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Research Article

New Inequalities and Uncertainty Relations on Linear Canonical Transform Revisit

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The uncertainty principle plays an important role in mathematics, physics, signal processing, and so on. Firstly, based on definition of the linear canonical transform (LCT) and the traditional Pitt's inequality, one novel Pitt's inequality in the LCT domains is obtained, which is connected with the LCT parameters a and b. Then one novel logarithmic uncertainty principle is derived from this novel Pitt's inequality in the LCT domains, which is associated with parameters of the two LCTs. Secondly, from the relation between the original function and LCT, one entropic uncertainty principle and one Heisenberg's uncertainty principle in the LCT domains are derived, which are associated with the LCT parameters a and b. The reason why the three lower bounds are only associated with LCT parameters a and b and independent of c and d is presented. The results show it is possible that the bounds tend to zeros.

1. Introduction

The uncertainty principle is one elementary principle in signal processing [1–10] and physics [11–13]. For one given function $f(t) \in L^1(R) \cap L^2(R)$ (without loss of generalization, assuming $\|f(t)\|_2 = 1$ in the following of this paper) and its Fourier transform (FT) $F(u)$, the product has the lowest bound

$$\Delta t^2 \cdot \Delta u_{cl}^2 = \int_{-\infty}^{+\infty} \left|(t-t_0)f(t)\right|^2 dt \cdot \int_{-\infty}^{+\infty} \left|(u-u_0)F(u)\right|^2 du \geq \frac{1}{4}, \tag{1}$$

where $t_0 = \int_{-\infty}^{+\infty} t|f(t)|^2 dt$, $u_0 = \int_{-\infty}^{+\infty} u|F(u)|^2 du$, Δt^2 is time spread, and Δu_{cl}^2 is frequency spread. Let $t_0 = 0$ (in our paper for given $f(t)$ we assume $t_0 \equiv 0$) and $u_0 = 0$, and the essence of uncertainty principle will not change [1–10] . However (1) can be written as

$$\Delta t^2 \cdot \Delta u_{cl}^2 = \int_{-\infty}^{+\infty} \left|tf(t)\right|^2 dt \cdot \int_{-\infty}^{+\infty} \left|uF(u)\right|^2 du \geq \frac{1}{4}. \tag{2}$$

In this paper we will give three uncertainty principles in the LCT domains: one logarithmic uncertainty principle based on Pitt's inequality [14–16]; one entropic uncertainty principle; one Heisenberg's uncertainty principle. Note that some of our results of this article are the extension and generality of our recent works [17–19], and it is likely that there is part of similarity in the process of derivation. However, the results of this paper and most of the derivation are different and novel. First, Heisenberg's uncertainty in the recent works, such as [18–22], has been involved. However, the results of [18, 22] only hold true for the real signals (not for complex signals). In addition, the result of [22] is only the first one of the three cases in [18]. In [19], Pitt's inequality and logarithmic uncertainty principle on LCT have not been involved. Moreover, the derivations here are different from that in [19]. On the other hand, the results in [20, 21] are only some special cases of those in [18, 19, 22] for special parameters.

The LCT is taken as the generalization of the FRFT and the Fresnel transform and has been widely studied and applied [9, 23–27] up till now. As a generalization of the traditional FT and the FRFT, the LCT has some properties with its transformed parameter. For more details, see [9, 23–27] and so forth. We now briefly review its definition and some basic properties.

For given function $f(t) \in L^1(R) \cap L^2(R)$ and $\|f(t)\|_2 = 1$ (in this article supposing this always holds), its definition of the LCT [9] is

$$F_{(a,b,c,d)}(u) = F_{(a,b,c,d)}(f(t)) = \int_{-\infty}^{\infty} f(t) K_{a,b,c,d}(u,t)\,dt$$
$$= \begin{cases} \sqrt{\dfrac{1}{i2\pi b}} \cdot e^{idu^2/2b} \displaystyle\int_{-\infty}^{\infty} e^{-iut/b} e^{iat^2/2b} f(t)\,dt & b \neq 0,\ ad - bc = 1 \\ \sqrt{d} \cdot e^{icdu^2/2} f(du), & b = 0, \end{cases} \tag{3}$$

where $a, b, c, d \in R$.

