Chebyshev Functions-Based New Designs of Halfband Low/Highpass Quasi-Equiripple FIR Digital Filters

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Chebyshev functions, which are equiripple in a certain domain, are used to generate equiripple halfband lowpass frequency responses. Inverse Fourier transformation is then used to obtain explicit formulas for the corresponding impulse responses. The halfband lowpass FIR digital filters designed in this way are quasi-equiripple, having performances very close to those of true equiripple filters, and are comparatively much simpler to design.

Keywords and phrases: digital filters, FIR, halfband, equiripple, Chebyshev functions.

1. INTRODUCTION

The simplest way of designing finite impulse response (FIR) digital filters (DFs) is to truncate the infinite Fourier series of the desired frequency responses, using a window of finite length [1]. These windows-based designs provide very simple formulas for the impulse responses (tap coefficients); however, truncation of the Fourier series results in large ripples on the frequency responses, especially close to the transition edges. This builds up a need for development of new design procedures of FIR DFs having better frequency responses.

One approach to a better frequency response leads to maximally flat (MAXFLAT) designs [2, 3], which have completely ripple-free frequency responses. However, a price is paid in terms of wider transition bands, which limits the applications of these otherwise excellent filters. Classical MAXFLAT designs have closed form expressions for the frequency responses, and inverse Fourier transformation is needed to find the corresponding impulse responses. Some recent developments [4, 5, 6, 7] have made MAXFLAT designs as simple as window-based designs by giving explicit formulas for the impulse responses.

An entirely different approach to better frequency response is to spread the ripple uniformly over the entire frequency band. This ensures the minimum of the maximum size of ripple for a certain set of design specifications. The Remez exchange algorithm [8] offers a very flexible design procedure for such equiripple filters, and gives excellent trade-off between the transition width and the ripple size. However, this procedure is relatively complex as it calculates the filter coefficients in an iterative manner and each iteration involves intensive search of extrema over the entire frequency band.

Several other filter design techniques can be found in literature [9, 10, 11, 12, 13, 14, 15, 16] and some of them allow quasi-equiripple frequency responses [11, 12, 13, 14] in order to pass up the complexity of true equiripple designs. Such a technique is presented in this paper for halfband low/highpass DFs which have received much attention of researchers [3, 5, 12, 14, 15, 16] due to their numerous applications, like in sampling rate alteration and signal splitting and reconstruction [1], and so forth. In this paper, we use Chebyshev functions to obtain halfband lowpass frequency responses and then use inverse Fourier transformation to obtain explicit formulas for the corresponding impulse responses. The resultant filters obtained in this way are not truly equiripple but simplicity of their design makes them quite attractive.

2. HALFBAND LOWPASS FREQUENCY RESPONSES

A Chebyshev function of order N,

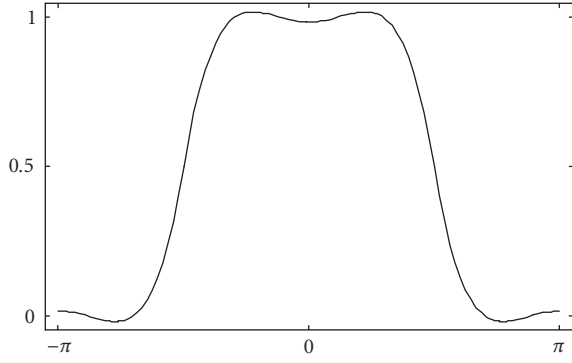$$f(\omega) = \cos\left[N\cos^{-1}\omega\right], \tag{1}$$

FIGURE 1: A Chebyshev functions-based halfband lowpass frequency response given by (2) for $N = 4$.

is an equiripple function of unit amplitude in the interval $|\omega| \leq 1$, and it increases sharply with ω for $|\omega| > 1$. The function $f(\omega)$ always has unit magnitude of opposite signs at $\omega = +1$ and $\omega = -1$ for odd values of N, and of the same sign for even values of N. For the latter case, $f(\omega)$ can be used to generate the frequency response of a halfband lowpass digital filter, as would be shown later in this section. From this point, N is assumed to be even in all the subsequent discussion.

It can be noted that $1 - \delta f(\omega)$, where $\delta = 0.5/f(\pi/2)$, represents the passband of an equiripple halfband lowpass filter for $|\omega| \leq \pi/2$. A complete halfband lowpass frequency response can be written as

$$H(\omega) = \begin{cases} \delta f(-\pi - \omega), & -\pi \leq \omega \leq -\dfrac{\pi}{2}, \\ 1 - \delta f(\omega), & -\dfrac{\pi}{2} \leq \omega \leq \dfrac{\pi}{2}, \\ \delta f(\pi - \omega), & \dfrac{\pi}{2} \leq \omega \leq \pi, \end{cases} \quad (2)$$

where

$$\delta = \frac{1}{2 \cos\left[N \cos^{-1}(\pi/2)\right]} \quad (3)$$

is the amplitude of the ripple on the frequency response.

