton formulae are applicable. It is indeed the latter formula (the trapezoidal) that has been chosen:

$$y_{i+1} = y_i + \frac{t_{i+1} - t_i}{2}\left[F_i + F_{i+1}\right]. \tag{5}$$

Previously, it has also been used for the identification of a continuous time autoregressive model [15]. Equations (2)–(5) lead to

$$\begin{aligned} y_{i+1} = {} & y_i + \mu(t_{i+1} - t_i) - \frac{(t_{i+1} - t_i)}{4}(\sigma_i^2 + \sigma_{i+1}^2) \\ & + \frac{1}{2}\sigma_i \Delta B_i + \frac{1}{2}\sigma_{i+1}\Delta B_{i+1}. \end{aligned} \tag{6}$$

The terms holding the Brownian increments ΔB have null expectations. Thus the following predictor $\widehat{y}_{i+1}$ of the observation y_t at $t = t_{i+1}$ is unbiased:

$$\widehat{y}_{i+1} = y_i + \mu(t_{i+1} - t_i) - \frac{(t_{i+1} - t_i)}{4}(\sigma_i^2 + \sigma_{i+1}^2). \tag{7}$$

The sense of this choice is that the best model will cause the drift to capture the main course line of the dynamics to the detriment of the diffusion part. Having such a predictor, the estimate of σ_{i+1} (σ_t at $t = t_{i+1}$) that minimizes the mean square prediction error is computed in a recursive way using a stochastic gradient algorithm, the so-called least mean squares algorithm abbreviated to LMS. In this context, the LMS minimizes at each discretization time the following criterion J:

$$J^{(i)} = (y_i - \widehat{y}_i)^2, \tag{8}$$

using a gradient optimization formula:

$$\widehat{\sigma}_{i+1} = \widehat{\sigma}_i - \lambda \frac{\partial J}{\partial \sigma_i}\bigg|_{\sigma_i = \widehat{\sigma}_i}. \tag{9}$$

The resulting formulae are ordered as follows:

$$\begin{aligned} \widehat{\sigma}_{i+1}^{(1)} &= \widehat{\sigma}_i^{(1)}\left(1 - \lambda(y_i - \widehat{y}_i)(t_{i+1} - t_i)\right), \\ \widehat{y}_{i+1} &= y_i + \mu(t_{i+1} - t_i) - \frac{(t_{i+1} - t_i)}{4}\left(\widehat{\sigma}_i^2 + \left[\widehat{\sigma}_{i+1}^{(1)}\right]^2\right), \\ \widehat{\sigma}_{i+1} &= \widehat{\sigma}_i\left(1 - \lambda(y_{i+1} - \widehat{y}_{i+1})(t_{i+1} - t_i)\right). \end{aligned} \tag{10}$$

Initial values $\widehat{y}_0$, $\widehat{\sigma}_0^{(1)}$ and $\widehat{\sigma}_0$ are taken nonstrictly null but arbitrarily small. As usual when using an LMS algorithm, it is the parameter λ that is responsible for the robustness and the right track [16].

3. ILLUSTRATION

In order to show the performance of the proposed method, different models for the volatility are considered. A constant volatility, for example, is useful in order to evaluate the performance in terms of residual error. A volatility sample path as a step function is interesting in order to evaluate the influence of the initial value on convergence. In addition, it has been widely documented that there is a systematic pattern in average volatility; where this is the case, we will show how estimation of the periodic component of the volatility is feasible. Furthermore, the volatility is modeled as a stochastic process, the solution for an SDE of Vasicek. Finally, we apply our method to real data: the German

