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Joint Source-Channel Coding Based on Cosine-Modulated Filter Banks for Erasure-Resilient Signal Transmission

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This paper examines erasure resilience of oversampled filter bank (OFB) codes, focusing on two families of codes based on cosine-modulated filter banks (CMFB). We first revisit OFBs in light of filter bank and frame theory. The analogy with channel codes is then shown. In particular, for paraunitary filter banks, we show that the signal reconstruction methods derived from the filter bank theory and from coding theory are equivalent, even in the presence of quantization noise. We further discuss frame properties of the considered OFB structures. Perfect reconstruction (PR) for the CMFB-based OFBs with erasures is proven for the case of erasure patterns for which PR depends only on the general structure of the code and not on the prototype filters. For some of these erasure patterns, the expression of the mean-square reconstruction error is also independent of the filter coefficients. It can be expressed in terms of the number of erasures, and of parameters such as the number of channels and the oversampling ratio. The various structures are compared by simulation for the example of an image transmission system.

Keywords and phrases: frames, filter banks, source coding, channel coding, erasure channels, Internet communication.

1. INTRODUCTION

The advent of multimedia communication over packet-switched (IP) networks is creating challenging problems in the area of coding. Due to the real-time nature of data streams, multimedia delivery usually makes use of unresponsive transport protocols, that is, User Datagram Protocol (UDP) and/or Real-Time Transport Protocol (RTP) [1]. In contrast with Transport Control Protocol (TCP), these protocols offer no control mechanism that would guarantee a level of QoS. The packets may be sent via different routes and may arrive at destination with a large delay or not arrive at all.

Traditional approaches to fight against erasures consist in sending redundant information along with the original information so that the lost data (or at least part of it) can be recovered from the redundant information. The design principles that have prevailed so far stem from Shannon's source and channel separation theorem which states that the source and channel optimum performance bounds can be approached as closely as desired by independently designing the source and channel coding strategies. However, this holds only under asymptotic conditions where both codes are allowed infinite length and complexity. If the design of the system is heavily constrained in terms of complexity or delay, the separate (also called tandem) approach can be largely suboptimal. This observation has motivated the consideration of joint source and channel coding (JSCC) design in practical systems (e.g., [2, 3]).

Among various JSCC techniques, JSCC based on oversampled transform codes (OTCs) has recently gained a lot of attention [2, 4, 5, 6, 7]. This is a fundamentally different approach whereby the error control coding and the signal decomposition are integrated in a single block by using an oversampled filter bank (OFB). The error protection in this approach is introduced before the quantization allowing in addition to suppress some quantization noise effects (which is not the case of traditional tandem approaches). So far, the research in this area has mostly focused on the investigation of (OTC) which are filter banks (FB) codes with polyphase filter orders equal to zero. The error-correcting capability of various OTCs has been studied in [6, 7, 8]. As in the case of the error-correcting codes over the finite field, it is desirable that the generator matrix of the OTC codes possesses a structure which facilitates the derivation of the decoding algorithms as well as the performance evaluation. For example, the generator matrices of OTC in [6, 7, 8] are constructed from the discrete Fourier transform (DFT) and direct cosine transform (DCT) matrices. In [8, 9], it has been shown that DFT codes have a Bose-Chaudhuri-Hocquenghem (BCH) code property and that the algorithms derived for BCH codes over the finite field can be used for decoding DFT codes in the absence of quantization noise. A performance analysis of erasure recovery with quantized DFT codes is presented in [8]. Considering the similar—but more general—problem of interpolation, the author in [10, 11, 12, 13] analyzes the stability of reconstruction using the eigenanalysis of the interpolation matrix operator. The approach applies to similar problems in various other fields as well (e.g., spectrum analysis, fault-tolerant computing).

Increasing the generator's polyphase matrix order of the OTC gives extra freedom in the transform design. The PR synthesis FB for a given analysis OFB [14, 15, 16, 17, 18] is indeed not unique and can thus be optimized for different application-related criteria. OFB have in particular received attention for noise reduction in subband coding applications [19]. A signal decomposition with an OFB is actually a frame expansion in $\ell^2(\mathbb{Z})$ [4, 15, 16, 20]. Frames are generalizations of a basis for an overcomplete system, or in other words, frames represent sets of vectors that span a Hilbert space but contain more numbers of vectors than a basis. Therefore signal representations using frames are known as overcomplete frame expansions. Because of their inbuilt redundancies, such representations can be useful for providing robustness to signal transmission over error-prone communication media.

The use of quantized OFB-based frame expansions to achieve resilience to erasures of compressed signals has also been considered in [4, 16, 21, 22]. The authors show in particular that, if the frame is uniform and tight, the mean-square reconstruction error is minimized. However, when used as joint source-channel codes, the frame property may be verified only for some erasure patterns. The performance analysis as well as the derivation of the reconstruction filters, which are dependent on the erasure patterns, are in addition rendered difficult in the general case of OFB due to the increased order of the generator-polyphase matrix. To proceed with the performance analysis for various types of erasure patterns and with the design of a practical system, we consider OFB codes with generator-polyphase matrices constructed from polyphase matrices of critically sampled cosine modulated filter banks (CMFB) [14, 23]. CMFBs have a simple structure. Hence, constructing the OFB code generator matrix from the polyphase matrices of the CMFB simplifies the code design as well as the performance analysis. We consider two OFB codes: codes, referred to as OCMFB, obtained from critically sampled CMFB by reducing the downsampling factors and codes, referred to as CMFB-OFB, which have a structure similar to that of DFT codes [8]. The study of OCMFB codes is motivated by the fact that it can be easily integrated in compression systems and is therefore of potential practical interest. The CMFB-OFB codes are considered in order to improve the performance of DFT codes [8]. That is, since CMFB-OFB codes have a similar structure as DFT codes but higher-order polyphase filters, we expect that it has a better performance than a DFT code.

The rest of the paper is organized as follows. In Section 3, OFB are reviewed in light of the frame, FB, and channel coding theory. In particular, the results on the equivalence between the projection receiver and the syndrome decoding derived for DFT codes in [5] are extended to the case of OFB codes. That is, for paraunitary FBs, it is shown that the signal reconstruction methods derived from the FB theory and from the coding theory are equivalent even in presence of quantization noise. The structures of OFB based on CMFBs that are considered in the sequel are described in Section 4, together with the corresponding frame properties and packetization schemes. In Section 5, the PR and the erasure recovery properties of the two families of codes considered are analyzed. Even though OFBs based on CMFB have simple structures, it is difficult to analytically verify the PR property for all erasure patterns. For some particular erasure patterns, we show that PR depends only on the structure of the code and not on the filter coefficients. Section 6 considers the problem of reconstruction in presence of quantization noise. The mean square error (MSE) performance bounds under particular quantization noise distribution assumptions are provided. It is shown that, in presence of quantization noise, the MSE for some erasure patterns does not depend on the filter coefficients. Section 7 provides performance results in terms of mean-square reconstruction error obtained with the codes studied here in comparison with a DFT code.

2. NOTATIONS

In the following, bold letters denote matrices. The expressions $\mathbf{X}^*$, $\mathbf{X}^T$, $\mathbf{X}^H$, and $\tilde{\mathbf{X}} = \mathbf{X}^H(1/z^*)$ denote the conjugate, the transpose, the transpose conjugate, and the paraconjugate of $\mathbf{X}$, respectively. The matrices $\mathbf{I}_N$ and $\mathbf{J}_N$ stand for the $[N \times N]$ identity and reverse identity matrices, respectively.

3. OFB AS CHANNEL CODES: FRAMEWORK AND BACKGROUND

Critically sampled FBs have been widely used in compressions systems based on subband signal decomposition. Oversampling has been considered mainly for reasons related to easier filter design (higher number of degrees of freedom) and/or for noise suppression [23, 24]. In this paper, we consider using oversampling for protection against signal degradation due to both quantization and transmission errors. In particular, we consider scenarios where channel errors occur due to packet losses in a transmission over packet-switched networks. The loss of a packet is referred to as an erasure.

3.1. General framework and problem statement

Consider an FB as shown in Figure 1. In this FB, an input signal $x(n)$ is split into N signals $y_k(n)$, $k = 0, \ldots, N-1$. The sequence $y_k(n)$ is obtained by downsampling the output of the filter k with a factor K, where $K \leq N$. The sequences $y_k(n)$ are then quantized. A single or a group of signals $y_k(n)$

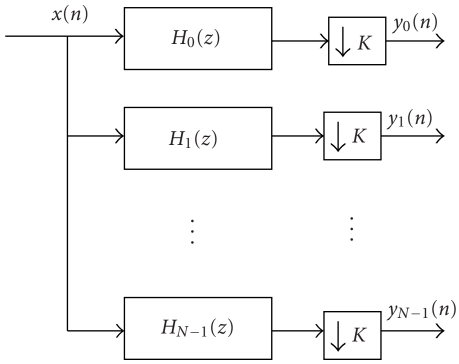

FIGURE 1: Block diagram of an N-channel FB with downsampling factors K.

is placed in one packet. Due to network congestion, some of the packets do not arrive at destination. The task of the receiver is to combine the received signals into a single signal $\hat{x}(n)$ which, in absence of quantization, is identical to the signal $x(n)$, and which, in presence of quantization, is as close as possible to $x(n)$ in the MSE sense. Due to redundant signal representation ($K < N$), PR is possible even if some of the signals $y_k(n)$ are lost. The packets in this model can be viewed as multiple descriptions of the signal [16]. Reception of a certain number of packets allows signal reconstruction with a certain MSE, while reception of additional packets improves the reconstructed signal quality in the MSE sense. The frame theory offers a general way of analyzing signal expansions [16, 17], while signal resilience to channel impairments is a field of study of coding theory. Here, we revise some concepts of OFB in light of the frame theory and give the analogy between OFB and channel codes. Since we consider using OFB for erasure recovery, we also refer to OFB as oversampled filter bank codes (OFBCs).

3.2. Frame-theoretic analysis

3.2.1. Definitions

A set of vectors $\Phi = \{\boldsymbol{\phi}_i\}_{i\in\mathbb{Z}}$ in a Hilbert space $\mathbb{H}$ of square summable sequences $\ell^2(\mathbb{Z})$ is a frame if for any $\mathbf{x} \in \ell^2(\mathbb{Z})$,

$$A \parallel \mathbf{x} \parallel^2 \leq \sum_{i\in\mathbb{Z}} \left| \langle \mathbf{x}, \boldsymbol{\phi}_i \rangle \right|^2 \leq B \parallel \mathbf{x} \parallel^2, \tag{1}$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ denotes the inner product of $\mathbf{x}$ and $\mathbf{y}$, and $A > 0$ and $B < \infty$ are constants called frame bounds. If Φ is a frame, there exists another frame $\Gamma = \{\boldsymbol{\gamma}_i\}_{i\in\mathbb{Z}}$ such that any signal $\mathbf{x} \in \mathbb{H}$ can be represented in a numerically stable way as $\mathbf{x} = \sum_{i\in\mathbb{Z}} \langle \mathbf{x}, \boldsymbol{\phi}_i \rangle \boldsymbol{\gamma}_i$ [17]. For an OFB with N channels and a downsampling factor K, the vectors constituting a frame are given by the translated versions of N elementary waveforms [17]

$$\Phi = \{\boldsymbol{\phi}_{i,j} : \phi_{i,j}(n) = \phi_i(n - jK)\ i = 0, 1, \ldots, N-1,\ j \in \mathbb{Z}\}, \tag{2}$$

where $\phi_{i,j}(n)$ is related to the filter impulse response as $h_i(n) = \phi_i^*(-n)$, $i = 0, \ldots, N-1$ [17, 24]. The inner products of the input signal $\mathbf{x}$ with vectors in a frame Φ are thus obtained at the output of an N-channel FB as

$$\begin{bmatrix} \vdots \\ y_{N-1}(n-1) \\ \vdots \\ y_0(n-1) \\ y_{N-1}(n) \\ \vdots \\ y_0(n) \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \cdots & \\ \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{L_V} & \mathbf{0} & \cdots & \cdots & \cdots & \\ \cdots & \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{L_V} & \mathbf{0} & \cdots & \cdots & \\ \cdots & \cdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \cdots & \\ \cdots & \cdots & \vdots & \mathbf{0} & \mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{L_V} & \mathbf{0} & \cdots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \end{bmatrix}}_{\mathbf{H}} \times \begin{bmatrix} \vdots \\ x((n-2)K-1) \\ \vdots \\ x((n-1)K) \\ x((n-1)K-1) \\ \vdots \\ x(nK) \\ \vdots \end{bmatrix}, \tag{3}$$

where $x(n)$ denotes the nth element of the input sequence $\mathbf{x}$ and $y_i(n)$ denotes the nth element of the sequence at the output of the ith filter. The quantity L_V is given by $\lceil L_P/K \rceil - 1$, where L_P denotes the largest filter length. The matrix $\mathbf{H}_i$ is given by

$$\begin{bmatrix} h_0((L_V+1)K - iK - 1) & \cdots & h_0((L_V+1)K - (i+1)K) \\ & \cdots & \\ h_{N-1}((L_V+1)K - iK - 1) & \cdots & h_{N-1}((L_V+1)K - (i+1)K) \end{bmatrix}. \tag{4}$$