From the definition, it is easily found that

$$F_{(a_2,b_2,c_2,d_2)}\{F_{(a_1,b_1,c_1,d_1)}(f(t))\} = F_{(a,b,c,d)}(f(t)), \tag{4}$$

where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, and i is complex unit.

For traditional FT which is a special case of $(a, b, c, d) = (0, 1, -1, 0)$, we have

$$F_{(0,1,-1,0)}(u) = F(u) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iut}\,dt,$$
$$f(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F(u) e^{iut}\,du. \tag{5}$$

This paper is organized as follows. Section 2 yields the novel Pitt's inequality and the logarithmic uncertainty principle in the LCT domains. In Section 3 one novel entropic uncertainty principle is derived. In Section 4 Heisenberg's uncertainty principle is obtained. Finally, Section 5 concludes our paper.

2. New Pitt's Inequality and Logarithmic Uncertainty Principle on LCT

Inequalities [3, 14–16, 28, 29] are a basic tool in the study of Fourier analysis or information theory, and many important theorems or principles are derived from them. One of them is the Pitt's inequality by Beckner [14–16]:

$$\int_{-\infty}^{\infty} |u|^{-\lambda} |F(u)|^2\,du \leq M_\lambda \int_{-\infty}^{\infty} |t|^{\lambda} |f(t)|^2\,dt, \tag{6}$$

where $M_\lambda = [\Gamma((1-\lambda)/4)/\Gamma((1+\lambda)/4)]^2$, $0 \leq \lambda < 1$, $F(u) = \sqrt{1/2\pi} \int_{-\infty}^{\infty} f(t) e^{-iut}\,dt$.

First we assume $a_l, b_l, c_l, d_l \in \mathbf{R}$ and $b_l \neq 0$ $(l = 1, 2, 3)$.

Set

$$G(u) = F_{(a_1,b_1,c_1,d_1)}(u) \exp\left(-i\frac{d_3 u^2}{2b_3}\right),$$
$$F_{(a_1,b_1,c_1,d_1)}(u) = F_{(a_1,b_1,c_1,d_1)}(f(t)), \tag{7}$$
$$g(t) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(u) e^{iut}\,du.$$

Noting the fact that $|F_{(a_1,b_1,c_1,d_1)}(u)\exp(-id_3u^2/2b_3)| = |F_{(a_1,b_1,c_1,d_1)}(u)|$ holds, we easily obtain

$$\int_{-\infty}^{\infty} |u|^{-\lambda} |G(u)|^2\,du = \int_{-\infty}^{\infty} |u|^{-\lambda} \left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2 du. \tag{8}$$

From (6) and (8), we have

$$\int_{-\infty}^{\infty} |u|^{-\lambda} \left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2 du \leq M_\lambda \int_{-\infty}^{\infty} |t|^{\lambda} |g(t)|^2\,dt. \tag{9}$$

Noting $g(t)$, we have

$$\int_{-\infty}^{\infty} |t|^{\lambda} |g(t)|^2\,dt = \int_{-\infty}^{\infty} \left|\frac{t}{b_3}\right|^{\lambda} \left|g\left(\frac{t}{b_3}\right)\right|^2 d\frac{t}{b_3}$$
$$= \frac{1}{|b_3|^{\lambda+1}} \int_{-\infty}^{\infty} |t|^{\lambda} \left|g\left(\frac{t}{b_3}\right)\right|^2 dt. \tag{10}$$

Here from the definition of FT we have

$$\left|g\left(\frac{t}{b_3}\right)\right|^2 = \left|\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(u) e^{iut/b_3}\,du\right|^2. \tag{11}$$

