

Optimized Irregular Low-Density Parity-Check Codes for Multicarrier Modulations over Frequency-Selective Channels

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This paper deals with optimized channel coding for OFDM transmissions (COFDM) over frequency-selective channels using irregular low-density parity-check (LDPC) codes. Firstly, we introduce a new characterization of the LDPC code irregularity called “irregularity profile.” Then, using this parameterization, we derive a new criterion based on the minimization of the transmission bit error probability to design an irregular LDPC code suited to the frequency selectivity of the channel. The optimization of this criterion is done using the Gaussian approximation technique. Simulations illustrate the good performance of our approach for different transmission channels.

Keywords and phrases: orthogonal frequency division multiplexing, frequency-selective channel, optimized channel coding, irregular LDPC codes, density evolution.

1. INTRODUCTION

In this paper, we address the problem of designing codes for transmissions over frequency-selective channels when orthogonal frequency division multiplexing (OFDM) modulation technique is used. Multicarrier modulations are good candidates for the emerging high rate transmissions, either wired, wireless, single, or multiuser. Several standards have chosen OFDM modulation because it allows a very simple mitigation of intersymbol interference (ISI), which could be very destructive when the information rate is high [1, 2]. The OFDM modulator transforms a frequency-selective channel into a set of flat-fading channels, which are easier to *equalize*. The problem of channel coding for OFDM systems (Coded-OFDM or COFDM) has been already addressed [3]. Based on the emerging capacity approaching coding schemes, we propose an alternative coding structure for COFDM. In some

applications, like wired xDSL transmissions, there exists a backward channel that propagates some information from the receiver back to the transmitter. Properly used, this information can give to the transmitter an estimation of the channel that is going to be crossed. We propose in this paper to make use of this information to design a code that is *adapted* to a frequency-selective OFDM channel. Although we assume perfect channel state information (CSI) at the transmitter, we will see that partial CSI is sufficient for our code design.

In 1948, Shannon [4] characterized the optimal performance theoretically reachable for coded transmission over a noisy channel. Since then, the construction of capacity approaching codes has been the main challenge of coding research. Turbo codes [5] and Gallager low-density parity-check (LDPC) codes [6, 7] are the two competing families of pseudorandom codes that could achieve the capacity for

various kind of channels. It has been shown that irregular LDPC codes are especially interesting because one can optimize the parameters that characterize their irregularity in order to find the codes that are the closest to the capacity for various types of channels. The optimization of these codes has been done for the binary erasure channel (BEC) [8] and for the AWGN channel [9]. The optimization of LDPC codes for various nonstandard channels has also been addressed in the literature [10, 11].

We propose in this paper to optimize the LDPC code irregularity for OFDM frequency-selective channels. After the OFDM demodulation, the signal that feeds the channel decoder is seen as coming from a set of Gaussian flat-fading channels, each one having a different noise power. So, this channel can be interpreted as a nonstationary Gaussian channel for which the optimization of LDPC codes is not a direct generalization of existing work. Indeed, such an optimization requires a finer characterization of the LDPC code irregularity, that we call *irregularity profile* [12]. This paper is organized as follow. In Section 2, we define the irregularity profile and make some recalls about OFDM signaling, LDPC codes, and their decoding algorithm. Section 3 describes our optimization algorithm suited to OFDM frequency-selective channels, the results are presented in Section 4, and a conclusion is given in Section 5.

2. PARAMETERIZATION OF LDPC CODED OFDM

In this section, we introduce the main concepts and notations about LDPC codes that we will use for the optimization. As well as turbo codes, LDPC codes can achieve reliable transmission for a signal-to-noise ratio (SNR) extremely close to the Shannon limit on the AWGN channel [13]. Moreover, these codes present some advantages, such as a simple description of their structure, the easiness to make them *irregular*, and a fully parallelizable decoding implementation [14].

2.1. Irregularity profile

In this section, we propose to generalize the parameterization of LDPC irregularity in order to cope with nonstationary channels. This new parameterization is called irregularity profile [12]. LDPC block codes are defined by a sparse *parity-check matrix* $H(M \times N)$, where N denotes the codeword length and M the number of parity checks (in this work, we use only full-rank parity-check matrix, so, if R is the code rate we have $M = (1 - R)N$). An LDPC code can also be represented by its factor graph which is a bipartite graph with two kinds of nodes: data nodes representing the codeword bits and function nodes representing the parity checks [15]. The n th data node and the k th check node are connected by an edge if and only if $H_{k,n}$ is equal to 1.