The infinite matrix $\mathbf{H}$ is the frame operator associated with the FB frame. It assigns to each input sequence $\mathbf{x}$ a sequence of products $\langle \mathbf{x}, \boldsymbol{\phi}_{i,j} \rangle$. PR is possible if and only if there exists a matrix $\mathbf{F}$ such that $\mathbf{FH} = \mathbf{I}_\infty$. For an OFB, the solution for synthesis filters is not unique and it can be expressed as [24]

$$\mathbf{F} = \hat{\mathbf{F}} + \mathbf{P}[\mathbf{I}_\infty - \mathbf{H}\hat{\mathbf{F}}], \tag{5}$$

where $\mathbf{P}$ is an arbitrary matrix and $\hat{\mathbf{F}}$ is the pseudoinverse of $\mathbf{H}$ given by $\hat{\mathbf{F}} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H$.

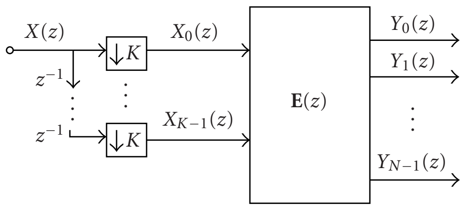

FIGURE 2: Polyphase implementation of the analysis FB of an $N \times K$ OFB code.

3.2.2. *Polyphase representation*

The FB structure of the frame allows to represent operations performed by an FB in a more compact way by using the concepts of polyphase signal decomposition. An FB output signal $\mathbf{Y}(z) = [Y_0(z) \cdots Y_{N-1}(z)]^T$ can indeed be represented as

$$\mathbf{Y}(z) = \mathbf{E}(z)\mathbf{X}(z), \tag{6}$$

where $\mathbf{X}(z)$ is the decomposed input signal $X(z)$ into K type-I polyphase components as

$$\begin{gathered} X(z) = \sum_{k=0}^{K-1} z^{-k} X_k(z^K), \\ \mathbf{X}(z) = [X_0(z) \cdots X_{K-1}(z)]^T, \end{gathered} \tag{7}$$

and $\mathbf{E}(z)$ is an $[N \times K]$ type-I polyphase analysis matrix with elements

$$\begin{gathered} E_{i,j}(z) = \sum_{k=0}^{L_V} h_i(Kk + j) z^{-k}, \\ i = 0, \ldots, N-1, \quad j = 0, \ldots, K-1. \end{gathered} \tag{8}$$

The polyphase implementation is depicted in Figure 2.

3.2.3. *Review of the main properties*

The properties of OFB-based frame expansions depend on the properties of the frame operator which is an infinite matrix. They can be defined in terms of the properties of the $[N \times K]$ polyphase matrix $\mathbf{E}(z)$ [17].

Proposition 1 (see [17, Theorem 1 and Corollary 1]). *An analysis polyphase matrix* $\mathbf{E}(z)$ *implements a frame expansion if and only if* $\mathbf{E}(z)$ *is of full rank on the unit circle, or equivalently, if and only if there exists a matrix of stable rational functions which is a left inverse of* $\mathbf{E}(z)$.

Therefore, if an FB implements a frame, PR is possible and the synthesis polyphase matrix $\mathbf{R}(z)$ is given by the left inverse of the analysis polyphase matrix $\mathbf{E}(z)$. For an OFB, the solution for the synthesis polyphase matrix is not unique. The general solution can be written as [15, 24]

$$\mathbf{R}(z) = cz^{-m_0}\{\hat{\mathbf{R}}(z) + \mathbf{U}(z)[\mathbf{I} - \mathbf{E}(z)\hat{\mathbf{R}}(z)]\}, \tag{9}$$

where $\mathbf{U}(z)$ is an arbitrary $[N \times K]$ matrix with $|\mathrm{U}_{i,j}(e^{j\omega})| < \infty$. The matrix $\hat{\mathbf{R}}(z)$ is the parapseudoinverse of $\mathbf{E}(z)$ given by $\hat{\mathbf{R}}(z) = [\tilde{\mathbf{E}}(z)\mathbf{E}(z)]^{-1}\tilde{\mathbf{E}}(z)$. However, it has been shown in [16] that, if the output of an OFB is corrupted by quantization error which can be modeled by additive white noise, and if the noise sequences in different channels are pairwise uncorrelated, then the parapseudoinverse is the best linear reconstruction operator in the MSE sense. We therefore consider only the parapseudoinverse receiver.

An FB implements a tight-frame expansion if and only if its polyphase analysis matrix $\mathbf{E}(z)$ is paraunitary, that is, $\tilde{\mathbf{E}}(z)\mathbf{E}(z) = c\mathbf{I}$, where c is a constant ($c \neq 0$) [17, 24]. Tight OFBs have the nice property that the parapseudoinverse is given by $\hat{\mathbf{R}}(z) = (1/c)\tilde{\mathbf{E}}(z)$. A frame implemented by an OFB is uniform if $\|h_i(n)\| = 1$, $i = 1, \ldots, N$ [16].

3.3. *Analogy with channel codes*

The $[N \times K]$ polyphase matrix $\mathbf{E}(z)$ can be considered as the generator filter matrix of an (N, K) OFB code. Similarly, an $[N - K, N]$ parity check filter $\mathbf{P}(z)$ can be defined as

$$\mathbf{P}(z)\mathbf{E}(z) = \mathbf{0}. \tag{10}$$

For example, a parity check filter matrix can be obtained from the Smith McMillan decomposition of the analysis polyphase matrix [25].

It was shown in [16] that when encoding with an FB implementing a uniform frame and decoding with the pseudoinverse receiver for the additive white noise model with the pairwise uncorrelated noise sequences in two different channels, the MSE is minimum if and only if the frame is tight. In the rest of the paper, we consider OFBs for which the original frame (without erasures) is tight.

It was shown in [17, 26] that paraunitary FBs can be factorized as

$$\mathbf{E}(z) = \mathbf{V}_D(z)\mathbf{V}_{D-1}(z) \cdots \mathbf{V}_1(z)\mathbf{V}_0, \tag{11}$$

where the paraunitary elementary building blocks $\mathbf{V}_i(z)$ are given by

$$\mathbf{V}_i = \mathbf{I} - \mathbf{v}_i\mathbf{v}_i^H + z^{-1}\mathbf{v}_i\mathbf{v}_i^H. \tag{12}$$

The vector $\mathbf{v}_i$ is an $[N \times 1]$ unit norm vector and $\mathbf{V}_0$ is an $[N \times K]$ matrix of scalars with $\mathbf{V}_0^T\mathbf{V}_0 = \text{const} \times \mathbf{I}$.

The polyphase analysis matrix $\mathbf{E}(z)$ can be further represented as

$$\mathbf{E}(z) = \mathbf{U}(z)\begin{bmatrix} \mathbf{\Lambda} \\ \mathbf{0} \end{bmatrix}\mathbf{W}, \tag{13}$$

where $\mathbf{U}(z) = \mathbf{V}_D(z)\mathbf{V}_{D-1}(z) \cdots \mathbf{V}_1(z)\mathbf{A}$, $\mathbf{\Lambda} = \mathbf{B}$, $\mathbf{W}(z) = \mathbf{C}$ and $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ are matrices obtained by singular value decomposition of $\mathbf{V}_0$. In this representation, the matrices $\mathbf{U}(z)$ and $\mathbf{W}$ are square matrices with $\tilde{\mathbf{U}}(z)\mathbf{U}(z) = \mathbf{I}_N$ and $\mathbf{W}^H\mathbf{W} = \mathbf{I}_K$, and $\mathbf{\Lambda}$ is a $[K \times K]$ nonsingular diagonal matrix.

Now, the parity check matrix $\mathbf{P}(z)$ can be found as

$$\begin{aligned}\mathbf{P}(z) &= \left[\mathbf{0}_{N-K\times K} \quad \mathbf{I}_{N-K}\right]\mathbf{U}(z)^{-1}\\ &= \left[\widetilde{\mathbf{U}}_{K,N-K}(z) \quad \widetilde{\mathbf{U}}_{N-K,N-K}(z)\right].\end{aligned} \tag{14}$$

We can observe that filtering any sequence $\mathbf{Y}(z)$ which was generated by a generator filter matrix $\mathbf{E}(z)$ yields zero syndromes. On the other hand, if the encoded signal is corrupted by quantization noise $\mathbf{N}(z)$, we have

$$\mathbf{S}(z) = \mathbf{P}(z)\big(\mathbf{Y}(z)+\mathbf{N}(z)\big) = \mathbf{P}(z)\widehat{\mathbf{Y}}(z) = \mathbf{P}(z)\mathbf{N}(z), \tag{15}$$

where $\widehat{\mathbf{Y}}(z)$ denotes a quantized version of $\mathbf{Y}(z)$.

We consider system conditions where the received signal differs from an encoded signal $\mathbf{Y}(z)$ due to both quantization noise and erasures. Here, we assume without loss of generality that the packet i contains a single quantized sequence $y_i(n)$. In this case, the received signal is denoted by $\widehat{\mathbf{Y}}_R(z)$ and can be written as

$$\begin{aligned}\widehat{\mathbf{Y}}_R(z) &= \left[Y_{i_1}(z)+N_{i_1}(z) \quad \cdots \quad Y_{i_R}(z)+N_{i_R}(z)\right]^T\\ &= \mathbf{Y}_R+\mathbf{N}_R(z) = \mathbf{E}_{R,K}(z)\mathbf{X}(z)+\mathbf{N}_R(z),\end{aligned} \tag{16}$$

where $i_1,\ldots,i_R$ are the indices of R received packets. $\mathbf{E}_{R,K}(z)$ is an $[R\times K]$ submatrix of $\mathbf{E}(z)$ corresponding to the received components $\widehat{\mathbf{Y}}_R(z)$ and $\mathbf{X}(z)$ is the z transform of a blocked input signal. Similarly, the vector of erased components is expressed as

$$\begin{aligned}\widehat{\mathbf{Y}}_E(z) &= \left[Y_{j_1}(z)+N_{j_1}(z) \quad \cdots \quad Y_{j_E}(z)+N_{j_E}(z)\right]^T\\ &= \mathbf{Y}_E(z)+\mathbf{N}_E(z) = \mathbf{E}_{E,K}(z)\mathbf{X}(z)+\mathbf{N}_E(z),\end{aligned} \tag{17}$$

where $j_1,\ldots,j_E$ are the indices of E erased packets.