Substituting $F_{(a_1,b_1,c_1,d_1)}(u) e^{-id_3u^2/2b_3}$ for $G(u)$ in (11) and using definition (3), we get

$$\left|g\left(\frac{t}{b_3}\right)\right|^2$$
$$= \left|\sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F_{(a_1,b_1,c_1,d_1)}(u) e^{-id_3u^2/2b_3} e^{iut/b_3}\,du\right|^2$$
$$= \left|\frac{\sqrt{-1/2ib_3\pi} \int_{-\infty}^{\infty} F_{(a_1,b_1,c_1,d_1)}(u)\, e^{-id_3u^2/2b_3} e^{iut/b_3} e^{-ia_3t^2/2b_3}\,du}{\exp(-ia_3t^2/2b_3)\sqrt{-1/ib_3}}\right|^2$$
$$= |b_3| \left|\sqrt{\frac{-1}{2ib_3\pi}} \int_{-\infty}^{\infty} F_{(a_1,b_1,c_1,d_1)}(u) e^{-id_3u^2/2b_3} e^{iut/b_3} e^{-ia_3t^2/2b_3}\,du\right|^2$$
$$= |b_3| \left|F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(t)\right|^2. \tag{12}$$

Thus we obtain

$$\int_{-\infty}^{\infty} |t|^{\lambda} |g(t)|^2\,dt$$
$$= \frac{1}{|b_3|^{\lambda}} \int_{-\infty}^{\infty} |t|^{\lambda} \left|F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(t)\right|^2 dt. \tag{13}$$

Set $t = \nu$, namely,

$$\int_{-\infty}^{\infty} |u|^{-\lambda} \left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2 du$$
$$\leq \frac{M_\lambda}{|b_3|^{\lambda}} \int_{-\infty}^{\infty} |\nu|^{\lambda} \left|F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(\nu)\right|^2 d\nu. \tag{14}$$

Let $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} d_3 & -b_3 \\ -c_3 & a_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ have

$$\begin{gathered} F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}(v)\right) = F_{(a_2,b_2,c_2,d_2)}(v) \\ b_3 = -a_1 b_2 + a_2 b_1. \end{gathered} \tag{15}$$

Comparing (14) with (15), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |u|^{-\lambda} \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &\quad \le \frac{M_\lambda}{|a_1 b_2 - a_2 b_1|^\lambda} \int_{-\infty}^{\infty} |v|^\lambda \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 dv. \end{aligned} \tag{16}$$

We can draw the conclusion that (16) is one extended Pitt's inequality in the LCT domains. It is easily found that this inequality is associated with LCT parameter a, b. Why do not the parameters c, d have relation with the extended Pitt's inequality in the LCT domains? From definition (3) of the LCT, we find that the parameters c, d only play the role of scaling and modulation. That the modulation has no effect on our (16) has been found from (8) and (12) directly. From the property $F_{(a,b,c,d)}(\sqrt{\rho} f(t/\rho)) = F_{(a\rho,b/\rho,c\rho,d/\rho)}(f(t))$, we can easily find that scaling also has no effect on (16).

From definition (1) when $(a_1, b_1, c_1, d_1) = (0, 1, -1, 0)$ and $(a_2, b_2, c_2, d_2) = (1, 0, 0, 1)$, (16) reduces to (6). When $(a_1, b_1, c_1, d_1) = (1, 0, 0, 1)$ and $(a_2, b_2, c_2, d_2) = (0, 1, -1, 0)$, (16) reads

$$\int_{-\infty}^{\infty} |t|^{-\lambda} \left| f(t) \right|^2 dt \le M_\lambda \int_{-\infty}^{\infty} |u|^\lambda |F(u)|^2 du. \tag{17}$$

Clearly, (17) is the other version of traditional Pitt's inequality. This is easily explained from the fact that $f(t)$ is also the FT of $F(u)$.

Particularly, if $\lambda = 0$, from (16) we can get Parseval's equality [9] associated with the LCT:

$$\int_{-\infty}^{\infty} \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du = \int_{-\infty}^{\infty} \left| F_{(a_2,b_2,c_2,d_2)}(u) \right|^2 dt. \tag{18}$$

In the following, we will achieve one logarithmic uncertainty principle in the LCT domains.