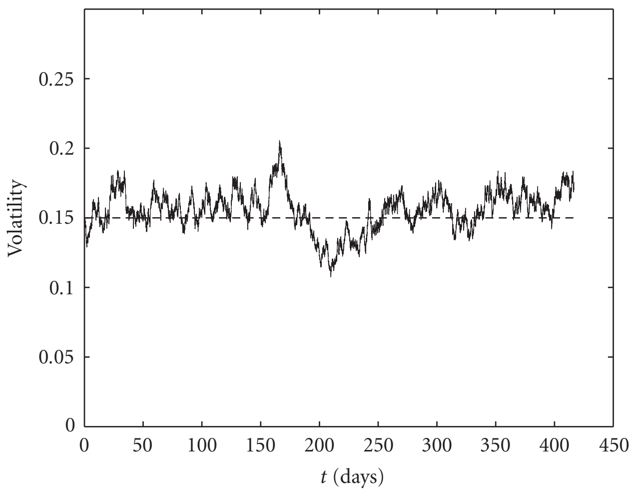

FIGURE 1: True constant volatility (dashed) versus its estimate (continuous).

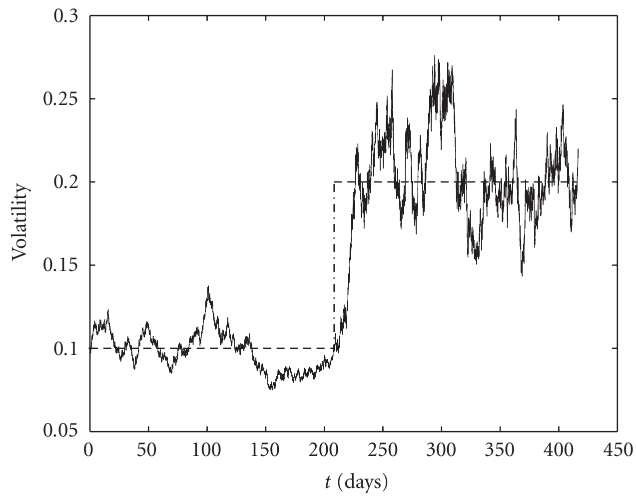

FIGURE 2: True volatility (dashed) versus its estimate (continuous).

electricity price observed each hour from the 1st of July 2000 to the 30th of June 2001 and the daily price of the Hang Seng index of the Hong Kong market from 1995 to 2007. It is worth noting that in all illustrative synthetic examples of this paragraph, the parameters can be chosen arbitrarily. The only essential thing to account for are realistic values of the volatility.

The proposed method is compared with the UKF for, first, the case of a periodic function of time, and second the case of a "synthetic" stochastic process. UKF is based on a state which has the unobservable volatility process as one of its components. UKF equations of the time and measurement updates for the first moment μ of the conditional density are, respectively,

$$\begin{aligned}\mu(t_{i+1}|t_i) &= \mu(t_i|t_i) + \mathbb{E}(F_i)(t_{i+1} - t_i),\\ \mu(t_{i+1}|t_{i+1}) &= \mu(t_{i+1}|t_i),\end{aligned} \tag{11}$$

$\mathbb{E}$standing for the mathematical expectation. UKF, thus, does not update prior estimates $\mu(t_{i+1}|t_i)$, and consequently it is not able to track time-varying volatility. Similar behavior is exhibited in [10, 11].

Next, a comparison is made between the above method and an offline estimation of the volatility. The latter was proposed in [12] which deals with the construction of a black-box continuous time model for the squared volatility process in the form of a stochastic differential equation. The starting point in this construction is a parametric form for the covariance function of this process. The model frame derives from a control theory technique known as the shaping filter. We give a brief account of the work presented in [12] and show that our present study outperforms it.

As regards observations, they are made according to both periodic and nonperiodic sampling schemes. For instance, the case of jitter sampling, as in [15], is considered in Section 3.2. The obtained performance is as good as that of a periodic sampling scheme.

3.1. Constant volatility

The observations are simulated with a volatility of 0.15. The initial value of the volatility, in the proposed method, is deliberately taken equal to the true value (= 0.15) so that we evaluate the residual estimation error. A periodic sampling scheme has been used. The result is reported in Figure 1. Both the mean value and the standard deviation of the relative error of estimation are about 1% and 6%, respectively. They are calculated by time averaging since the volatility value is constant along its trajectory.