If we assume (without loss of generality) that the first R packets are received, the syndromes $\mathbf{S}(z)$ are given by

$$\begin{aligned}\mathbf{S}(z) &= \mathbf{P}(z)_{N-K,N}\begin{bmatrix}\widehat{\mathbf{Y}}_R(z)\\ \mathbf{Y}_E(z)-\mathbf{Y}_E(z)\end{bmatrix} = \mathbf{P}(z)_{N-K,R}\widehat{\mathbf{Y}}_R(z)\\ &= -\mathbf{P}(z)_{N-K,E}\mathbf{Y}_E(z)+\mathbf{P}(z)_{N-K,R}\mathbf{N}_R(z),\end{aligned} \tag{18}$$

where $\mathbf{P}(z)_{N-K,R}$ and $\mathbf{P}(z)_{N-K,E}$ are matrices consisting of the first R and the last E columns of $\mathbf{P}(z)_{N-K,N}$.

There are two ways to reconstruct a signal from the received samples. We can either estimate $\mathbf{X}(z)$ from (16) or first estimate the erased signals $\mathbf{Y}_E(z)$ from (18), and then reconstruct the signal as if there were no erasures. We refer to the first approach as reconstruction by projection on the signal space, in short as projection decoding, and the second approach is referred to as syndrome decoding.

3.4. Equivalence between projection decoding and syndrome decoding

The reconstruction methods based on the parapseudoinverse of the analysis matrix after erasures and the methods based on syndromes are equivalent even in presence of quantization noise. That is, we can write

$$\begin{aligned}&\big(\widetilde{\mathbf{E}}(z)\mathbf{E}(z)\big)^{-1}\widetilde{\mathbf{E}}(z)\begin{bmatrix}\widehat{\mathbf{Y}}_R(z)\\ \bar{\mathbf{Y}}_E(z)\end{bmatrix}\\ &\quad= \big(\widetilde{\mathbf{E}}_{R,K}(z)\mathbf{E}_{R,K}(z)\big)^{-1}\widetilde{\mathbf{E}}_{R,K}(z)\widehat{\mathbf{Y}}_R(z),\end{aligned} \tag{19}$$

where $\widehat{\mathbf{Y}}_R(z)$ denotes a vector with received quantized signal components, and $\bar{\mathbf{Y}}_E(z)$ represents a vector of erased components estimated from the syndrome equations. Assuming that the matrix $\mathbf{P}_{N-K,E}(z)$ is of rank E on the unit circle, the erased components are estimated from (18) by using the parapseudoinverse of $\mathbf{P}_{N-K,E}(z)$. That is,

$$\begin{aligned}\bar{\mathbf{Y}}_E(z) =& \big(\widetilde{\mathbf{P}}_{N-K,E}(z)\mathbf{P}_{N-K,E}(z)\big)^{-1}\widetilde{\mathbf{P}}_{N-K,E}(z)\mathbf{S}(z)\\ &- \big(\widetilde{\mathbf{P}}_{N-K,E}(z)\mathbf{P}_{N-K,E}(z)\big)^{-1}\widetilde{\mathbf{P}}_{N-K,E}(z)\mathbf{P}_{N-K,R}(z)\widehat{\mathbf{Y}}_R(z),\\ \bar{\mathbf{Y}}_E(z) =& -\big(\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{E,N-K}(z)\big)^{-1}\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{R,N-K}(z)\widehat{\mathbf{Y}}_R(z),\end{aligned} \tag{20}$$

where $\mathbf{U}(z)$ is partitioned as

$$\mathbf{U}(z) = \begin{bmatrix}\mathbf{U}_{R,K}(z) & \mathbf{U}_{R,N-K}(z)\\ \mathbf{U}_{E,K}(z) & \mathbf{U}_{E,N-K}(z)\end{bmatrix}. \tag{21}$$

We first note that, whenever (in the absence of quantization noise) PR based on the parapseudoinverse of $\mathbf{E}_{R,K}(z)$ is possible, it is also possible to reconstruct $\mathbf{Y}_E(z)$ from the syndromes in (18). This can be shown by observing that [27]

$$\begin{aligned}&\det\left\{\begin{bmatrix}\mathbf{U}_{K,K}(z) & \mathbf{U}_{K,N-K}(z)\\ \mathbf{0} & \mathbf{I}_{N-K}\end{bmatrix}\begin{bmatrix}\mathbf{U}_{K,K}(z) & \mathbf{U}_{K,N-K}(z)\\ \mathbf{U}_{N-K,K}(z) & \mathbf{U}_{N-K,N-K}(z)\end{bmatrix}^{-1}\right\}\\ &\quad= \det\left\{\begin{bmatrix}\mathbf{I}_K & \mathbf{0}_{K,N-K}\\ \widetilde{\mathbf{U}}_{K,N-K}(z) & \widetilde{\mathbf{U}}_{N-K,N-K}(z)\end{bmatrix}\right\}.\end{aligned} \tag{22}$$

From the above equation, it follows that

$$\det\left\{\mathbf{U}_{K,K}(z)\right\}\times \text{const} = \det\left\{\widetilde{\mathbf{U}}_{N-K,N-K}(z)\right\}, \tag{23}$$

assuming that the first K components have been received and that the last $N-K$ components have been erased. The same can be shown for any other selection of lost and received samples. Therefore, whenever it is possible to perfectly reconstruct $\mathbf{X}(z)$ from (16), it is also possible to perfectly reconstruct $\mathbf{Y}_E(z)$ from the syndrome equations in (18). We further show that (19) is valid even in presence of quantization noise. From the signal decomposition given in (13), the parapseudoinverse can be calculated as [28]

$$\big(\widetilde{\mathbf{E}}(z)\mathbf{E}(z)\big)^{-1}\widetilde{\mathbf{E}}(z) = \mathbf{W}^H\left[\boldsymbol{\Lambda}^{-1} \quad \mathbf{0}\right]\widetilde{\mathbf{U}}(z). \tag{24}$$

By combining (14), (24), and (19), we get

$$\begin{aligned}
&\mathbf{W}^H\boldsymbol{\Lambda}^{-1}\left[\widetilde{\mathbf{U}}_{R,K}(z)\quad \widetilde{\mathbf{U}}_{E,K}(z)\right]\\
&\quad\times\begin{bmatrix}\mathbf{I}_R(z)\\ -\left(\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{E,N-K}(z)\right)^{-1}\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{R,N-K}(z)\end{bmatrix}\widehat{\mathbf{Y}}_R\\
&= \mathbf{W}^H\boldsymbol{\Lambda}^{-1}\left(\widetilde{\mathbf{U}}_{R,K}(z)\mathbf{U}_{R,K}(z)\right)^{-1}\widetilde{\mathbf{U}}_{R,K}(z)\widehat{\mathbf{Y}}_R(z).
\end{aligned} \tag{25}$$

From the paraunitary condition $\mathbf{U}(z)\widetilde{\mathbf{U}}(z) = \widetilde{\mathbf{U}}(z)\mathbf{U}(z) = \mathbf{I}_N$, we have the following equalities:

$$\begin{aligned}
&\left(\widetilde{\mathbf{U}}_{R,K}(z)\mathbf{U}_{R,K}(z)\right)^{-1}\\
&\qquad= \left(\mathbf{I}_K - \widetilde{\mathbf{U}}_{E,K}(z)\mathbf{U}_{E,K}(z)\right)^{-1}\\
&\qquad= \mathbf{I}_K + \widetilde{\mathbf{U}}_{E,K}(z)\left(\mathbf{I}_E - \mathbf{U}_{E,K}(z)\widetilde{\mathbf{U}}_{E,K}(z)\right)^{-1}\mathbf{U}_{E,K}(z), \quad (26)\\
&\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{R,N-K}(z) = -\mathbf{U}_{E,K}(z)\widetilde{\mathbf{U}}_{R,K}(z),\\
&\left(\mathbf{U}_{E,N-K}(z)\widetilde{\mathbf{U}}_{E,N-K}(z)\right)^{-1} = \left(\mathbf{I}_E - \mathbf{U}_{E,K}(z)\widetilde{\mathbf{U}}_{E,K}(z)\right)^{-1}.
\end{aligned}$$

By substituting these equalities in (25), we see that the matrices multiplying $\widehat{\mathbf{Y}}_R(z)$ at both sides of (25) are equal. This proves that, in presence of both erasures and quantization noise, the reconstruction method based on the syndrome filters and the reconstruction method based on the parapseudoinverse are equivalent. As both methods are equivalent, in the sequel, we consider only reconstruction based on the parapseudoinverse of the analysis matrix after erasures.

4. STRUCTURES BASED ON CMFBS

Oversampling increases both the design and implementation computational cost. For this reason, we consider OFBs based on CMFBs which have low design and implementation complexity. In particular, we consider OCMFBs with an integer oversampling ratio, and OFBs with the analysis polyphase matrix composed of two critically sampled CMFB polyphase matrices.

4.1. Critically sampled CMFB

In an N-channel CMFB, the analysis filters $h_k(n)$ are obtained by cosine modulation of the prototype $p(n)$ as [26]

$$h_k(n) = 2p(n)\cos\left(\frac{\pi}{N}(k+0.5)\left(n-\frac{D}{2}\right)+\phi_k\right), \quad n = 0,1,\ldots,L_p-1, \tag{27}$$

where D denotes the overall delay of the analysis-synthesis system and $\phi_k = (-1)^k\pi/4$. The $2N$ polyphase components of a prototype $p(n)$ with length $L_p = 2mN$ are given by

$$P_j(z) = \sum_{l=0}^{m-1} p(2lN+j)z^{-l}. \tag{28}$$

The analysis polyphase matrix for the critically sampled case, that is, for the oversampling ratio $L = N/K = 1$, is given by

$$\mathbf{E}^{(1)}(z) = \mathbf{T}_a\begin{bmatrix}\mathbf{P}_0(z^2)\\ z^{-1}\mathbf{P}_1(z^2)\end{bmatrix} = \mathbf{T}_a\mathbf{P}(z), \tag{29}$$

where

$$\begin{aligned}
\left[\mathbf{T}_a\right]_{k,j} &= 2\cos\left[\frac{\pi}{N}\left(k+\frac{1}{2}\right)\left(j-\frac{D}{2}\right)+\phi_k\right],\\
&\qquad k = 0,\ldots,N-1,\ j = 0,\ldots,2N-1,\\
\mathbf{P}_0(z) &= \operatorname{diag}\left[P_0(-z^2),P_1(-z^2),\ldots,P_{N-1}(-z^2)\right],\\
\mathbf{P}_1(z) &= \operatorname{diag}\left[P_N(-z^2),P_{N+1}(-z^2),\ldots,P_{2N-1}(-z^2)\right].
\end{aligned} \tag{30}$$

We consider paraunitary CMFBs with finite impulse response (FIR), linear phase prototype filters of length $L_p = 2mN$, where m is an even integer. When $L_p = 2mN$ and m is an even integer, the $\mathbf{T}_a$ matrix can be written as

$$\mathbf{T}_a = \sqrt{N}\boldsymbol{\Lambda}_c\mathbf{C}\left[(\mathbf{I}_N-\mathbf{J}_N)-(\mathbf{I}_N+\mathbf{J}_N)\right], \tag{31}$$

where $\mathbf{C}$ is a (type 4) DCT matrix given by

$$[\mathbf{C}]_{k,n} = \sqrt{\frac{2}{N}}\cos\frac{\pi}{N}(k+0.5)(n+0.5) \tag{32}$$

and $\boldsymbol{\Lambda}_c$ is a diagonal matrix with $[\boldsymbol{\Lambda}_c]_{k,k} = \cos(\pi(k+0.5)m)$. The analysis polyphase matrix is given by

$$\mathbf{E}^{(1)}(z) = \sqrt{N}\boldsymbol{\Lambda}_c\mathbf{C}\left[(\mathbf{I}_N-\mathbf{J}_N)-(\mathbf{I}_N+\mathbf{J}_N)\right]\begin{bmatrix}\mathbf{P}_0(z^2)\\ z^{-1}\mathbf{P}_1(z^2)\end{bmatrix}. \tag{33}$$