A regular (N, t_c, t_r) LDPC code has a parity-check matrix with exactly t_c ones per column and t_r ones per row. When the data nodes and the check nodes have unequal connection degrees (number of edges connected to a node), the LDPC code is irregular. The irregularity is conveniently specified by two polynomials: $\lambda(x) = \sum_{i=2}^{t_c \max} \lambda_i x^{i-1}$ and $\rho(x) = \sum_{j=2}^{t_r \max} \rho_j x^{j-1}$,

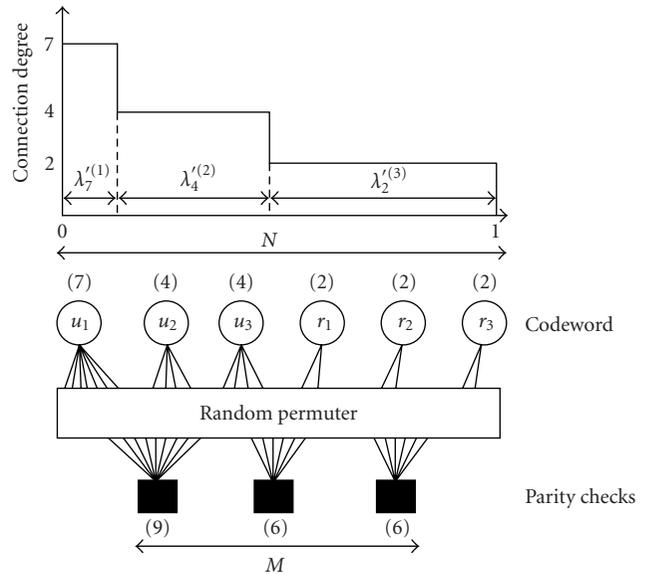


FIGURE 1: Factor graph of a rate $R = 1/2$ irregular LDPC code. The connection degree of each node is specified in braces. We have drawn the irregularity profile according to the fraction of data nodes.

where λ_i is the fraction of edges which are connected to a degree i data nodes and ρ_j is the fraction of edges which are connected to a degree j check nodes. $t_{c \max}$ and $t_{r \max}$ represent the maximal data and check node connection degrees and a degree i data node (resp., a degree j check node) is a node connected to exactly i (resp., j) edges. These two polynomials are related by $(1 - R) \sum_{i=2}^{t_{c \max}} \lambda_i / i = \sum_{j=2}^{t_{r \max}} \rho_j / j$. It is also useful to use the following dual polynomial representation: $\lambda'(x) = \sum_{i=2}^{t_{c \max}} \lambda'_i x^{i-1}$ and $\rho'(x) = \sum_{j=2}^{t_{r \max}} \rho'_j x^{j-1}$ with λ'_i being the fraction of data nodes with a connection degree i and ρ'_j being the fraction of degree j check nodes. (λ_i, ρ_j) and (λ'_i, ρ'_j) are related by $\lambda'_i = (\lambda_i / i) / \sum_{k=2}^{t_{c \max}} \lambda_k / k$ and $\rho'_j = (\rho_j / j) / \sum_{k=2}^{t_{r \max}} \rho_k / k$. Using these irregularity parameters, the optimization of LDPC codes has already been performed for various channels, including BEC [8, 13], AWGN [9], and Rayleigh channels [16]. The optimization method is based on the study of the asymptotic behavior of the LDPC codes during the decoding steps. For more details about irregular codes, we refer the reader to [13].

In the above described parameterization, $(\lambda'(x), \rho'(x))$ represents the distribution functions of the node degrees, but a node with a given degree could be placed anywhere within the codeword. This is not an issue for memoryless channels, but when the channel is not stationary or has memory, the order in which the nodes are placed in the codeword matters. That is why we introduce a more general description of irregular LDPC codes that we call irregularity profile. This parameterization includes the location p_i of the set of data nodes with a given connection degree i . For example, the code described in Figure 1 is defined by $\lambda'(x) = \sum_{i=2}^{t_{c \max}} \lambda'_i x^{i-1} = (1/6)^{(1)} x^6 + (1/3)^{(2)} x^3 + (1/2)^{(3)} x$ and $\rho'(x) = (1/3) x^8 + (2/3) x^5$. Note that the position of the

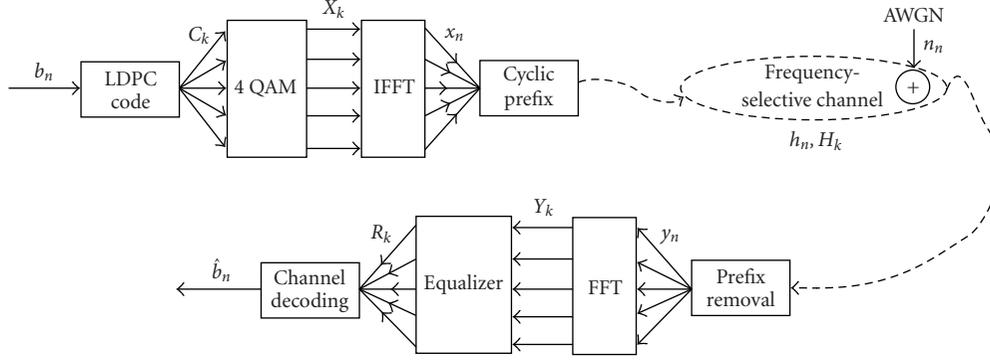


FIGURE 2: OFDM transmission with LDPC channel coding over frequency-selective channels.

check nodes is arbitrary since the parity checks are not influenced by the channel. This irregularity description provides a good framework in order to optimize the LDPC parameters for a wide range of channels, including nonstationary channels. We will denote the irregularity profile $(\underline{\lambda}, \underline{p}, \rho)$. The vectors $\underline{\lambda}$ and \underline{p} collect the coefficients of the two polynomials $(\lambda(x), \rho(x))$ while the vector \underline{p} indicates the positions of the different groups of nodes. Using this parameterization, we assume that the nodes with the same degree are located in the same channel neighborhood. Even if better codes that do not fulfil the last assumption may exist, in order to complete the optimization of the irregularity profile in a “reasonable time,” we have decided to restrict the number of parameters. We will see in the simulations that, although restrictive, this definition of the irregularity profile yields a significant performance improvement.