3.2. Volatility as a step function

In order to illustrate the convergence behavior of the proposed method, a step function with the initial value of 0.1 and the final value of 0.2 is taken as the volatility sample path. The proposed method is implemented with an initial value of 0.1 for the volatility. A jitter sampling scheme has been used with maximum value of half the sampling period. Many simulations have been carried out with different values of λ; the value 0.04 for λ makes a good tradeoff between robustness and right track. The result is reported in Figure 2; it shows the capability of the algorithm to follow rapid variations even for nonuniformly sampled data. Both the mean value and the standard deviation of the relative error of estimation are about 1% and 10%, respectively. Here again they are calculated by time averaging; this is legitimate since there is piecewise repetition of the volatility value along its trajectory. To explore further the performance evaluation of this result, we have computed the Theil index. It is approximately $3\cdot 10^{-5}$. The Theil index formula is

$$\text{Theil} = \frac{1}{N}\sum_{\text{samples}} \frac{\sigma_{\text{est}}}{\sigma_{\text{ref}}} \log\left(\frac{\sigma_{\text{est}}}{\sigma_{\text{ref}}}\right). \tag{12}$$

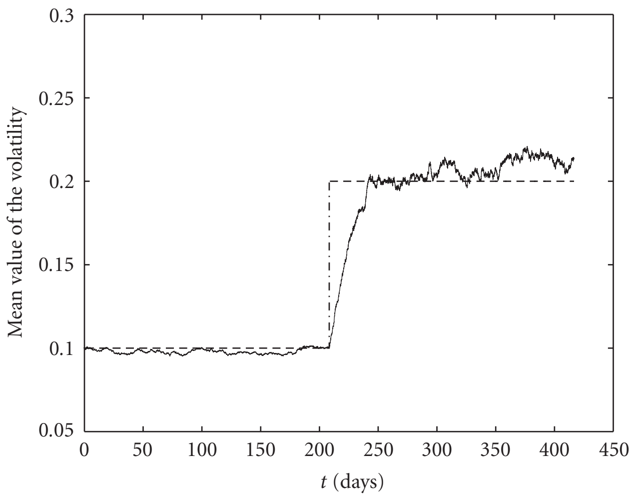

Figure 3: True volatility (dashed) versus the mean for 100 of its estimates (continuous).

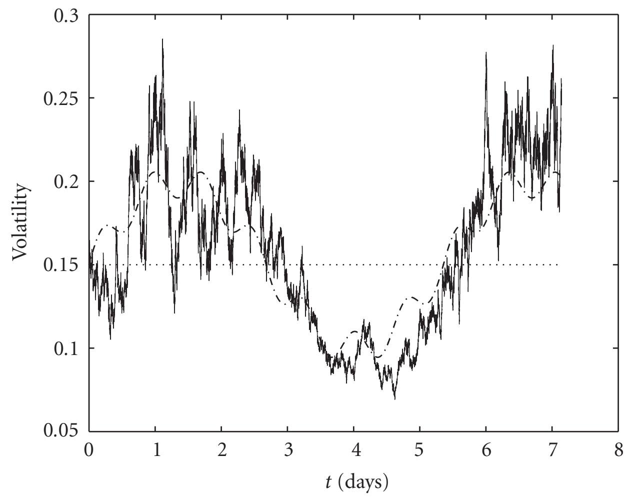

Figure 4: True (dashed) versus estimated volatility: proposed method (continuous), UKF (dotted).

Here N is the number of samples in the reference trajectory to be estimated. σ_{est} is the estimate of the volatility σ_t at $t = t_{i+1}$, denoted by $\hat{\sigma}_{i+1}$ in Section 2, and σ_{ref} is the reference: the (true) volatility σ_t at $t = t_{i+1}$, denoted σ_{i+1} $(i = 0, \dots, N-1)$.