The synthesis polyphase matrix can be written as

$$\begin{aligned}
\mathbf{R}^{(1)}(z) &= z^{-(2m-1)}\widetilde{\mathbf{R}}^{(1)}(z)\\
&= \left[z^{-1}\mathbf{J}_N\mathbf{P}_1(z^2)\mathbf{J}_N\quad \mathbf{J}_N\mathbf{P}_0(z^2)\mathbf{J}_N\right]\mathbf{T}_a^T.
\end{aligned} \tag{34}$$

4.2. Oversampled CMFB code

As current signal compression systems already use critically sampled FBs for signal decomposition, the most straightforward way to introduce redundancy at this point in the system is to use a subsampling factor which is smaller than the number of channels. Classification of the oversampled CMFBs with PR and paraunitary conditions for OCMFBs have been considered in [14]. The same authors studied OCMFBs frame properties and the design and implementation issues. OCMFB with integer oversampling ratio have been considered in [23], essentially for obtaining less restrictive constraints for the analysis and synthesis prototypes. In this paper, we are interested in using the redundancy for erasure recovery. The prototype filters are optimized for the N-channel critically sampled CMFB. Redundant signal representation is obtained by replacing subsampling factors N with subsampling factors $K = N/L$, where L is an integer. An $[N\times K]$ analysis polyphase matrix of this OFB can be expressed as [23]

$$\begin{aligned}
\mathbf{E}^{(L)}(z) &= \mathbf{E}^{(1)}(z^L)\left[\mathbf{I}_K\quad z^{-1}\mathbf{I}_K\quad\cdots\quad z^{-(L-1)}\mathbf{I}_K\right]^T\\
&= \mathbf{E}^{(1)}(z^L)\mathbf{S}^L(z),
\end{aligned} \tag{35}$$

where $\mathbf{E}(z)$ is given in (33). The synthesis polyphase filters are given by

$$\mathbf{R}^{(L)}(z) = \sqrt{\frac{K}{N}} z^{-(L-1)} \widetilde{\mathbf{S}}^L(z)\mathbf{R}^{(1)}(z^L), \tag{36}$$

where $\mathbf{R}^{(1)}$ is given in (34). It has been shown in [23] that if the critically sampled FB is paraunitary, the OFB is also paraunitary. That is,

$$\begin{aligned}\widetilde{\mathbf{E}}^{(L)}(z)\mathbf{E}^{(L)}(z) &= \widetilde{\mathbf{S}}^L(z)\widetilde{\mathbf{E}}^{(1)}(z^L)\mathbf{E}^{(1)}(z^L)\mathbf{S}^L(z)\\ &= \widetilde{\mathbf{S}}^L(z)\mathbf{S}^L(z) = \frac{N}{K}\mathbf{I}_K.\end{aligned} \tag{37}$$

4.2.1. Packetization

To carry out the performance analysis of these codes in presence of erasures, one has first to design the transport or packetization structure of the different code samples. The most natural choice for the packetization scheme in the system with an OCMFB code is to put consecutive samples at the output of each filter in a different packet. This is because every Lth sample at the output of each filter is an output of a critically sampled FB and therefore, we can expect that PR will be possible even after loosing some packets. Another reason for packetizing the samples in this way is that we avoid the possibility of loosing a signal in an entire subband, which is not desirable in source coding applications where various subbands have different importance. We have therefore adopted the following packetization scheme. There are N_P packets per image. Each of the N_P consecutive coefficients in a subband is placed in a different packet. Packetization for the ith filter in an OCMFB is shown in Figure 3. For example, loosing the packet k means that every N_Pth subband coefficient starting from the kth is lost in all subbands.

4.2.2. Polyphase representation and analysis of an OCMFB code

In order to facilitate the analysis of an OFB code performance in presence of erasures, it is convenient to represent a polyphase matrix of an OFB code in such a way that an erasure corresponds to loosing samples generated by a single or a group of rows in this matrix. For this reason, we represent an $[N \times K]$ analysis polyphase matrix in (35) by a $[N_P N \times N_P K] = [N' \times K']$ polyphase matrix which is obtained as follows. The filters of the N-channel FB are represented in an expanded form as

$$\mathbf{h}^E = \begin{bmatrix}\mathbf{h}_0^E\\ \mathbf{h}_1^E\\ \vdots\\ \mathbf{h}_{NN_P-1}^E\end{bmatrix} = \begin{bmatrix} h_0(0) & \cdots & h_0(L_P-1) & \mathbf{0}_{1\times(N_P-2)K} & \mathbf{0}_{1\times K}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ h_{N-1}(0) & \cdots & h_{N-1}(L_P-1) & \mathbf{0}_{1\times(N_P-2)K} & \mathbf{0}_{1\times K}\\ \mathbf{0}_{1\times K} & h_0(0) & \cdots & h_0(L_P-1) & \mathbf{0}_{1\times(N_P-2)K}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0}_{1\times K} & h_{N-1}(0) & \cdots & h_{N-1}(L_P-1) & \mathbf{0}_{1\times(N_P-2)K}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0}_{1\times K} & \mathbf{0}_{1\times(N_P-2)K} & h_0(0) & \cdots & h_0(L_P-1)\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ \mathbf{0}_{1\times K} & \mathbf{0}_{1\times(N_P-2)K} & h_{N-1}(0) & \cdots & h_{N-1}(L_P-1)\end{bmatrix}, \tag{38}$$

where $h_i(j)$ is a jth element in the ith filter impulse response. The elements of the polyphase matrix are given by

$$\begin{aligned}E_{i,j}(z) &= \sum_{l=0}^{d-1} h_i^E(K'l+j)z^{-l}, \quad d = \left\lceil \frac{(N_P-1)K+L_P}{N_P K}\right\rceil,\\ &\quad j = 0,\ldots,K'-1,\ i = 0,\ldots,N'-1.\end{aligned} \tag{39}$$

The polyphase matrix can be written as

$$\begin{aligned}\mathbf{E}(z) &= \left[\mathbf{E}_0^T(z)\ \ \mathbf{E}_2^T(z)\ \ \cdots\ \ \mathbf{E}_{N_P-1}^T(z)\right]^T\\ &= \left[\left(\mathbf{E}^S(z)\right)^T\left(\mathbf{E}^S(z)\mathbf{T}_1(z)\right)^T\ \ \cdots\ \ \left(\mathbf{E}^S(z)\mathbf{T}_{N_P-1}(z)\right)^T\right]^T,\end{aligned} \tag{40}$$

where $\mathbf{E}_j(z) = \mathbf{E}^S(z)\mathbf{T}_{(j)}(z)$,

$$\begin{aligned}\mathbf{T}_i(z) &= \begin{bmatrix}\mathbf{0}_{(K'-Ki)\times Ki} & \mathbf{I}_{K'-Ki}\\ z^{-1}\mathbf{I}_{Ki} & \mathbf{0}_{Ki\times(K'-Ki)}\end{bmatrix},\\ \mathbf{T}_{i+j}(z) &= \mathbf{T}_i(z)\mathbf{T}_j(z), \quad i+j < N_P,\\ \mathbf{E}^S(z) &= \left[\mathbf{A}_0\ \ \mathbf{A}_1\ \ \cdots\ \ \mathbf{A}_{2m-1}\right]\\ &\quad \times \mathbf{P}\left[\mathbf{I}_{N_PK}\ \ z^{-1}\mathbf{I}_{N_PK}\ \ \cdots\ \ z^{-(d-1)}\mathbf{I}_{N_PK}\right]^T\\ &= \left[\mathbf{B}_0\ \ \mathbf{B}_1\ \ \cdots\ \ \mathbf{B}_{2m}\right]\\ &\quad \times \left[\mathbf{I}_{N_PK}\ \ z^{-1}\mathbf{I}_{N_PK}\ \ \cdots\ \ z^{-(d-1)}\mathbf{I}_{N_PK}\right]^T\end{aligned} \tag{41}$$

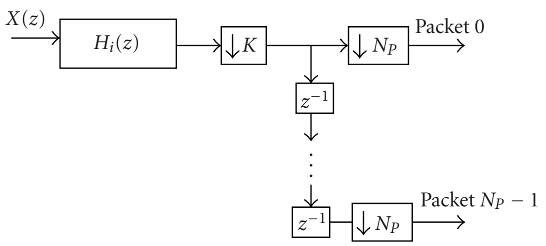

Figure 3: Packetization scheme for the ith subband in an OCMFB system.

with

$$\mathbf{A}_{2i} = (-1)^i \sqrt{N} \mathbf{\Lambda}_c \mathbf{C}(\mathbf{I}_N - \mathbf{J}_N), \quad i = 0, 1, \ldots, m-1, \tag{42}$$

$$\mathbf{A}_{2i+1} = -(-1)^i \sqrt{N} \mathbf{\Lambda}_c \mathbf{C}(\mathbf{I}_N + \mathbf{J}_N), \quad i = 0, 1, \ldots, m-1, \tag{43}$$

$$\mathbf{P} = \left[\operatorname{diag}\{p(0), p(1), \ldots, p(L_P - 1)\} \quad \mathbf{0}_{L_P \times (dN_P K - L_P)}\right], \tag{44}$$

$$\mathbf{B}_j = \mathbf{A}_j\left[\operatorname{diag}\{p(jN), \ldots, p(jN + N - 1)\}\right], \quad j = 0, 1, \ldots, 2m-1, \tag{45}$$

$$\mathbf{B}_{2m} = \mathbf{0}_{N \times (dN_P K - L_P)}. \tag{46}$$

An erasure k corresponds to a loss of coefficients generated by the rows of $\mathbf{E}_k(z)$. The term consecutive erasures or a burst of erasures denotes a loss of coefficients corresponding to rows of $\left[\mathbf{E}_k^T(z) \;\; \mathbf{E}_{k+1}^T(z) \;\; \cdots \;\; \mathbf{E}_{k+L_B}^T(z)\right]^T$, where L_B is referred to as a burst length. A burst of L_B erasures denotes a periodic erasure pattern where, out of N_P consecutive samples, L_B consecutive samples are lost in all subbands and $N_P - L_B$ consecutive samples are received in all subbands.

Proposition 2. *For $N_P K = iN$, where i is an integer, the polyphase matrix $\mathbf{E}(z)$ defined by* (40)–(46) *implements a strongly uniform tight-frame signal expansion.*

Proof. We first note that the matrix $\mathbf{E}(z)\widetilde{\mathbf{E}}(z)$ has the matrices $\mathbf{E}^S(z)\widetilde{\mathbf{E}}^S(z)$ along the main diagonal. For $N_P K = N$, $\mathbf{E}^S(z)$ represents a polyphase matrix of an N channel critically-sampled CMFB with a polyphase matrix as in (33). That is,

$$\begin{aligned}
\mathbf{F}(z) &= \mathbf{E}^S(z)\widetilde{\mathbf{E}}^S(z) \\
&= \left(\mathbf{B}_0 + z^{-1}\mathbf{B}_1 + \cdots + z^{-(2m-1)}\mathbf{B}_{2m-1}\right) \\
&\quad \times \left(\mathbf{B}_0^T + z\mathbf{B}_1^T + \cdots + z^{(2m-1)}\mathbf{B}_{2m-1}^T\right) \\
&= z^{-(2m-1)}\mathbf{B}_{2m-1}\mathbf{B}_0^T + \cdots + \left(\mathbf{B}_0\mathbf{B}_0^T + \cdots + \mathbf{B}_{2m-1}\mathbf{B}_{2m-1}^T\right) \\
&\quad + z^{-1}\left(\mathbf{B}_1\mathbf{B}_0^T + \cdots + \mathbf{B}_{2m-1}\mathbf{B}_{2m-2}^T\right) + \cdots + z^{2m-1}\mathbf{B}_0\mathbf{B}_{2m-1}^T \\
&= z^{-(2m-1)}\mathbf{F}_{-(2m-1)} + \cdots + \mathbf{F}_0 + \cdots + z^{(2m-1)}\mathbf{F}_{2m-1},
\end{aligned} \tag{47}$$

where all the terms $\mathbf{F}_i$ for $i \neq 0$ are equal to zero and $\mathbf{F}_0 = \mathbf{I}_N$.