A typical halfband lowpass response obtained by (2), for $N = 4$, is shown in Figure 1.

3. THE IMPULSE RESPONSE

The impulse response of an FIR filter, corresponding to the frequency response given by (2), can be obtained as

$$\begin{aligned} h_n &= \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega) e^{jn\omega} d\omega \\ &= \frac{\delta}{2\pi} \left[\int_{-\pi}^{-\pi/2} f(-\pi - \omega) e^{jn\omega} d\omega - \int_{-\pi/2}^{\pi/2} f(\omega) e^{jn\omega} d\omega \right. \\ &\quad \left. + \int_{\pi/2}^{\pi} f(\pi - \omega) e^{jn\omega} d\omega \right] + \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{jn\omega} d\omega, \end{aligned}$$
$$(4)$$

where $f(\omega)$ takes only even values of N and is defined by (1).

Direct evaluation of the integrals in (4) seems impossible for arbitrary values of N. We evaluated them for a large set of different values of N and established the following relations:

$$\begin{aligned} \int f(\omega) e^{jn\omega} d\omega &= \int \cos\left[N \cos^{-1}\omega\right] e^{jn\omega} d\omega \\ &= e^{jn\omega} \sum_{k=0}^{N} a_k \omega^{N-k}, \\ \int f(\pi - \omega) e^{jn\omega} d\omega &= e^{jn\omega} \sum_{k=0}^{N} a_k (\omega - \pi)^{N-k}, \\ \int f(-\pi - \omega) e^{jn\omega} d\omega &= e^{jn\omega} \sum_{k=0}^{N} a_k (\omega + \pi)^{N-k}. \end{aligned}$$
$$(5)$$

Defining $\text{int}[x]$ as the maximum integer less than or equal to x and $j = \sqrt{-1}$, a_k can be written as

$$a_k = \frac{2^{N-1} j^{k-1} N}{n^{1+k}(N-k)!} \sum_{i=0}^{\text{int}[k/2]} \frac{(N-i-1)!}{i!} \left(\frac{n}{2}\right)^{2i}. \quad (6)$$

The above expressions for the integrals and a_k were established by looking at pattern of the results obtained by using different numerical values of N in (4). They have been verified for all even values of N below 30, and therefore we conjecture that they are true for all even values of N.

Using and simplifying these integrals in (4), we get

$$h_n = \frac{\sin[n\pi/2]}{n\pi} \left[1 - jN\delta \sum_{k=0}^{N} \left(\frac{\pi}{2}\right)^{N-k} a_k \left[1 + (-1)^{N-k}\right] \right]. \quad (7)$$

As N has only even values, the second term in (7) becomes zero for odd values of k and we obtain

$$\begin{aligned} h_n &= \frac{\sin[n\pi/2]}{n\pi} \\ &\times \left[1 - N\delta \sum_{\substack{k=0 \\ k=\text{even}}}^{N} \frac{(-1)^{k/2} \pi^{N-k}}{(N-k)!} \sum_{i=0}^{k/2} \frac{(N-i-1)!}{i!} \left(\frac{n}{2}\right)^{2i-k} \right]. \end{aligned}$$
$$(8)$$

The impulse response given by (8) is of infinite length and must be truncated beyond a finite number of terms to realize an FIR filter. This truncation, due to Gibbs phenomenon [1], would deform the shape of the ripple and result in nonequiripple frequency responses. However, it can be noted from (8) that the magnitude of h_n falls very sharply as n increases, and the truncated coefficients are relatively very small in magnitude. Therefore, the resulting frequency responses obtained from the remaining coefficients are very close to equiripple, as would be shown later in Section 4.

For an arbitrary even value of N, the number of peaks on the passband of the frequency response defined by (2) is $N - 1$. Furthermore, it is known that for an even value of M, a true equiripple halfband lowpass filter of length $2M + 1$ (in fact $2M - 1$, as two external coefficients are zeros) has

$M - 1$ peaks on the passband. To make our design as close as possible to a true equiripple, we truncate h_n in (8) beyond $n = N - 1$ ($h_n = 0$ for $n = N$ as well as all other even values of n). Here, it should be noted that keeping more terms beyond $n = N$ would certainly make the response closer to equiripple, but at the cost of increased filter length. On the other hand, increasing the length by using a higher value of N in (8) would reduce the overall size of the ripple on the entire frequency response.