In addition, Monte Carlo simulations have been carried out: the mean sample path for 100 estimated trajectories of the volatility is reported in Figure 3. The mean value and the standard deviation of its relative error of estimation are about 1% and 5%, respectively. This shows that the standard deviation of the estimation error drops significantly as the simulation number increases. That is, as expected, the empirical mean sample paths are to converge to the true mean.

As has been said in the introduction to Section 3, the parameters can be chosen arbitrarily within all synthetic examples. The only essential thing to take into consideration is the realistic values of the volatility. The general validity of our method should thus be studied by varying these parameters. They are the initial and the final values of the step function in the context of this subsection. Column 1 in Table 1 shows initial values of three different step functions; column 2 shows their corresponding final values. Columns 3 and 4 show the mean value and the standard deviation of the relative error of estimation obtained by Monte Carlo simulations (25 estimated trajectories of the volatility for each couple of parameters). The last two columns show the mean Theil index of the 25 estimated trajectories of the volatility using our method versus the Theil index of UKF for each couple of parameters.

3.3. Volatility as a deterministic periodic function of time

Whenever the volatility is subject to seasonality, we wish to recover the season(s) using our method. We consider the following deterministic function of time for the volatility trajectory:

$$\sigma(t) = a_0 + a_1 \sin(\omega_1 t) + a_2 \sin(\omega_2 t). \quad (13)$$

The pulsations ω_1 and ω_2 correspond to a one-week and a one-day seasonality; this is, for instance, the case of German electricity price treated in 3.6. a_0, a_1 and a_2 are chosen so as to have realistic values of the volatility. In the simulation of Figure 4, they are 0.15, 0.05, and 0.01, respectively.

Both the true volatility and its estimate for a periodic sampling scheme, and for λ of 0.07, are plotted in Figure 4. The estimated volatility using UKF is constant, yet the proposed method is able to track the volatility oscillations. The Theil index is about 10^{-3}; UKF yields a Theil index of 10^{-2}. The mean value and the standard deviation of the relative error are about 1% and 16%, respectively; they are calculated by time averaging. To justify this, we do check error ergodicity. This is done by fixing an instant, repeating again the simulation several times with respect to the same volatility trajectory till this instant. The mean value and the standard deviation of the relative error for this instant are obtained by averaging on simulations. Their values are in the order of what is given above. Besides, we proceed likewise in the following (the mean value and the standard deviation of the relative error are to be calculated by time averaging).

The mean trajectory of 100 estimated trajectories of the volatility is reported in Figure 5. The mean value and the standard deviation of its relative error of estimation are about 1% and 8%, respectively. In addition, the power spectral densities (PSD) for the true volatility sample path and the mean of its estimates are confronted in Figure 6; the two PSDs therein are clearly close to each other.

We furthermore vary the parameters a_1 and a_2 and perform Monte Carlo simulations (100 estimated trajectories of the volatility for each couple of parameters) so that we obtain the results in Table 2.

Table 1

Initial value	Final value	Relative error mean	Relative error StD	Theil index	Theil index UKF
0.1	0.2	$-0.2\cdot 10^{-5}$	0.05	$2\cdot 10^{-3}$	0.2
0.05	0.25	$0.6\cdot 10^{-4}$	0.1	$4\cdot 10^{-2}$	1.49
0.01	0.29	$-1.5\cdot 10^{-4}$	0.06	$1.5\cdot 10^{-2}$	20

Table 2

a_1	a_2	Relative error mean	Relative error StD	Theil index	Theil index UKF
0.05	0.01	0.015	0.08	10^{-4}	$3\cdot 10^{-2}$
0.1	0.01	$2\cdot 10^{-2}$	0.1	$2\cdot 10^{-3}$	0.4