For $N_P K = iN$ where i is an integer, we have

$$\begin{aligned}
\mathbf{F}(z) &= \Big(\left[\mathbf{B}_0 \cdots \mathbf{B}_{i-1}\right] + z^{-1}\left[\mathbf{B}_i \cdots \mathbf{B}_{2i-1}\right] \\
&\qquad + \cdots + z^{-(d-1)}\left[\mathbf{B}_{(d-1)i} \cdots \mathbf{B}_{(2m-1)}\mathbf{0}_{N \times (N(di-2m))}\right]\Big) \\
&\quad \times \Big(\left[\mathbf{B}_0 \cdots \mathbf{B}_{i-1}\right]^T + z\left[\mathbf{B}_i \cdots \mathbf{B}_{2i-1}\right]^T \\
&\qquad + \cdots + z^{(d-1)}\left[\mathbf{B}_{(d-1)i} \cdots \mathbf{B}_{(2m-1)}\mathbf{0}_{N \times (N(di-2m))}\right]^T\Big) \\
&= z^{-(d-1)}\mathbf{F}_{-(d-1)i} + \cdots + z^{-1}\mathbf{F}_{-i} \\
&\quad + \mathbf{F}_0 + z\mathbf{F}_i + \cdots + z^{(d-1)}\mathbf{F}_{(d-1)i} = \mathbf{I}_N,
\end{aligned} \tag{48}$$

where $d = \lceil 2m/i \rceil$. It is shown in [16] that strongly uniform frames are implemented by an $[N \times K]$ polyphase matrix $\mathbf{E}(\omega)$ with the following property:

$$\sum_{k=0}^{K-1} \left|\mathbf{E}_{n,k}(\omega)\right|^2 = 1 \tag{49}$$

for $n = 1, \ldots, N$ and $\omega \in [-\pi, \pi]$. This is equivalent to the property that all diagonal elements of $\mathbf{E}(\omega)\mathbf{E}^H(\omega)$ are equal to 1 [16]. Hence, for $N_P K = iN$, $N_P = iL$, the polyphase matrix $\mathbf{E}(z)$ defined by (40)–(46) implements a strongly uniform tight-frame signal expansion. □

Note that for $N_P K = iN$, that is, $N_P = iL$, where i is integer, every matrix $\langle \mathbf{E} \rangle^k(z)$ consisting of the following rows of $\mathbf{E}(z)$:

$$\langle \mathbf{E} \rangle^k(z) = \left[\mathbf{E}_k^T(z) \;\; \mathbf{E}_{k+L}^T(z) \;\; \cdots \;\; \mathbf{E}_{k+(i-1)L}^T(z)\right]^T, \tag{50}$$

where $0 < k \leq L-1$, is a square paraunitary polyphase matrix of the critically sampled CMFB. That is, we have

$$\langle \mathbf{E} \rangle^k(z)\langle \widetilde{\mathbf{E}} \rangle^k(z) = \langle \widetilde{\mathbf{E}} \rangle^k(z)\langle \mathbf{E} \rangle^k(z) = \mathbf{I}_{N_P K}, \tag{51}$$

$$\sum_{k=0}^{L-1} \langle \widetilde{\mathbf{E}} \rangle^k(z)\langle \mathbf{E} \rangle^k(z) = L\mathbf{I}_{N_P K}. \tag{52}$$

4.3. OFB codes composed of two CMFB polyphase matrices

The optimal design of an OFB in the rate-distortion sense and for an erasure channel is a difficult task [18] and requires radical changes of the system. Oversampling the outputs of an FB already used for subband decomposition is a simple way to add erasure resilience to the transmitted signal. However, in order to allow more freedom for the way the redundancy is introduced to the system, and possibly facilitate and/or improve the decoding performance, we also consider providing erasure resilience by adding an additional OFB after subband decomposition by a critically sampled FB.

For example, it was shown in [8] that an (N, K) DFT code with K even is robust to $N - K - 1$ erasures.

Here, we consider codes which are similarly structured as DFT [8] codes, but have higher-order polyphase matrices.

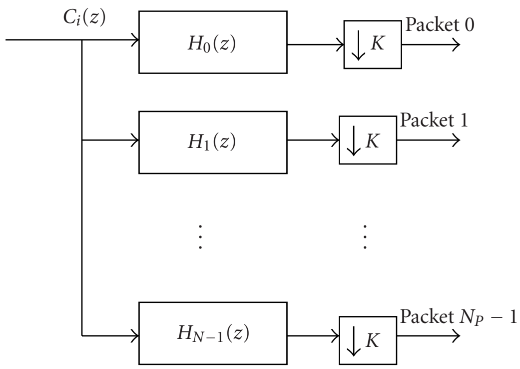

FIGURE 4: Packetization scheme for subband i in a CMFB-OFB system.

The polyphase matrix of such codes is given by

$$\mathbf{E}(z) = \sqrt{\frac{N}{K}}\mathbf{C}(z)\mathbf{B}\mathbf{A}(z), \tag{53}$$

where $\mathbf{C}(z)$ is a synthesis polyphase matrix for an N channel critically sampled CMFB, $\mathbf{A}(z)$ is an analysis polyphase matrix of a K channel critically sampled CMFB, and $\mathbf{B}$ is an $[N \times K]$ matrix given by $\mathbf{B} = [\mathbf{I}_K \ \ \mathbf{0}]^T$. Since matrices $\mathbf{C}(z)$ and $\mathbf{A}(z)$ are paraunitary, the polyphase matrix of this code is paraunitary as well. Indeed,

$$\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = \frac{N}{K}\widetilde{\mathbf{A}}(z)\mathbf{B}^T\widetilde{\mathbf{C}}(z)\mathbf{C}(z)\mathbf{B}\mathbf{A}(z) = \frac{N}{K}\mathbf{I}_K. \tag{54}$$

The system using this code is referred to as CMFB with an OFB code (CMFB-OFB).

Packetization

For the systems with a DFT code or a CMFB-OFB code, we assume that an input signal subband decomposition is obtained by a critically sampled FB. Each subband signal is arranged in a one-dimensional array and encoded by an (N, K) CMFB-OFB or a CMFB-DFT code. In these structures, the simultaneous outputs of the oversampled filters are placed in different packets. The packetization scheme for the ith subband signal $C_i(z)$ is shown in Figure 4. Loosing the packet k means that the kth CMFB-OFB output is completely lost in every subband.

5. CODING AND FRAME-THEORETIC PROPERTIES OF CMFB-BASED OFB CODES

As it was shown in [16, 17], PR after E erasures and no quantization noise is possible if and only if the analysis polyphase matrix after erasures, denoted by $\mathbf{E}_R(z)$, is of full rank on the unit circle. This is a quite general statement which does not give much insight into the erasure resilience of an OFB code. It is of interest to characterize an erasure-correcting code more precisely based on its structure and parameters such as filter lengths, number of channels, and decimation factors. Although OFB codes based on CMFB have a simple structure, it is difficult to analytically examine the rank of the analysis polyphase matrix after erasures for all erasure patterns. However, for some erasure patterns, the structure of the code guarantees PR in absence of quantization noise. In this section, we theoretically study PR properties of CMFB-based codes for erasure patterns for which we can show that PR is guaranteed by the general structure of the code and does not depend on particular prototype filters. The remarks on the correctability of some other erasure patterns are made based on experimentation results.

5.1. *Properties of OCMFB codes*

For the OCMFB code, we discuss bursty erasures and erasure patterns for which PR property is the straightforward consequence of the code structure.

5.1.1. *Bursty erasures*

The analysis matrix after E consecutive erasures with erasure indices $\{0, \ldots, E-1\}$ is given by

$$\begin{aligned}\mathbf{E}_R(z) &= \left[\mathbf{E}_E^T(z) \ \ \mathbf{E}_{E+1}^T(z) \ \ \cdots \ \ \mathbf{E}_{N_P-1}^T(z)\right]^T \\ &= \left[\mathbf{E}_0^T(z) \ \ \mathbf{E}_1^T(z) \ \ \cdots \ \ \mathbf{E}_{R-1}^T(z)\right]^T\mathbf{T}_E(z),\end{aligned} \tag{55}$$

where $\mathbf{E}_j(z)$ and $\mathbf{T}_i(z)$ are defined in (40). The total number of packets is given by $N_P = E + R$. PR is possible if and only if $\mathbf{E}_R(z)$ is of full rank on the unit circle.

Proposition 3. *Any erasure pattern consisting of E consecutive or circularly consecutive erasures is correctable if and only if the matrix* $\mathbf{E}_R(z)$ *in* (55) *is of full rank on the unit circle.*

Proof. Consider an erasure burst with erasure indices $\{k, \ldots, k+E-1\}$. We denote the analysis matrix after erasures for this erasure pattern by $\mathbf{E}_R^A$. This matrix can be written as

$$\mathbf{E}_R^A(z) = \left[\mathbf{E}_0^T(z) \ \ \cdots \ \ \mathbf{E}_{k-1}^T(z) \ \ \cdots \ \ \mathbf{E}_{k+E}^T(z) \ \ \cdots \ \ \mathbf{E}_{N_P-1}^T(z)\right]^T. \tag{56}$$

Then, we have

$$\begin{aligned}&\widetilde{\mathbf{E}}_R^A(z)\mathbf{E}_R^A(z) \\ &\quad = L\mathbf{I} - \widetilde{\mathbf{E}}_E(z)\mathbf{E}_E(z) \\ &\quad = L\mathbf{I} - \widetilde{\mathbf{E}}_k(z)\mathbf{E}_k(z) - \cdots - \widetilde{\mathbf{E}}_{k+E-1}(z)\mathbf{E}_{k+E-1}(z) \\ &\quad = \widetilde{\mathbf{T}}_k(z)\big(L\mathbf{I} - \widetilde{\mathbf{E}}_0(z)\mathbf{E}_0(z) - \cdots - \widetilde{\mathbf{E}}_{E-1}(z)\mathbf{E}_{E-1}(z)\big)\mathbf{T}_k(z) \\ &\quad = \widetilde{\mathbf{T}}_k(z)\big(\widetilde{\mathbf{E}}_E(z)\mathbf{E}(z) + \cdots + \widetilde{\mathbf{E}}_{N_P-1}(z)\mathbf{E}_{N_P-1}(z)\big)\mathbf{T}_k(z) \\ &\quad = \widetilde{\mathbf{T}}_k(z)\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z)\mathbf{T}_k(z),\end{aligned} \tag{57}$$

where $\mathbf{T}_k(z)$ is a paraunitary matrix defined in (40) and $\mathbf{E}_R(z)$ is as in (55). Since $\mathbf{T}_i(z)$ is paraunitary, it follows that $\mathbf{E}_R^A(z)$ is of full rank on the unit circle if and only if $\mathbf{E}_R(z)$ is of full rank. A similar case can be shown for an erasure burst consisting of circularly consecutive erasures with indices $\{0,\dots,k-1,k+R,\dots,N_P-1\}$. □

From Proposition 3, it follows that in the analysis of bursty erasures it is sufficient to consider the erasure burst with erasure indices $\{0,\dots,E-1\}$. Note that this proof does not depend on the fact that an FB is cosine modulated. It is therefore valid for any paraunitary FB.

Remark 1. The maximum number of consecutive erasures which can be corrected is limited by $E \le L_V$, where $L_V = L_P/K - 1 = 2mN/K - 1$ and L_P is the prototype filter length. For more than L_V erasures, the polyphase matrix $\mathbf{E}(z)$ in (40) and the encoding matrix $\mathbf{H}$ in (3) after removal of the rows corresponding to the erasure pattern contain all zero columns. The input samples corresponding to these columns cannot be recovered. Similarly, we can conclude that for an erasure pattern with exactly $E = L_V$ consecutive erasures, the necessary condition for PR is the reception of $R \ge \lceil ((L_V+1)K-K)/(N-K)\rceil = \lceil L_V/(L-1)\rceil$ packets. This follows from the fact that the analysis polyphase matrix after erasures has to have a sufficient number of rows in order to be of full rank.