Set $S(\lambda) = |a_1 b_2 - a_2 b_1|^\lambda \int_{-\infty}^{\infty} |u|^{-\lambda} |F_{(a_1,b_1,c_1,d_1)}(u)|^2 du - M_\lambda \int_{-\infty}^{\infty} |v|^\lambda |F_{(a_2,b_2,c_2,d_2)}(v)|^2 dv$.

Then we have

$$\begin{aligned} S'(\lambda) = {}& |a_1 b_2 - a_2 b_1|^\lambda \ln(|a_1 b_2 - a_2 b_1|) \\ &\times \int_{-\infty}^{\infty} |u|^{-\lambda} \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &- |a_1 b_2 - a_2 b_1|^\lambda \\ &\times \int_{-\infty}^{\infty} |u|^{-\lambda} \ln(|u|) \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &- M_\lambda \int_{-\infty}^{\infty} |v|^\lambda \ln(|v|) \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 dv \\ &- (M_\lambda)' \int_{-\infty}^{\infty} |v|^\lambda \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 dv, \end{aligned} \tag{19}$$

where $(M_\lambda)' = (-(1/2)\Gamma((1-\lambda)/4)\Gamma'((1-\lambda)/4)\Gamma^2((1+\lambda)/4) - (1/2)\Gamma((1+\lambda)/4)\Gamma'((1+\lambda)/4)\Gamma^2((1-\lambda)/4))/\Gamma^4((1+\lambda)/4)$.

Since $S(\lambda) \le 0$ when $0 \le \lambda < 1$ and the fact $S(0) = 0$ and $\int_{-\infty}^{\infty} |F_{(a_1,b_1,c_1,d_1)}(u)|^2 du = \int_{-\infty}^{\infty} |F_{(a_2,b_2,c_2,d_2)}(v)|^2 dv = 1$, we obtain the following inequality in mathematics [11, 30].

$$S'(0+) \le 0. \tag{20}$$

Namely,

$$\begin{aligned} &\int_{-\infty}^{\infty} \ln|u| \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &\quad + \int_{-\infty}^{\infty} \ln|v| \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 dv \\ &\quad \ge \ln|a_1 b_2 - a_2 b_1| + \frac{\Gamma'(1/4)}{\Gamma(1/4)}. \end{aligned} \tag{21}$$

From (21), we have

$$\begin{aligned} &\int_{-\infty}^{\infty} \ln|u|^2 \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &\quad + \int_{-\infty}^{\infty} \ln|v|^2 \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 dv \\ &\quad \ge \ln\left(|a_1 b_2 - a_2 b_1|^2\right) + \frac{2\Gamma'(1/4)}{\Gamma(1/4)}. \end{aligned} \tag{22}$$

Clearly, the bound of the inequality (21) (or (22)) is connected with the LCT parameters a and b and independent of c and d.

If

$$\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} \vartheta & \vartheta - 1 \\ 1 & 1 \end{bmatrix} \tag{23}$$

where

$$\vartheta = \sqrt{-\frac{2\Gamma'(1/4)}{\Gamma(1/4)}}, \tag{24}$$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \tag{25}$$

$$\ln\left(|a_1 b_2 - a_2 b_1|^2\right) + (2\Gamma'(1/4))/(\Gamma(1/4)) = 0. \tag{26}$$

It means that the bound of this inequality may be zero.

When $(a_1, b_1, c_1, d_1) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$ and $(a_2, b_2, c_2, d_2) = (\cos\beta, \sin\beta, -\sin\beta, \cos\beta)$, (22) reads

$$\begin{aligned} &\int_{-\infty}^{\infty} \ln|u|^2 |F_\alpha(u)|^2 du + \int_{-\infty}^{\infty} \ln|v|^2 \left| F_\beta(v) \right|^2 dv \\ &\quad \ge \ln\left(|\sin(\alpha - \beta)|^2\right) + \frac{2\Gamma'(1/4)}{\Gamma(1/4)}. \end{aligned} \tag{27}$$