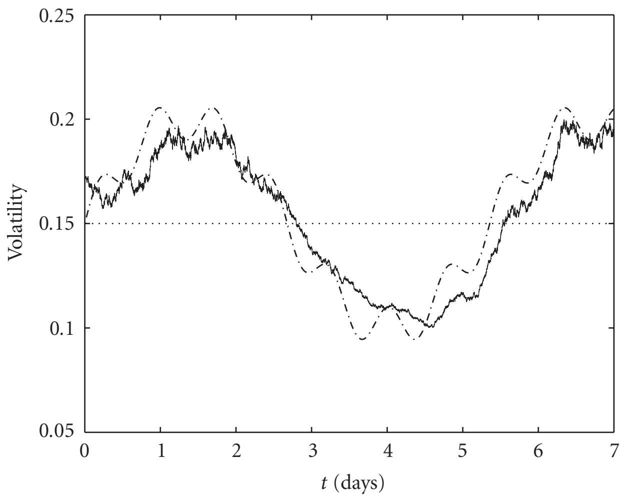

Figure 5: True volatility (dashed) versus the mean for 100 of its estimates (continuous).

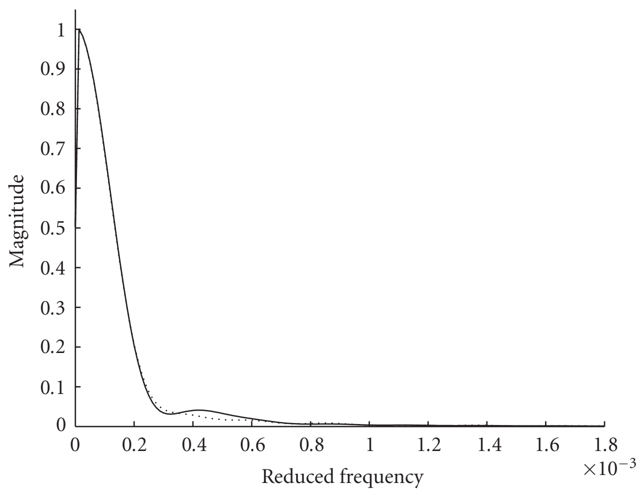

Figure 6: PSD of the true volatility (continuous) and that of its estimate (dotted).

3.4. Volatility as a stochastic process

To synthesize sample paths of the volatility process as well as the stock price, the following SDE of Vasicek is considered:

$$d\sigma_t = \alpha(\theta - \sigma_t)dt + \xi dB_t, \quad (14)$$

where $\alpha = 0.0001$, $\theta = 0.15$, and $\xi = 0.0007$. We assume the drift μ is known ($\mu = 0.015$). The true volatility sample path and the estimated one, using both the proposed method and UKF, are reported in Figure 7. The volatility is estimated at every half hour for 416 days. For this simulation, we choose the initial value of the volatility equal to $\theta(= 0.15)$. As above, the estimated volatility using UKF is constant. The proposed method, however, is able to track the volatility fluctuations.

The empirical distribution of the estimation error for the sample path in Figure 7 is reported in Figure 8. Like UKF, the proposed method is subject to bias, but the bias is clearly smaller. The standard deviation obtained with UKF is 0.033, whereas within the proposed method, it is 0.015.

3.5. Illustration using real data

Figure 9 shows the daily price of the Hang Seng index of the Hong Kong market from 1995 to 2007. This sample path exhibits a volatility clustering phenomenon: periods of high-price fluctuations are followed by periods of high fluctuations, and the same can be said about periods of low-price fluctuations. The implementation result on this sample path is shown in Figure 10. Notice the beginning of a period of high volatility around the 700th day; this corresponds to the Asian financial crisis of October 1997.