It should be noted that the second term in (8) can be written in a more understandable way in terms of matrices, and therefore an impulse response of length $2N - 1$, N = even, can be written as

$$h_{\pm n} = \begin{cases} 0.5, & n = 0, \\ \dfrac{(-1)^{(n-1)/2}}{n\pi}\left[1 - (\mathbf{B}\cdot\mathbf{C})_{(n+1)/2}\right], & n = \text{odd}, \ 0 < n < N, \\ 0, & n = \text{even}, \ 0 < n < N, \end{cases} \tag{9}$$

where $\mathbf{B}$ is a vector of length $N/2 + 1$ and is defined by

$$b_k = \frac{\delta N (-1)^{k-1} \pi^{N-2k+2}}{(N - 2k + 2)!}, \quad 1 \le k \le \frac{N}{2} + 1, \tag{10}$$

and $\mathbf{C}$ is an ($N/2 + 1 \times N/2$) matrix defined by

$$c_{k,l} = \sum_{i=0}^{k-1} \frac{(N - i - 1)!}{i!}\left(l - \frac{1}{2}\right)^{2(i-k+1)},$$

$$1 \le k \le \frac{N}{2} + 1, \ 1 \le l \le \frac{N}{2}. \tag{11}$$

It should be noted that $\mathbf{B} \cdot \mathbf{C}$ need to be calculated only once in (9). It should be also noted that the calculation of $\mathbf{B} \cdot \mathbf{C}$ involves high precision terms and calculations performed at low precision can lead to erroneous results. The lower indexed terms have relatively smaller magnitudes that decrease further as N increases, and therefore these terms are affected the most. However, a simple check on $\mathbf{B} \cdot \mathbf{C}$ allows performing the calculations at low precision. It is observed that for any value of N, the value of the elements of $\mathbf{B} \cdot \mathbf{C}$ increases with the index. If this is not the case, that is, the magnitude of an element of $\mathbf{B} \cdot \mathbf{C}$ is greater than the next element, then this is the indication that roundoff error has dominated and that particular element should be set to zero. This can be understood by the following example.

For $N = 20$, the elements of $\mathbf{B}$ have small magnitudes, as low as the order of 10^{-17}, and therefore a precision of at least 17 decimal points must be used; otherwise, the roundoff errors in the elements of $\mathbf{B}$ would accumulate in $\mathbf{B} \cdot \mathbf{C}$ and dominate its smaller valued elements. In this example, the true value of the first element of $\mathbf{B} \cdot \mathbf{C}$ is 0.003; used in (9), it gives $h_1 = 0.3173$. With a lower precision, for example, using 16 decimal points, the first element of $\mathbf{B} \cdot \mathbf{C}$ comes to be 0.3219; used in (9), it gives $h_1 = 0.2158$. If we use a much lower precision, say 7 decimal points, and then apply

the above check, that is, set the first element of $\mathbf{B} \cdot \mathbf{C}$ as zero, (9) gives $h_1 = 0.3183$.

Halfband highpass DFs can be designed by replacing $(-1)^{(n-1)/2}$ in (9) by $(-1)^{(n+1)/2}$.

4. COMPARISON WITH EQUIRIPPLE DESIGNS

It can be noted that if $\mathbf{B} \cdot \mathbf{C} = \mathbf{0}$, then (9) simply gives the impulse response of a rectangular-windows-based halfband lowpass filter which is notorious for large ripple closer to the band edges. This vector $\mathbf{B} \cdot \mathbf{C}$ tries to make the response equiripple by spreading the ripple uniformly on the entire frequency band. Therefore, $\mathbf{B} \cdot \mathbf{C}$, multiplied by the term outside the brackets in (9), can be defined as the impulse response corresponding to the error function (deviation from true equiripple) of a rectangular-windows-based halfband lowpass filter. It should however be noted that the presented designs are not truly equiripple due to the Gibbs phenomenon [1] that arises due to the truncation of the impulse response given by (8).

Amplitude responses of halfband lowpass DF designed using the presented procedure for $N = 10$ and $N = 20$ are shown in Figures 2 and 3, respectively. Clearly, they are very close to the equiripple responses of the same specifications obtained by the Remez algorithm, also shown in the figures for comparison. The smaller windows in the figures show details of the passbands. It can be noted that the presented filters have a ripple slightly larger than the Remez algorithm-based filters near the band edges; however, they appear to be more accurate in the rest of the bands.

5. A MODIFICATION IN THE DESIGN

It is well known that, in a frequency response, the ripple size and the transition bandwidth have an inverse relation. Remez exchange algorithm offers high flexibility such that any desired transition bandwidth can be obtained by suitably adjusting the ripple size, and vice versa.