Proposition 4. *Consider an $L = 2$-times oversampled N channel OCMFB with the polyphase matrix $\mathbf{E}(z)$ defined in* (40)–(46). *Reception of $R = L_V$ packets is a sufficient condition for PR of $E = L_V$ consecutive erasures, if and only if none of the first K coefficients of the prototype filter is zero.*

Proof. The analysis polyphase matrix after erasures with indices $\{L_V,\dots,2L_V-1\}$ is given by

$$\mathbf{E}_R(z) = [\mathbf{A} \;\; \mathbf{B}], \tag{58}$$

where $\mathbf{A}$ and $\mathbf{B}$ are matrices of scalars given by

$$\mathbf{A}_{NL_V\times KL_V} = \begin{bmatrix} \mathbf{H}_0 & \mathbf{H}_1 & \cdots & \mathbf{H}_{L_V-2} & \mathbf{H}_{L_V-1} \\ \mathbf{0} & \mathbf{H}_0 & \cdots & \mathbf{H}_{L_V-1} & \mathbf{H}_{L_V-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{H}_0 \end{bmatrix},$$
$$\mathbf{B}_{NL_V\times KL_V} = \begin{bmatrix} \mathbf{H}_{L_V} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{H}_{L_V-1} & \mathbf{H}_{L_V-2} & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{H}_1 & \mathbf{0} & \cdots & \mathbf{H}_{L_V-1} & \mathbf{H}_{L_V} \end{bmatrix}. \tag{59}$$

PR is possible if and only if this matrix is of full rank.

The analysis polyphase matrix of the OCMFB (without erasures) can be written as

$$\mathbf{E}(z) = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ z^{-1}\mathbf{B} & \mathbf{A} \end{bmatrix}. \tag{60}$$

From the paraunitary condition $\widetilde{\mathbf{E}}(z)\mathbf{E}(z) = L\mathbf{I}_K$, we have

$$\mathbf{B}^T\mathbf{A} = \mathbf{0}, \qquad \mathbf{A}^T\mathbf{B} = \mathbf{0}$$
$$\mathbf{B}^T\mathbf{B} + \mathbf{A}^T\mathbf{A} = L\mathbf{I}_K. \tag{61}$$

Since $\mathbf{B}^T\mathbf{A} = \mathbf{0}$, we have

$$\begin{aligned}\text{range}\{\mathbf{A}\} \cap \text{range}\{\mathbf{B}\} = \{\mathbf{0}\} &\Longrightarrow \text{rank}\{[\mathbf{A} \;\; \mathbf{B}]\} \\ &= \text{rank}\{\mathbf{A}\} + \text{rank}\{\mathbf{B}\}.\end{aligned} \tag{62}$$

Therefore the analysis matrix after erasures in (58) is invertible if and only if $\text{rank}\{\mathbf{A}\} = \text{rank}\{\mathbf{B}\} = KL_V$. That is, the matrices $\mathbf{A}$ and $\mathbf{B}$ have to be of full column rank. We further show that this is the case if and only if $\det([\text{diag}\{p(0),\dots,p(K-1)\}]) \ne 0$, where $p(j)$ is the jth prototype filter coefficient.

The matrices $\mathbf{H}_j$ can be written as

$$\begin{aligned}\mathbf{H}_{(i-1)L+(k-1)} &= \mathbf{B}_i\big[\mathbf{0}_{K\times(k-1)K} \;\; \mathbf{I}_K \;\; \mathbf{0}_{K\times(L-k)K}\big]^T, \\ &i = 1,\dots,2m,\; k = 1,\dots,L,\end{aligned} \tag{63}$$

where $\mathbf{B}_i$ is defined in (46). For example, the matrices $\mathbf{H}_0$ and $\mathbf{H}_{L_V}$ are given by

$$\mathbf{H}_0 = \sqrt{N}\boldsymbol{\Lambda}_c\mathbf{C}\begin{bmatrix}\mathbf{I}_K \\ -\mathbf{J}_K\end{bmatrix}\begin{bmatrix} p(0) & & \\ & \ddots & \\ & & p(K-1)\end{bmatrix},$$
$$\mathbf{H}_{L_V} = -(-1)^{m-1}\sqrt{N}\boldsymbol{\Lambda}_c\mathbf{C}\begin{bmatrix}\mathbf{I}_K \\ \mathbf{J}_K\end{bmatrix}\begin{bmatrix} p(K-1) & & \\ & \ddots & \\ & & p(0)\end{bmatrix}, \tag{64}$$

where $\boldsymbol{\Lambda}_c$ and $\mathbf{C}$ were specified in Section 4.1.

From the structure of $\mathbf{H}_j$ it can be easily concluded that, for $N = 2K$, the rank of this matrix is equal to the number of nonzero terms in $[\text{diag}\{p(jK),\dots,p(jK+K-1)\}]$.

Let $\det(\text{diag}\{p(0),\dots,p(K-1)\}) \ne 0$. Then, we have $\text{rank}\{\mathbf{H}_0\} = \text{rank}\{\mathbf{H}_{L_V}\} = K$ and from the block triangular structure of matrices $\mathbf{A}$ and $\mathbf{B}$, it follows that $\text{rank}\{\mathbf{A}\} = \text{rank}\{\mathbf{B}\} = KL_V$. That is, $\text{rank}\{[\mathbf{A} \;\; \mathbf{B}]\} = 2L_VK = L_VN$, which proves that the analysis matrix after erasures is invertible.

Let $p(i) = 0$, for some $0 \le i \le K-1$. Then the matrices $\mathbf{H}_0$ and $\mathbf{H}_{L_V}$ do not have full column rank. As the columns of $\mathbf{H}_0$ ($\mathbf{H}_{L_V}$) define the first K columns of $\mathbf{A}$ (the last K columns of $\mathbf{B}$), it follows that $\text{rank}(\mathbf{A}) < L_VK$ and $\text{rank}(\mathbf{B}) < L_VK$. Consequently, $\text{rank}\{[\mathbf{A} \;\; \mathbf{B}]\} < 2L_VK$. Hence, the analysis matrix is, in this case, not invertible. □

Remark for $N = 2$

For a 2-channel OFB with analysis filters lengths L_P, the sufficient condition for the PR of L_P-1 consecutive erasures is a successful reception of L_P-1 packets [20]. The proof is quite general and relies only on the fact that polynomials $H_0(z)$ and $H_1(z)$ representing the z transform of the 2-channel FB filter responses are relatively prime.

Remark for $N_P \ne 2L_V$

The experimental results show that reconstruction filters in the case of bursty erasures are, in general, of infinite impulse response (IIR). As the burst length increases, the zeros of the invariant factors in the Smith form of the analysis polyphase matrix after erasures move closer to the unit circle. For example, in a two-time oversampled 4-channel CMFB, with $N_P = 8$, the maximum number of consecutive erasures that can be corrected is 3. For this example and 4 consecutive erasures, the analysis polyphase matrix has 2 zeros on the unit circle.

Remark for $L > 2$

For $L > 2$, it is difficult to analytically prove PR for $E = L_V$ and $R = \lceil ((L_V + 1)K - K)/(N - K) \rceil = \lceil L_V/(L - 1) \rceil$. The experimental results with an $N = 4$-channel OFB with filter length $L_P = 16$ and oversampling $L = 4$ show that $E = L_V = 15$ consecutive erasures can be recovered, provided that the number of the received packets is $R = ((L_V + 1)K - K)/(N - K) = 5$.

5.1.2. Some other correctable erasure patterns

We now list some properties of an OCMFB which are the straightforward consequence of the fact that the OFB is obtained from a critically sampled FB by reducing the downsampling factors. We assume that $N_P = iL$ and that i is an integer.

Proposition 5. *PR is possible for any set of erasures for which the analysis matrix after erasures $\mathbf{E}_R(z)$ contains rows of $\mathbf{E}(z)$ given by $\langle \mathbf{E} \rangle^k(z)$ in* (50).

Proof. This proposition follows from the fact that $\langle \mathbf{E} \rangle^k(z)$ is a polyphase matrix of a critically sampled paraunitary N channel CMFB. If a finite set of channels has a subset that is a frame, then the original set of channels is also a frame [16]. Therefore, any larger set of received packets containing packets corresponding to $\langle \mathbf{E} \rangle^k(z)$ allows PR. □

Corollary 1. *For the considered packetization scheme, with $N_P K = iN$, one erasure can always be recovered.*

Proof. For $L > 1$, the analysis matrix after one erasure always contains rows of $\mathbf{E}(z)$ given by $\langle \mathbf{E} \rangle^k(z)$. Therefore, a single erasure is always correctable. □

Proposition 6. *For a single erasure, the reconstruction filters are FIR.*

Proof. For the erasure at position k, the analysis matrix after erasures is given by

$$\begin{aligned}
\mathbf{E}_R(z) &= \left[\mathbf{E}_0^T(z) \ \cdots \ \mathbf{E}_{k-1}^T(z) \ \mathbf{E}_{k+1}^T(z) \ \cdots \ \mathbf{E}_{N_P-1}^T(z)\right]^T, \\
\det\{\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z)\} &= \det\{L\mathbf{I}_{N_P K} - \widetilde{\mathbf{E}}_E(z)\mathbf{E}_E(z)\} \\
&= \det\{L\mathbf{I}_N - \mathbf{E}_k(z)\widetilde{\mathbf{E}}_k(z)\},
\end{aligned} \tag{65}$$

where $\mathbf{E}_k(z)$ is as in (40).

Since the rows in $\mathbf{E}_k(z)$ are pairwise orthogonal (51), we have

$$\det\{\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z)\} = \det\{(L-1)\mathbf{I}_N\} = \text{const}. \tag{66}$$

That is, the analysis matrix after erasures is paraunitary. It has been shown in [17] that, for a frame associated with an FIR FB with the polyphase analysis matrix $\mathbf{E}(z)$, its dual frame (frame corresponding to the parapseudoinverse of $\mathbf{E}(z)$) consists of finite-length vectors if and only if $\widetilde{\mathbf{E}}(z)\mathbf{E}(z)$ is unimodular. Hence, the reconstruction filters are FIR. □

Let S_k be a set of the erasure indices given by $S_k = \{k, k+L, \ldots, k+(i-1)L\}$, $k = 0, \ldots, L-1$. The rows of the analysis polyphase matrix $\mathbf{E}(z)$ corresponding to the erasure pattern with erasure indices given by $S_o \subset S_k$ are pairwise orthogonal. This can be seen from (51). That is, for j orthogonal erasures, we have

$$\begin{aligned}
\mathbf{E}_E(z) &= \left[\mathbf{E}_{k+i_1L}^T(z) \ \mathbf{E}_{k+i_2L}^T(z) \ \cdots \ \mathbf{E}_{k+i_jL}^T(z)\right]^T, \\
\mathbf{E}_E(z)\widetilde{\mathbf{E}}_E(z) &= \mathbf{I}_{jN},
\end{aligned} \tag{67}$$

where $0 \le i_k < i$.

Similarly, from (52), we can observe that for the rows of $\mathbf{E}(z)$ corresponding to the erasure pattern with erasure indices given by a set $S_t = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_j}$, $0 < i_k \le L-1$, we have

$$\begin{aligned}
\mathbf{E}_E(z) &= \left[\langle \mathbf{E} \rangle^{i_1}(z)^T \ \langle \mathbf{E} \rangle^{i_2}(z)^T \ \cdots \ \langle \mathbf{E} \rangle^{i_j}(z)^T\right]^T, \\
\widetilde{\mathbf{E}}_E(z)\mathbf{E}_E(z) &= j\mathbf{I}_{N_P K},
\end{aligned} \tag{68}$$

where $\langle \mathbf{E} \rangle^{i_k}(z)$ is as in (50). That is, the rows of $\mathbf{E}(z)$ corresponding to the erasure pattern with erasure indices given by a set S_t form a tight frame.

Proposition 7. *For the erasure patterns having erasure indices given by the sets S_o, S_t, and $S_{to} = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_j} \cup S_o$, $0 < j \le L-2$, the reconstruction filters are FIR.*

This can be shown by following the same procedure as in the proof of Proposition 2, and by using (67) and (68).