3.6. Comparison with offline estimation of the volatility

We assume prior information about the unknown process $(\sigma_t)^2$: its stationarity in the large sense and a parametric model for its covariance function. Let the covariance function of the process $(\sigma_t)^2$ be given by the following formula:

$$k(\tau) = De^{-\alpha|\tau|} \quad \alpha > 0, \quad (15)$$

where D is the process variance. This type of covariance function allows one to fit the observed time dependence in the returns. Such a covariance function includes memory in the correlation pattern of the volatility. The spectral density of $(\sigma_t)^2$ is then given by the formula

$$s(\omega) = \frac{1}{2\pi}\frac{2D\alpha}{\omega^2 + \alpha^2}. \quad (16)$$

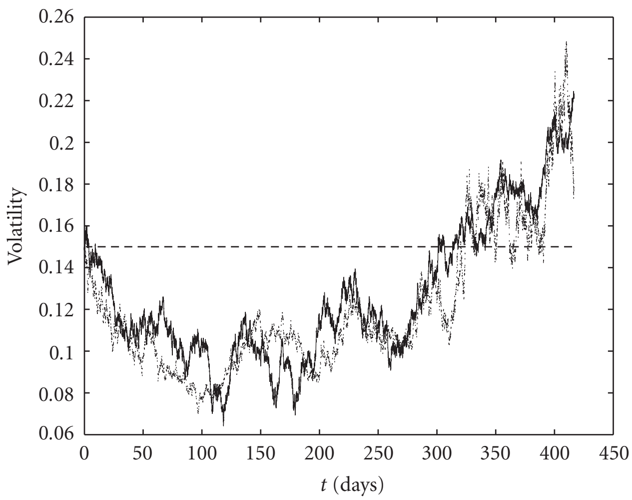

FIGURE 7: True (continuous) versus estimated volatility sample path: proposed method (dotted), UKF (dashed).

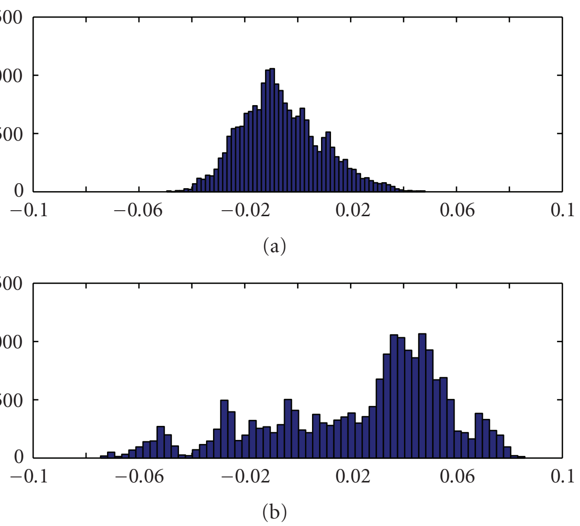

FIGURE 8: Empirical distribution for the estimation error. (a) The proposed method, (b) UKF.

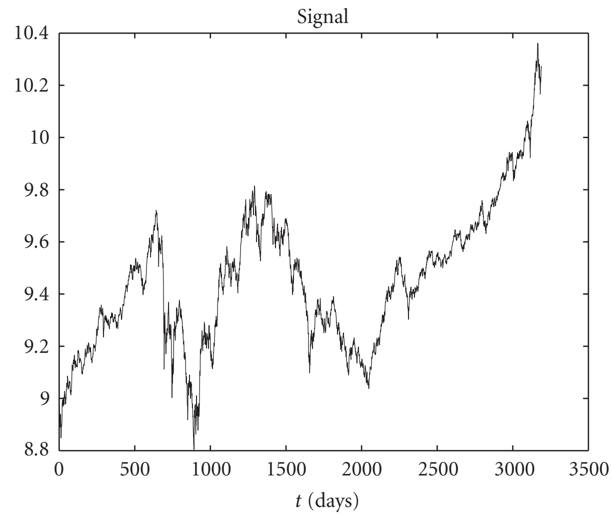

FIGURE 9: Log-price of the Hang Seng index.