In comparison with Heisenberg's uncertainty principle (28) in two fractional Fourier transform domains [1, 5, 7]:

$$\begin{aligned} &\int_{-\infty}^{\infty} |u|^2 |F_\alpha(u)|^2 du \int_{-\infty}^{\infty} |v|^2 \left| F_\beta(v) \right|^2 dv \\ &\quad \ge \frac{|\sin(\alpha - \beta)|^2}{4} \end{aligned} \tag{28}$$

we find that there is one common term $|\sin(\alpha-\beta)|^2$ in (27) and (28). This tells us that in new transformed domains the new uncertainty principles have relations with the transform parameters. When $(a_1,b_1,c_1,d_1)=(1,0,0,1)$ and $(a_2,b_2,c_2,d_2)=(0,1,-1,0)$, (22) reads $\int_{-\infty}^{\infty}\ln|t||f(t)|^2dt+\int_{-\infty}^{\infty}\ln|u||F(u)|^2du \geq \Gamma'(1/4)/\Gamma(1/4)$, which is the traditional logarithmic uncertainty principle by Beckner [16].

3. Entropy and Entropic Uncertainty Principle on LCT

The entropy is introduced by Shannon [31], and it has become one of the most important measures in information theory. The entropy has been widely used in many fields such as physics, communication, mathematics, signal analysis, and so forth.

The entropy is defined [31, 32] by

$$E(\rho(x)) = -\int_{-\infty}^{\infty}\rho(x)\ln\rho(x)dx, \tag{29}$$

where $\rho(x)$ is the probability density function of the variablex.

The entropic uncertainty principle plays one important role in signal processing and information theory. They are the extensions of traditional Heisenberg's uncertainty principle from time-frequency analysis to information theory and physical quantum. The traditional entropic uncertainty principle have been discussed in many papers such as [6, 10–13]. However, up till now there is no published paper covering the entropic uncertainty principle connected with the LCT. The traditional entropic uncertainty principle is described [6, 11–13] as

$$-\int_{-\infty}^{\infty}|f(t)|^2\ln|f(t)|^2dt-\int_{-\infty}^{\infty}|F(u)|^2\ln|F(u)|^2du \geq \ln(\pi e). \tag{30}$$

In the following, based on (30), the entropic uncertainty principle in two LCT domains is derived.

First, similarly we assume $a_l,b_l,c_l,d_l\in\mathbf{R}$ and $b_l\neq 0$ $(l=1,2,3)$.

Set

$$\begin{gathered}G(u)=F_{(a_1,b_1,c_1,d_1)}(u)\exp\left(-i\frac{d_3u^2}{2b_3}\right),\\ F_{(a_1,b_1,c_1,d_1)}(u)=F_{(a_1,b_1,c_1,d_1)}(f(t)),\\ g(t)=\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}G(u)\,e^{iut}du.\end{gathered} \tag{31}$$

Noting the fact that the equation

$$\left|F_{(a_1,b_1,c_1,d_1)}(u)\exp\left(-i\frac{d_3u^2}{2b_3}\right)\right| = |F_{(a_1,b_1,c_1,d_1)}(u)| \tag{32}$$

holds, we easily get

$$\begin{aligned}\int_{-\infty}^{\infty}&|G(u)|^2\ln|G(u)|^2du\\ &=\int_{-\infty}^{\infty}|F_{(a_1,b_1,c_1,d_1)}(u)|^2\ln|F_{(a_1,b_1,c_1,d_1)}(u)|^2du.\end{aligned} \tag{33}$$

From (30) and (33), we have

$$\begin{aligned}&-\int_{-\infty}^{\infty}|g(t)|^2\ln|g(t)|^2dt\\ &-\int_{-\infty}^{\infty}|F_{(a_1,b_1,c_1,d_1)}(u)|^2\ln|F_{(a_1,b_1,c_1,d_1)}(u)|^2du\\ &\geq\ln(\pi e).\end{aligned} \tag{34}$$