5.2. Properties of the code composed of two CMFB polyphase matrices

Since the encoding by a CMFB-OFB consists of similar operations as in the case of DFT codes, we may expect that this code has similar performance in terms of PR and an improved performance in terms of mean-square reconstruction error due to the higher order of the polyphase filters.

Here, we define symmetric erasures as erasure patterns with one or more pairs of erasures with indices of the form $\{k, N_P - 1 - k\}$. That is, symmetric erasure patterns with two erasures are given by the set of erasure indices $\{k, N_P - 1 - k\}$, where $0 \le k \le \lfloor (N-K)/2 \rfloor + 1$.

Proposition 8. *For an (N, K) OFB code composed of two CMFB polyphase matrices as in* (40)*, any set of $E \le (N-K)$ symmetric erasures is correctable, and the parapseudoinverse reconstruction filters are FIR.*

Proof. For symmetric erasure patterns, $\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z)$ is given by

$$\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z) = \sqrt{\frac{N}{K}}\widetilde{\mathbf{A}}(z)[\mathbf{I}_K \quad \mathbf{0}]\mathbf{C}^T \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}}_{\mathbf{D}} \times \mathbf{C}\begin{bmatrix} \mathbf{I}_K \\ \mathbf{0} \end{bmatrix}\mathbf{A}(z)\sqrt{\frac{N}{K}}, \tag{69}$$

where $\mathbf{D}$ is a diagonal matrix with R nonzero elements in positions corresponding to the indices of the R received channels. That is we, can write

$$\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z) = \frac{N}{K}\widetilde{\mathbf{A}}(z)\mathbf{C}_{K\times R}^T\mathbf{C}_{R\times K}\mathbf{A}(z). \tag{70}$$

Since $\mathbf{A}(z)$ is paraunitary, for $R \geq K$, we have $\det\{\widetilde{\mathbf{E}}_R(z)\mathbf{E}_R(z)\} = \det\{(N/K)\mathbf{C}_{K\times R}^T\mathbf{C}_{R\times K}\} = \text{const}$. This proves both PR for E symmetric erasures, and that reconstruction filters are FIR. □

Corollary 2. *Any erasure pattern with $E \leq \lfloor (N-K)/2 \rfloor + 1$ is correctable.*

Remark 2. In general, except for the symmetric erasures, the synthesis filters are of IIR and noncausal. In contrast with OCMFB codes, the reconstruction filters may be IIR even for the case of one erasure. It has been observed from the experiments that, as the number of erasures increases, the poles of the IIR synthesis filters approach the unit circle.

6. RECONSTRUCTION IN PRESENCE OF QUANTIZATION NOISE

Apart from verifying PR in presence of erasures, it is necessary to evaluate the performance of the OFB codes in presence of quantization error. The mean-square reconstruction error is the main performance criteria for source coding systems. For the N-dimensional quantization noise process $[q_0[n] \cdots q_{N-1}[n]]^T$ of $[N \times N]$ power spectral matrix $\mathbf{S}_q(z) = \sum_{l=-\infty}^{\infty} \mathbf{C}_q(l)z^{-l}$, where $\mathbf{C}_q(l) = E\{\mathbf{q}(n)\mathbf{q}(n-l)^H\}$ and $E\{\cdot\}$ is the expectation operator, the MSE is given by [19]

$$\text{MSE} = \frac{\sigma^2}{2\pi K}\int_{-\pi}^{\pi} \text{trace}\left\{\mathbf{R}(\omega)\mathbf{S}_q(\omega)\mathbf{R}^H(\omega)\right\}d\omega, \tag{71}$$

where $\mathbf{R}(\omega)$ is the synthesis polyphase matrix. For the uncorrelated white quantization noise model with the same variances $\sigma^2 = E\{|q_k(n)|^2\}$, the power spectral matrix is given by $\mathbf{S}_q(z) = \sigma^2\mathbf{I}_N$. For this noise model and the parapseudoinverse receiver, the MSE due to quantization is given by [16]

$$\text{MSE} = \frac{\sigma^2}{2\pi K}\int_{-\pi}^{\pi} \text{trace}\left\{\left[\mathbf{E}_R^H(\omega)\mathbf{E}_R(\omega)\right]^{-1}\right\}d\omega, \tag{72}$$

where $\mathbf{E}_R(\omega)$ is the analysis polyphase matrix after erasures. Since all considered OFB codes implement tight-frame signal expansion and the MSE in absence of erasures is equal to

$$\begin{aligned} \text{MSE} &= \frac{\sigma^2}{2\pi K}\int_{-\pi}^{\pi} \text{trace}\left\{\left[\mathbf{E}^H(\omega)\mathbf{E}(\omega)\right]^{-1}\right\}d\omega \\ &= \frac{\sigma^2}{2\pi K}\int_{-\pi}^{\pi} \text{trace}\left\{\left(\frac{N}{K}\mathbf{I}_K\right)^{-1}\right\}d\omega = \frac{K}{N}\sigma^2. \end{aligned} \tag{73}$$

Remark 3. For $L > 1$, the assumption of uncorrelated white noise is not justified [19]. For correlated noise, the expression for the MSE depends on the noise power spectral matrix $\mathbf{S}_q(\omega)$ [19]. However, if one assumes simple additive white noise model, and that the noise sequences generated by two different channels are pairwise uncorrelated, one can derive simple expressions for the MSE for certain erasure patterns.

6.1. MSE in the system with an OCMFB code

In general, the MSE depends on the filter coefficients and has to be calculated as in (72). However, for the pairwise orthogonal erasures or erasures for which erased rows of the analysis polyphase matrix form a tight frame, the MSE is independent of the filter coefficients. In addition to this, it has been proven in [16] that, if the original frame is strongly uniform, the MSE is minimum for these erasure patterns.

We assume uncorrelated white noise model with $E\{|q_k(n)|^2 = \sigma^2\}$, $N_P = iL$, and i integer.

For erasure patterns corresponding to j, $0 < j \leq i$, pairwise orthogonal erasures, the MSE is given by

$$\begin{aligned} \text{MSE}_o &= \frac{\sigma^2}{2\pi N_P K}\int_{-\pi}^{\pi} \text{trace}\left\{\left(\mathbf{E}_R^H(\omega)\mathbf{E}_R(\omega)\right)^{-1}\right\}d\omega \\ &= \frac{\sigma^2}{2\pi N_P K}\int_{-\pi}^{\pi} \text{trace}\left\{\left(L\mathbf{I}_{N_P K} - \left[\mathbf{E}_{k+i_1 L}^H(\omega)\cdots\mathbf{E}_{k+i_j L}^H(\omega)\right] \times \left[\left(\mathbf{E}_{k+i_1 L}\right)^T(\omega)\cdots\left(\mathbf{E}_{k+i_j L}\right)^T(\omega)\right]^T\right)^{-1}\right\}d\omega. \end{aligned} \tag{74}$$

Using the matrix inversion lemma, the fact that the rows corresponding to the erasures are pairwise orthogonal and that the original frame is uniform, we get

$$
\begin{aligned}
&\mathrm{MSE}_o \\
&= \frac{\sigma^2}{2\pi N_P K} \int_{-\pi}^{\pi} \operatorname{trace}\left\{ \left(\frac{1}{L} \mathbf{I}_{N_P K} + \frac{1}{L^2} \left[\mathbf{E}_{k+i_1 L}^H(\omega) \cdots \mathbf{E}_{k+i_j L}^H(\omega) \right] \right.\right. \\
&\qquad \left.\left. \times \frac{L}{(L-1)} \left[\mathbf{E}_{k+i_1 L}^T(\omega) \cdots \mathbf{E}_{k+i_j L}^T(\omega) \right]^T \right) \right\} d\omega \\
&= \frac{\sigma^2}{N_P K} \left(\frac{1}{L} N_P K + \frac{1}{L(L-1)} jN \right) = \sigma^2 \frac{K}{N} \left(1 + \frac{jK}{i(N-K)} \right).
\end{aligned} \tag{75}
$$

Further, for erasure patterns with erasure indices given by $S_t = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_j}$, $0 < j \le L-1$, the MSE is given by

$$
\begin{aligned}
\mathrm{MSE}_t &= \frac{\sigma^2}{2\pi N_P K} \int_{-\pi}^{\pi} \operatorname{trace}\left\{ \left(L \mathbf{I}_{N_P K} \right.\right. \\
&\qquad \left.\left. - \langle \mathbf{E} \rangle^{i_1 - i_j H}(\omega) \langle \mathbf{E} \rangle^{i_1 - i_j}(\omega) \right)^{-1} \right\} d\omega \\
&= \frac{\sigma^2}{2\pi N_P K} \int_{-\pi}^{\pi} \operatorname{trace}\left\{ \left((L-j) \mathbf{I}_{N_P K} \right)^{-1} \right\} d\omega \\
&= \frac{\sigma^2}{L-j} = \sigma^2 \frac{K}{N} \frac{1}{1 - jK/N},
\end{aligned} \tag{76}
$$

where $\langle \mathbf{E} \rangle^{i_1 - i_j}(\omega) = [[\langle \mathbf{E} \rangle^{i_1}(\omega)]^T \cdots [\langle \mathbf{E} \rangle^{i_j}(\omega)]^T]^T$.

Similarly, for erasure patterns with erasure indices given by $S_{to} = S_{i_1} \cup S_{i_2} \cup \cdots \cup S_{i_{j_2}} \cup S_o$, $0 < j_2 \le L-2$, and $S_o = \{k + m_1 L, k + m_2 L, \ldots, k + m_{j_1} L\}$, $0 < j_1 \le i$, the MSE is given by $\mathrm{MSE}_{t+o} = (\sigma^2/(L-j_1))(1 + j_2/(L-j_1-1)i)$.

6.2. *MSE in a system with a code composed of two CMFB polyphase matrices*

For symmetric erasures, the MSE is dependent on the positions of the erasures. However, it does not depend on the prototype filter coefficients. The MSE in this case is given by

$$
\mathrm{MSE}_{\mathrm{sym}} = \frac{\sigma^2}{N} \operatorname{trace}\left\{ \left(\mathbf{C}_{R\times K}^T \mathbf{C}_{R\times K} \right)^{-1} \right\}, \tag{77}
$$

where $\mathbf{C}_{R\times K}$ is a matrix obtained from $\mathbf{C}_{N\times K}$ by removing the rows which correspond to the erasure positions.

7. SIMULATION RESULTS

In this section, we evaluate the performance of the described OFB codes by simulation for the example of an image transmission system. The parameters of the simulated codes are as follows. A CMFB used for signal decomposition is an $N = 4$-channel FB formed from a prototype filter of length 16. The image subband decomposition is obtained by applying filtering first on the columns and then on the rows of the image. The oversampling ratio is $L = 2$. In the OCMFB system, the redundancy is introduced in the horizontal filtering stage. The polyphase matrix of the CMFB-OFB code is built from the polyphase matrices of the 8- and the 4-channel CMFBs with prototype filter lengths 32 and 16, respectively. The DFT code is the (8, 4) DFT code from [8]. There are $N_P = 8$ packets per image. The packets are formed as explained in Section 4. We consider uniform scalar quantization with a following mapping of subband coefficients $y^i[n]$ to quantized symbols $\hat{y}^i[n] : \hat{y}^i[n] = \delta^i \operatorname{round}(y^i[n]/\delta^i)$, where δ^i is the quantization step size in subband i. The quantizers in the different subbands are different in the sense that they employ different quantization step sizes. The set of optimal quantizers step size is chosen from the set of admissible quantizers step sizes by optimizing the rate-distortion performance [29]. In this optimization, we have assumed first-order Markov model for the subband coefficients. The parameters of the Markov model are estimated by simulation. The rate has been estimated based on the entropy per symbol for the two-symbol block [30]. The optimization is performed for the system with no erasures.