The presented design can be also made little more flexible by multiplying vector $\mathbf{B} \cdot \mathbf{C}$ by a nonnegative factor β. As described earlier, $\mathbf{B} \cdot \mathbf{C}$ tends to spread the ripple of a rectangular-window-based filter over the entire frequency band. Therefore, a value of $\beta = 0$ gives the rectangular-window-based design with shortest transition bandwidth and large ripple. A value of $\beta = 1$ gives the presented design, in which ripple is spread over the entire band at the expense of relatively wider transition bands. However, as it can be seen in Figures 2 and 3, the designed filters still have ripple of relatively larger size near the transition edges. From this, we get the idea that using β slightly greater than 1 would further reduce the ripple size, and as an obvious consequence, transition band would be widened. It should however be noted that if we increase β beyond a certain value, the actual shape of the frequency response would start getting deformed. Based on our experience, we suggest that a value of $\beta > 2$ should not be used, and further reduction in the ripple size should be achieved by increasing the length of the filter.
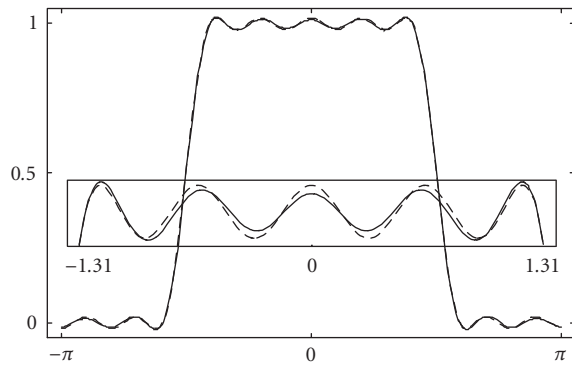
FIGURE 2: Amplitude responses of halfband lowpass FIR filters designed with the presented procedure (solid line) and the Remez algorithm (dotted line) for $N = 10$. The smaller window shows the passband details.
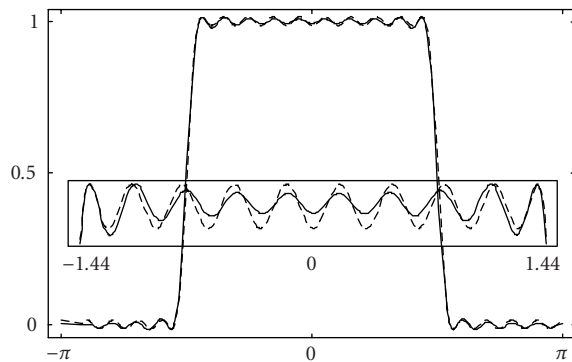

FIGURE 3: Amplitude responses of halfband lowpass FIR filters designed with the presented procedure (solid line) and the Remez algorithm (dotted line) for $N = 20$. The smaller window shows the passband details.
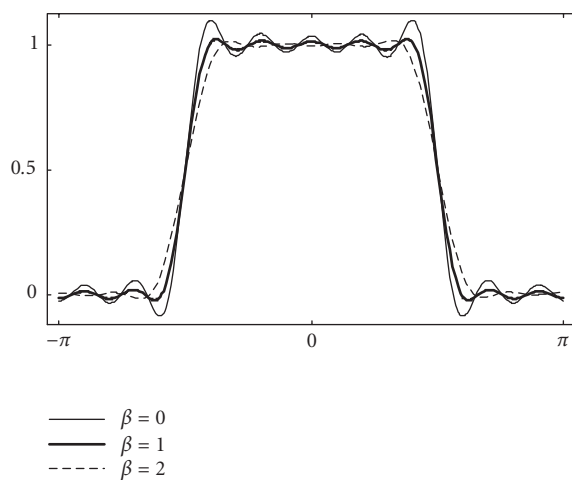

FIGURE 4: Amplitude responses of halfband lowpass FIR filters designed with the modified procedure for $N = 10$ and $\beta = 0, 1, 2$. A value of $\beta = 0$ gives a rectangular-window-based design, $\beta = 1$ gives the presented design, and a higher value of β further smoothens the frequency response.

In Figure 4, the magnitude responses of a filter designed for $N = 10$ and $\beta = 0, 1, 2$ are shown.

6. CONCLUSIONS

New designs of Chebyshev functions-based halfband low/highpass FIR DFs have been presented with explicit formulas for the impulse response coefficients. These formulas are similar to the windows-based formulas with an additional term that attempts to uniformly spread the ripple over the entire frequency band, and thus obtains nearly equiripple frequency responses. Explicit formulas for impulse responses make the presented designs much simpler as compared to the available equiripple and quasi-equiripple designs.

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