Note the property of scaling:

$$\begin{aligned}\int_{-\infty}^{\infty}&|g(t)|^2\ln|g(t)|^2dt\\ &=\frac{1}{|b_3|}\int_{-\infty}^{\infty}\left|g\left(\frac{t}{b_3}\right)\right|^2\ln\left|g\left(\frac{t}{b_3}\right)\right|^2dt.\end{aligned} \tag{35}$$

Thinking about the definition of FT.

$$\left|g\left(\frac{t}{b_3}\right)\right|^2=\left|\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}G(u)e^{iut/b_3}du\right|^2. \tag{36}$$

Similarly with (12), substituting $F_{(a_1,b_1,c_1,d_1)}(u)e^{-id_3u^2/2b_3}$ for $G(u)$ in (36) and using definition (3), we get

$$\begin{aligned}\left|g\left(\frac{t}{b_3}\right)\right|^2&=\left|\sqrt{\frac{1}{2\pi}}\int_{-\infty}^{\infty}F_{(a_1,b_1,c_1,d_1)}(u)e^{-id_3u^2/2b_3}e^{iut/b_3}du\right|^2\\ &=|b_3|\,|F_{(d_3,-b_3,-c_3,a_3)}(F_{(a_1,b_1,c_1,d_1)})(t)|^2.\end{aligned} \tag{37}$$

Thus we obtain

$$\begin{aligned}\int_{-\infty}^{\infty}&|g(t)|^2\ln|g(t)|^2dt\\ &=\frac{1}{|b_3|}\int_{-\infty}^{\infty}\left(|b_3|\,|F_{(d_3,-b_3,-c_3,a_3)}(F_{(a_1,b_1,c_1,d_1)})(t)|^2\right)\\ &\quad\times\ln\left(|b_3|\,|F_{(d_3,-b_3,-c_3,a_3)}(F_{(a_1,b_1,c_1,d_1)})(t)|^2\right)dt.\end{aligned} \tag{38}$$

Set $t=\nu$, then

$$\begin{aligned}&-\frac{1}{|b_3|}\int_{-\infty}^{\infty}\left(|b_3|\,|F_{(d_3,-b_3,-c_3,a_3)}(F_{(a_1,b_1,c_1,d_1)})(\nu)|^2\right)\\ &\times\ln\left(|b_3|\,|F_{(d_3,-b_3,-c_3,a_3)}(F_{(a_1,b_1,c_1,d_1)})(\nu)|^2\right)d\nu\\ &-\int_{-\infty}^{\infty}|F_{(a_1,b_1,c_1,d_1)}(u)|^2\ln|F_{(a_1,b_1,c_1,d_1)}(u)|^2du\geq\ln(\pi e).\end{aligned} \tag{39}$$

Set $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} d_3 & -b_3 \\ -c_3 & a_3 \end{bmatrix} \cdot \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, then we have

$$F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}(v)\right) = F_{(a_2,b_2,c_2,d_2)}(v), \quad b_3 = -a_1 b_2 + a_2 b_1. \tag{40}$$

Comparing (39) with (40), we have

$$\begin{aligned} &-\int_{-\infty}^{\infty} \left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 \ln\left(\left| F_{(a_2,b_2,c_2,d_2)}(v) \right|^2 \right) dv \\ &-\int_{-\infty}^{\infty} \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 \ln \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \\ &\geq \ln\left(\pi e |a_1 b_2 - a_2 b_1|\right). \end{aligned} \tag{41}$$

Namely, $E(|F_{(a_1,b_1,c_1,d_1)}(u)|^2) + E(|F_{(a_2,b_2,c_2,d_2)}(v)|^2) \geq \ln(\pi e|a_1 b_2 - a_2 b_1|)$.