For the considered system parameters, PR is verified numerically. It has been found that the considered codes can correct any erasure pattern with three erasures, or less. For more than three erasures, there are erasure patterns for which the analysis matrix after erasures is either singular or very close to singular on the unit circle. The synthesis filters are calculated based on the parapseudoinverse of the analysis matrix after erasures. The impulse responses of the reconstruction filters which are infinite are truncated. The results are obtained for the gray-scale [512×512] Lena image. The MSE for the system with CMFB and no error protection is 26.6417. The overall rate is equal to 0.448 bits/sample. The rate in the systems with OCMFB, CMFB-OFB, and CMFB-DFT is 0.446, 0.444, and 0.448 bits/sample, respectively.

Table 1 shows the MSE averaged over all erasure patterns, consecutive and circularly consecutive erasures, and over non-consecutive erasures, for the various JSCC approaches. From Table 1, we can observe that in the case of no erasures or one erasure the MSE in the systems with OFB codes is lower than that in an uncoded system. That is, in the systems with OFB codes, a part of the quantization noise is corrected. For two erasures, the average MSE in the systems with OFBs is comparable to that of an uncoded system. However, as the number of erasures increases, the differences between the MSE for various erasure patterns increase. As in the case of DFT codes [8], the largest MSE is obtained in the case of consecutive and circularly consecutive erasures in all structures. For these erasure patterns, a degradation can be visually observed in the reconstructed images. The visual degradation is less pronounced in the case of two than in the case of three erasures. Up to two erasures, the average MSE is minimum for the OCMFB. For three erasures, the CMFB-OFB outperforms the OCMFB code. Both OFB codes outperform the DFT code. Figures 5 and 6 illustrate the visual impact of erasures in the OCMFB and CMFB-OFB systems, respectively. These figures show the reconstructed images for which the erasure patterns yield worst MSE.

The above results give a flavor of how the performance of various JSCC schemes compares with the performance of the classical tandem JSCC. That is, We consider the system where the packets are protected by an (N, K) Reed-Solomon code. Then the perfect recovery of the quantized coefficients

TABLE 1: MSE in a system with a subband decomposition by a CMFB and various FB codes.

OFB structure	No. erasures	MSE	MSE cons.	MSE noncons.
OCMFB	0	16.2643	16.2643	16.2643
CMFB-DFT	0	22.6748	22.6748	22.6748
CMFB-OFBC	0	21.7332	21.7332	21.7332
OCMFB	1	18.1042	18.1042	18.1042
CMFB-DFT	1	24.6557	24.6557	24.6557
CMFB-OFBC	1	23.8859	23.8859	23.8859
OCMFB	2	25.1636	37.8657	20.0828
CMFB-DFT	2	28.2648	31.1946	27.0929
CMFB-OFBC	2	28.1666	32.9071	26.2704
OCMFB	3	49.7256	96.3684	41.9518
CMFB-DFT	3	53.3442	169.7165	33.9488
CMFB-OFBC	3	43.6894	99.0508	34.4625

FIGURE 5: Reconstructed image for the erasure pattern $(1, 2, 8)$ and OCMFB code, MSE = 99.3577.

FIGURE 6: Reconstructed image for the erasure pattern $(2, 3, 4)$ and CMFB-OFB code, MSE = 143.2362.

is possible for any $N-K$ erasures. For all erasure patterns, the MSE is the same and equal to the MSE after reconstruction by the synthesis filters of the critically sampled CMFB in the absence of erasures. It can be concluded that the JSCC approaches with OFBs can be of interest when the number of erasures is not high with respect to the erasure-correcting capability of the code. This is due to the fact that OFBC and OTC reduce the MSE in case of few erasures.

8. CONCLUSIONS

In this paper, we have studied erasure resilience of OFBs in the context of multiple description coding. We have discussed the analogies between OFBs and channel codes and showed that signal reconstruction methods derived from the FB theory and coding theory are equivalent even in presence of quantization error. We have further presented a semianalytical analysis of the two OFB structures based on CMFBs. That is, we have pointed out the erasure patterns for which PR is guaranteed by the general structure of OFB code and does not depend on particular prototype filters. It has been shown that, with a suitable choice of the parameter $N_P = iL$, where i is an integer, the polyphase matrix of an L-times oversampled CMFB implements a strongly uniform frame and is robust to one erasure. With this choice of the parameter $N_P = iL$, there is a set of erasure patterns for which the conditions for PR by an OCMFB code are automatically fulfilled since the received packets contain coefficients generated by the critically sampled CMFB. It is also shown that L_V consecutive erasures can be recovered by a two-times OFB, with $N_P \geq 2L_V$. For (N, K) OFB code composed of two CMFB polyphase matrices, we have shown that all erasure patterns with up to $\lfloor (N-K)/2 \rfloor + 1$ erasures and symmetric erasure patterns with up to $N-K$ erasures can be corrected. As PR could not be verified analytically for all erasure patterns, we have examined it numerically. We have further discussed the properties of CMFB-based OFBs in terms of the mean-square reconstruction error, which is the main criterion for JSCC applications. We have given expressions for the MSE for particular erasure patterns for which the MSE is independent of the prototype filter coefficients. The comparison of the performance of various OFB codes is verified by simulation for the example of an image transmission system. The results indicate that the system with OFB codes performs better than a classical system in terms of MSE when the number of erasures is not high with respect to the erasure-correcting capability of the code. For further work, it would be interesting to look at the reconstruction methods when PR is not possible. When PR is possible, the uniqueness of the para-pseudoinverse may be a disadvantage, since there is less flexibility for adjustment of the synthesis filters parameters such as filter lengths and delays. Therefore, examining the tradeoffs for using some other possible synthesis filters may also be an interesting issue to look at.

REFERENCES

[1] H. Schulzrinne, S. Casner, R. Frederick, and V. Jacobson, *RTP: A Transport Protocol for Real-Time Applications*, IETF Network Working Group, January 1996.

[2] A. Gabay, O. Rioul, and P. Duhamel, "Joint source-channel coding using structured oversampled filters banks applied to image transmission," in *Proc. IEEE Int. Conf. Acoustics, Speech, Signal Processing (ICASSP '01)*, vol. 4, pp. 2581–2584, Salt Lake City, Utah, USA, May 2001.

[3] L. P. Kondi, F. Ishtiaq, and A. K. Katsaggelos, "Joint source-channel coding for motion-compensated DCT-based SNR scalable video," *IEEE Trans. Image Processing*, vol. 11, no. 9, pp. 1043–1052, 2002.

[4] V. K. Goyal, J. Kovacevic, and M. Vetterli, "Quantized frame expansions as source-channel codes for erasure channels," in *Proc. IEEE Data Compression Conference (DCC '99)*, pp. 326–335, Snowbird, Utah, USA, March 1999.

[5] G. Rath and C. Guillemot, "Frame-theoretic analysis of DFT codes with erasures," *IEEE Trans. Signal Processing*, vol. 52, no. 2, pp. 447–460, 2004.

[6] J.-L. Wu and J. Shin, "Discrete cosine transform in error control coding," *IEEE Trans. Commun.*, vol. 43, no. 5, pp. 1857–1861, 1995.

[7] J. K. Wolf, "Redundancy, the discrete Fourier transform, and impulse noise cancellation," *IEEE Trans. Commun.*, vol. 31, no. 3, pp. 458–461, 1983.

[8] G. Rath and C. Guillemot, "Performance analysis and recursive syndrome decoding of DFT codes for bursty erasure recovery," *IEEE Trans. Signal Processing*, vol. 51, no. 5, pp. 1335–1350, 2003.

[9] A. Gabay, P. Duhamel, and O. Rioul, "Spectral interpolation coder for impulse noise cancellation over a binary symmetric channel," in *Proc. European Signal Processing Conference (EUSIPCO '00)*, Tampere, Finland, September 2000.

[10] P. J. S. G. Ferreira, "The stability of a procedure for the recovery of lost samples in band-limited signals," *IEEE Trans. Signal Processing*, vol. 40, no. 2-3, pp. 195–205, 1994.

[11] P. J. S. G. Ferreira, "The eigenvalues of matrices that occur in certain interpolation problems," *IEEE Trans. Signal Processing*, vol. 45, no. 8, pp. 2115–2120, 1997.

[12] P. J. S. G. Ferreira, "Stability issues in error control coding in the complex field, interpolation, and frame bounds," *IEEE Signal Processing Lett.*, vol. 7, no. 3, pp. 57–59, 2000.

[13] P. J. S. G. Ferreira, "Mathematics for multimedia signal processing II—discrete finite frames and signal reconstruction," in *Signal Processing for Multimedia*, J. S. Byrnes, Ed., pp. 35–54, IOS Press, Amsterdam, The Netherlands, 1999, presented at the NATO Advanced Study Institute "Signal Processing for Multimedia," Il Ciocco, Italy, July 1998.

[14] H. Bölcskei and F. Hlawatsch, "Oversampled cosine modulated filter banks with perfect reconstruction," *IEEE Transactions Circuits Systems II*, vol. 45, no. 8, pp. 1057–1071, 1998.

[15] H. Bölcskei, F. Hlawatsch, and H. G. Feichtinger, "Frame-theoretic analysis of oversampled filter banks," *IEEE Trans. Signal Processing*, vol. 46, no. 12, pp. 3256–3268, 1998.

[16] J. Kovacevic, P. L. Dragotti, and V. Goyal, "Filter bank frame expansions with erasures," *IEEE Trans. Inform. Theory*, vol. 48, no. 6, pp. 1439–1450, 2002.

[17] Z. Cvetkovic and M. Vetterli, "Oversampled filter banks," *IEEE Trans. Signal Processing*, vol. 46, no. 5, pp. 1245–1255, 1998.

[18] P. L. Dragotti, S. D. Servetto, and M. Vetterli, "Optimal filter banks for multiple description coding: analysis and synthesis," *IEEE Trans. Inform. Theory*, vol. 48, no. 7, pp. 2036–2052, 2002.

[19] H. Bölcskei and F. Hlawatsch, "Noise reduction in oversampled filter banks using predictive quantization," *IEEE Trans. Inform. Theory*, vol. 47, no. 1, pp. 155–172, 2001.

[20] R. Motwani and C. Guillemot, "Tree-structured oversampled filter banks as joint source-channel codes: application to image transmission over erasure channels," *IEEE Trans. Signal Processing*, vol. 52, no. 9, pp. 2584–2599, 2004.

[21] P. L. Dragotti, J. Kovacevic, and V. Goyal, "Quantized oversampled filter banks with erasures," in *Proc. IEEE Data Compression Conference (DCC '01)*, pp. 173–182, Snowbird, Utah, USA, March 2001.

[22] S. Mehrotra and P. A. Chou, "On optimal frame expansions for multiple description quantization," in *Proc. IEEE International Symposium on Information Theory (ISIT '00)*, Sorrento, Italy, June 2000.

[23] J. Kliewer and A. Mertins, "Oversampled cosine-modulated filter banks with arbitrary system delay," *IEEE Trans. Signal Processing*, vol. 46, no. 4, pp. 941–955, 1998.

[24] H. Bölcskei, *Oversampled filter banks and predictive subband coders*, Ph.D. thesis, Vienna University of Technology, Vienna, Austria, November 1997.

[25] F. Labeau, *Strategies for exploiting residual redundancy in joint source-channel coding*, Ph.D. thesis, Université Catholique de Louvain, Louvain-la-Neuve, Belgium, September 2000.

[26] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*, Prentice Hall, Englewood Cliffs, NJ, USA, 1993.

[27] G. D. Forney Jr., "Convolutional codes I: Algebraic structure," *IEEE Trans. Inform. Theory*, vol. 16, no. 6, pp. 720–738, 1970.

[28] A. Varga, "On computing generalized inverse systems using matrix pencil methods," in *Proc. Mathematical Theory of Networks and Systems (MTNS '00)*, Perpignan, France, June 2000.

[29] K. Ramchandran and M. Vetterli, "Best wavelet packet bases in a rate-distortion sense," *IEEE Trans. Image Processing*, vol. 2, no. 2, pp. 160–175, 1993.

[30] J. G. Proakis, *Digital Communications*, McGraw-Hill, New York, NY, USA, 1995.

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