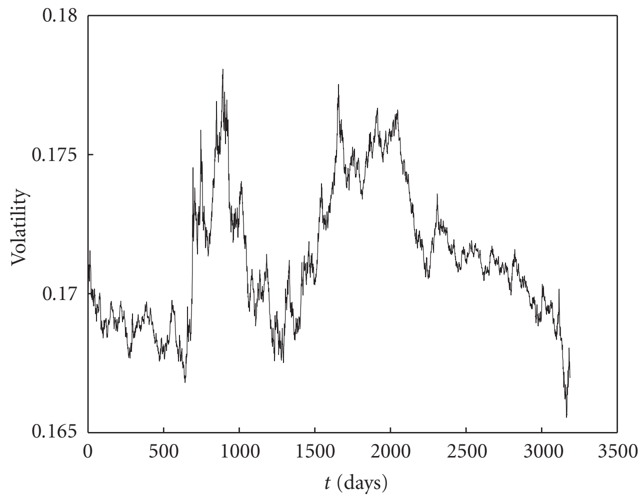

FIGURE 10: Estimated volatility sample path.

The spectral density $s(\omega)$ is rewritten as

$$s(\omega) = \frac{1}{2\pi}\left|\frac{H(j\omega)}{F(j\omega)}\right|^2, \quad \omega \in \mathbb{R}, \tag{17}$$

where

$$H(j\omega) = \sqrt{2D\alpha}, \qquad F(j\omega) = j\omega + \alpha. \tag{18}$$

Now

$$\Phi(s) = \frac{H(s)}{F(s)}, \quad s \in C, \tag{19}$$

represents the transfer function of a stationary linear system; the system is, furthermore, stable as the root of $F(s)$ is in the left half-plane of the complex variable s. Recalling that $1/2\pi$ is the spectral density of a white noise of intensity 1, we come to the following conclusion. $(\sigma_t)^2$ may be considered as the response of the filter whose transfer function is $\Phi(s)$ to a white noise with unit intensity. From the ordinary differential equation describing such a filter, we obtain the following stochastic differential equation as a model for the squared volatility process $(\sigma_t)^2$. This is the first state component denoted by X_t^1:

$$\begin{aligned} dX_t^1 &= X_t^2 dt, \\ dX_t^2 &= -\alpha X_t^1 dt - \sqrt{2D\alpha} dW_t. \end{aligned} \tag{20}$$

Here, W is a stochastic process with independent and stationary increments of intensity 1. If we suppose that W starts at 0 and that its trajectories are continuous in probability, then we can give it the name Lévy process. We suppose further the existence of stationary solutions to the SDE when W has positive increments so as to assure the

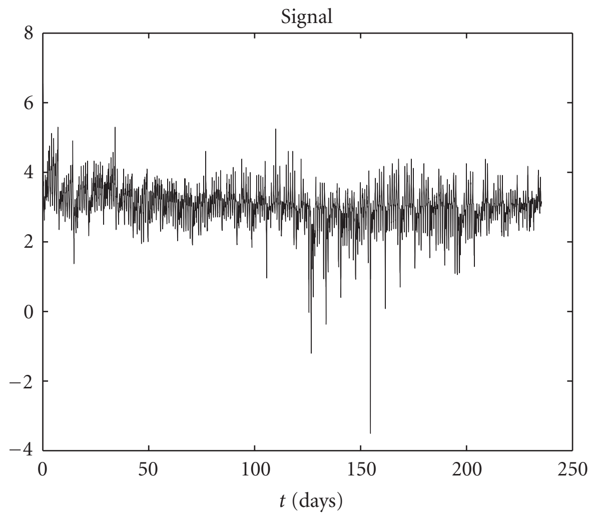

Figure 11: Log-price of the electricity.