Clearly, the entropic uncertainty principle in the LCT domains (see (41)) is connected with the LCT parameters a and b and independent of c and d. Why do not the parameters c, d have relation with the entropic uncertainty principle in the LCT domains? From definition (3) of the LCT, we find that the parameters c, d only play the role of scaling and modulation. That the modulation has no effect on our inequality (41) has been found from (33) and (37) directly. From the property $F_{(a,b,c,d)}(\sqrt{\rho} f(t/\rho)) = F_{(a\rho,b/\rho,c\rho,d/\rho)}(f(t))$, we can easily find that scaling also has no effect on (41) as well as above shown. Similarly, if $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} 1/\pi e & 1/\pi e & -1 \\ 1 & 1 & \end{bmatrix}$, $\ln(\pi e|a_1 b_2 - a_2 b_1|) = 0$. It means that the bound of this entropic uncertainty principle may be zero.

When $(a_1, b_1, c_1, d_1) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$ and $(a_2, b_2, c_2, d_2) = (\cos\beta, \sin\beta, -\sin\beta, \cos\beta)$, (41) reads

$$\begin{aligned} &-\int_{-\infty}^{\infty} |F_\alpha(v)|^2 \ln\left(|F_\alpha(v)|^2\right) dv \\ &-\int_{-\infty}^{\infty} \left|F_\beta(u)\right|^2 \ln \left|F_\beta(u)\right|^2 du \\ &\geq \ln(\pi e \left|\sin(\alpha - \beta)\right|). \end{aligned} \tag{42}$$

Clearly, (42) is the entropic uncertainty principle in the fractional Fourier transform domains.

When $(a_1, b_1, c_1, d_1) = (1, 0, 0, 1)$ and $(a_2, b_2, c_2, d_2) = (0, 1, -1, 0)$, (41) reduces to the traditional case (30).

4. Heisenberg's Uncertainty Principle on LCT

As (1), (2) showing, Heisenberg's uncertainty principle mainly discusses the product of time spread and frequency spread. In the same manner as Section 3, in this section, Heisenberg's uncertainty principle in the LCT domains is derived. Without loss of generality, assuming the mean values of the variables are zeros, namely,

$$\int_{-\infty}^{+\infty} |t|^2 \left| f(t) \right|^2 dt \cdot \int_{-\infty}^{+\infty} |u|^2 |F(u)|^2 du \geq \frac{1}{4}. \tag{43}$$

First, similarly we assume $a_l, b_l, c_l, d_l \in \mathbf{R}$ and $b_l \neq 0$ $(l = 1, 2, 3)$.

Set

$$\begin{aligned} G(u) &= F_{(a_1,b_1,c_1,d_1)}(u) \exp\left(-i \frac{d_3 u^2}{2b_3}\right), \\ F_{(a_1,b_1,c_1,d_1)}(u) &= F_{(a_1,b_1,c_1,d_1)}(f(t)), \\ g(t) &= \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(u) e^{iut} du. \end{aligned} \tag{44}$$

Noting the fact that the equation

$$\left| F_{(a_1,b_1,c_1,d_1)}(u) \exp\left(-i \frac{d_3 u^2}{2b_3}\right) \right| = \left| F_{(a_1,b_1,c_1,d_1)}(u) \right| \tag{45}$$

holds, we easily obtain

$$\int_{-\infty}^{+\infty} |u|^2 |G(u)|^2 du = \int_{-\infty}^{+\infty} |u|^2 \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du. \tag{46}$$

From (43) and (46), we have

$$\int_{-\infty}^{+\infty} |t|^2 \left| g(t) \right|^2 dt \cdot \int_{-\infty}^{+\infty} |u|^2 \left| F_{(a_1,b_1,c_1,d_1)}(u) \right|^2 du \geq \frac{1}{4}. \tag{47}$$

Through variable's scaling, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} |t|^2 \left| g(t) \right|^2 dt &= \int_{-\infty}^{+\infty} \left| \frac{t}{b_3} \right|^2 \left| g\left(\frac{t}{b_3}\right) \right|^2 d\left(\frac{t}{b_3}\right) \\ &= \frac{1}{|b_3|^3} \int_{-\infty}^{+\infty} |t|^2 \left| g\left(\frac{t}{b_3}\right) \right|^2 dt. \end{aligned} \tag{48}$$

Meanwhile noting

$$\left| g\left(\frac{t}{b_3}\right) \right|^2 = \left| \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} G(u) e^{iut/b_3} du \right|^2. \tag{49}$$

Similarly with (12), substituting $F_{(a_1,b_1,c_1,d_1)}(u) e^{-id_3 u^2/2b_3}$ for $G(u)$ in (49) and using definition (3), we get

$$\begin{aligned} \left| g\left(\frac{t}{b_3}\right) \right|^2 &= \left| \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{\infty} F_{(a_1,b_1,c_1,d_1)}(u) e^{-id_3 u^2/2b_3} e^{iut/b_3} du \right|^2 \\ &= |b_3| \left| F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(t) \right|^2. \end{aligned} \tag{50}$$

Thus we obtain

$$\begin{aligned} &\int_{-\infty}^{+\infty} |t|^2 \left| g(t) \right|^2 dt \\ &= \frac{1}{|b_3|^2} \int_{-\infty}^{+\infty} |t|^2 \left(\left| F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(t) \right|^2 \right) dt. \end{aligned} \tag{51}$$

Set $t = v$, then get

$$\frac{1}{|b_3|^2}\int_{-\infty}^{+\infty}|v|^2\left(\left|F_{(d_3,-b_3,-c_3,a_3)}\left(F_{(a_1,b_1,c_1,d_1)}\right)(v)\right|^2\right)dv \cdot \int_{-\infty}^{+\infty}|u|^2\left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2du \geq \frac{1}{4}. \quad (52)$$

From (15) and (40), compared (51) with (52), we have

$$\int_{-\infty}^{+\infty}|u|^2\left|F_{(a_1,b_1,c_1,d_1)}(u)\right|^2du \cdot \int_{-\infty}^{+\infty}|v|^2\left(F_{(a_2,b_2,c_2,d_2)}(v)^2\right)dv \geq \frac{|a_1b_2-a_2b_1|^2}{4}. \quad (53)$$

Clearly, Heisenberg's uncertainty principle in the LCT domains (see (53)) is only connected with the LCT parameters a and b and independent of c and d. Why do not the parameters c, d have relation with the entropic uncertainty principle in the LCT domains? The reasons are the same as those in Sections 2 and 3. When $a_1b_2 - a_2b_1 \to 0$, the bound of (53) tends to be zero.

When $(a_1, b_1, c_1, d_1) = (\cos\alpha, \sin\alpha, -\sin\alpha, \cos\alpha)$ and $(a_2, b_2, c_2, d_2) = (\cos\beta, \sin\beta, -\sin\beta, \cos\beta)$, (53) reads

$$\int_{-\infty}^{+\infty}|u|^2|F_\alpha(u)|^2du \cdot \int_{-\infty}^{+\infty}|v|^2\left(F_\beta(v)^2\right)dv \geq \frac{|\sin(\alpha-\beta)|^2}{4}. \quad (54)$$

However (54) is the Heisenberg's uncertainty principle in the fractional Fourier transform domains [1, 5, 7, 17]. When $(a_1, b_1, c_1, d_1) = (1, 0, 0, 1)$ and $(a_2, b_2, c_2, d_2) = (0, 1, -1, 0)$, (53) reduces to the traditional case (43).

5. Conclusions

Three uncertainty principles associated with the LCT are presented in this paper. Firstly, from definition of LCT and the traditional Pitt's inequality, one novel Pitt's inequality in the LCT domains is obtained, which is connected with the LCT parameters a and b and independent of the LCT parameters c and d. Then one novel logarithmic uncertainty principle is derived from this novel Pitt's inequality in two LCT domains. Secondly, based on the relation between one original function and LCT, the entropic uncertainty principle in two LCT domains is proposed. Thirdly, from the relation between one original function and its LCT, Heisenberg's uncertainty principle in two LCT domains is obtained. Note that the three lower bounds are only associated with LCT parameters a and b and independent of c and d. In addition, the reasons are given. Moreover, one clear observation is that our three uncertainty principles hold for both real and complex signals. Our future work includes finding out how these cases can be generalized to discrete and multidimensional signals.

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