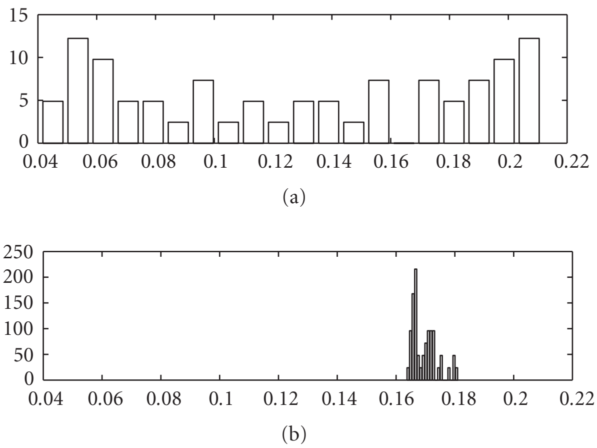

Figure 12: Histograms of online volatility estimate (b) and an offline one (a).

positivity of X_t^1. According to the above notation, (2) is rewritten in the form

$$dy_t = \left(\mu - \frac{X_t^1}{2}\right)dt + \sqrt{X_t^1}\,dB_t. \tag{21}$$

We suppose that the condition in the proposition of paragraph 4 of [12] applies, which ensures that (20) has stationary solutions. We then calibrate the model (20)-(21) on the observations from which seasonality has been removed. The calibration is based upon stochastic calculus and the Lévy processes theory.

First, we apply the above offline method to electricity price; observations of the German market for each hour from the 15th of June 2000 to the 31st of December 2003 are processed. Figure 11 shows the asset log-price trajectory. The obtained variance D and rate α amount to around $2.98 \cdot 10^{-6}$ and 0.03, respectively. Figure 12 displays two histograms: at the top is the histogram of the sample path of the volatility process obtained from the above method, at the bottom is

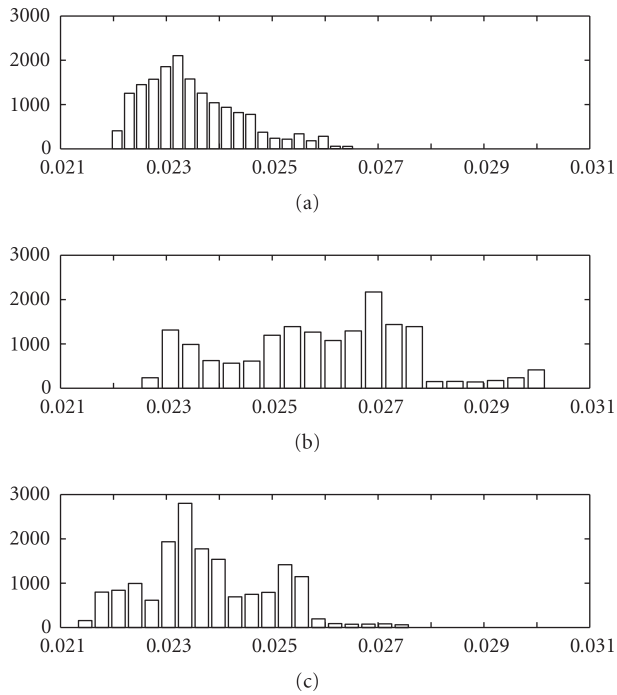

Figure 13: Histogram of sample paths for the true volatility (a), histogram of the offline volatility estimate (b), and histogram of the online volatility estimate (c).

the histogram of the volatility sample path estimated by the main method of the paper. Second, since volatility is actually impossible to observe, showing only an application of the online method on real data is not ideal for a comparison with the offline method of this subsection. We compare the two methods on the synthetic stochastic process of Section 3.4; this is shown in Figure 13 below.

4. CONCLUSION

Evidence to date suggests that stochastic volatility models for market prices are likely to be useful in practice. A real-time estimation algorithm of the volatility when observing the market asset price is proposed. The obtained estimate shows a clear improvement of precision when compared with the unscented Kalman filter. The proposed method inherits a low computational cost from LMS algorithms. Our algorithm has a complexity of 9 elementary operations per sample. It outperforms the offline method inasmuch as it does not require any effort to transform data, for example, to take seasonality off. This, on the other hand, was necessary in the method of the previous subsection.

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