Our goal is to optimize the irregularity profile for OFDM frequency-selective channels, which can be interpreted, at the OFDM demodulator output, as a *no-stationary flat-fading Gaussian* channel. Now, we will briefly present the OFDM transmission scheme and the derivation of the likelihood expression which are needed to initialize the LDPC decoder.

2.2. OFDM communication system

The OFDM system consists in dividing the available spectrum into many carriers, each one being modulated by a low-rate data stream. The structure of the communication system is shown in Figure 2. The information bits b_n are encoded by an LDPC code and the resulting codeword is sent to the OFDM transmitter. After a serial-to-parallel conversion the bits are mapped into a 4-QAM constellation on the N_c subcarriers to obtain a block of symbols X_k ($k = 1, \dots, N_c$). Then, this block is transformed into a time-domain sequence by the inverse discrete Fourier transform (IDFT). In wired (xDSL) transmissions, the signal is baseband and therefore real valued. Although the optimization of LDPC codes does not require a baseband channel, we have decided to restrict the derivation of the formula to this case. In order to transmit a real signal, we build a block of symbols with a Hermitian symmetry which leads us to a real signal by IDFT. The trans-

mit signal is $x_n = \sum_{k=1}^{2N_c} S_k \cos(2\pi k f_0 n + \phi_k)$ with $X_k = S_k e^{i\phi_k}$. A cyclic prefix longer than the channel memory is used as a preamble and the signal x_n is sent through the frequency-selective channel. After removing the cyclic prefix, the received signal can be written as $y_n = \sum_{j=0}^{L-1} h_j x_{n-j} + n_n$, with h_j ($j = 0, \dots, L-1$) representing the coefficients of the channel impulse response and n_n being the AWGN with zero mean and variance σ_n^2 . The DFT-transforms the time-domain sequence y_n into a frequency-domain sequence Y_k and the frequency-selective channel becomes a set of N_c Gaussian ISI free channels with fading H_k :

$$Y_k = H_k X_k + N_k \quad \forall k = 1, \dots, N_c, \quad (1)$$

with H_k being the k th channel spectrum coefficient, X_k the k th symbol, and N_k the Gaussian noise with zero mean and variance σ_n^2 . The equalization corrects the channel distortion, and is easily done in the frequency domain with a simple multiplication by a coefficient K_k (for all $k = 1, \dots, N_c$) on each subcarrier: $R_k = K_k Y_k$ for all $k = 1, \dots, N_c$. We have used the zero forcing (ZF) equalizer and, so, the k th equalizer coefficient is $K_k = H_k^* / |H_k|^2$. Note that the choice of the equalizer type is not important in our case since any of the usual equalizers used in OFDM transmission (ZF, MMSE, maximum likelihood) would lead to the same expressions of the messages feeding the LDPC decoder. For the sake of simplicity, we then chose the ZF equalizer. Using the equalizer output R_k and the channel model (1), we obtain the expressions of the observed log-likelihood ratios u_{0k} (LLRs):

$$\begin{aligned} u_{02k} &= \log \frac{p(R_k | C_{2k} = 1)}{p(R_k | C_{2k} = 0)} = \log \frac{\sum_{X_k/C_{2k}=1} p(R_k | X_k)}{\sum_{X_k/C_{2k}=0} p(R_k | X_k)} \\ &= \frac{4 \operatorname{Re} \{R_k K_k^* H_k^*\}}{|K_k|^2 \sigma_n^2}, \\ u_{02k+1} &= \log \frac{p(R_k | C_{2k+1} = 1)}{p(R_k | C_{2k+1} = 0)} = \log \frac{\sum_{X_k/C_{2k+1}=1} p(R_k | X_k)}{\sum_{X_k/C_{2k+1}=0} p(R_k | X_k)} \\ &= \frac{4 \operatorname{Im} \{R_k K_k^* H_k^*\}}{|K_k|^2 \sigma_n^2} \quad \forall k = 1, \dots, N_c, \end{aligned} \quad (2)$$

where (C_{2k}, C_{2k+1}) are the 2 codeword bits used to form the k th 4-QAM symbol X_k , and $\text{Re}\{\cdot\}$ and $\text{Im}\{\cdot\}$ denote the real and the imaginary parts, respectively. Using these LLRs as initialization messages, we finally iteratively decode the noisy codeword to obtain an estimate \hat{u}_n of the input sequence. The decoding algorithm will be presented in the next section.

2.3. Decoding LDPC codes using belief propagation

LDPC codes are easily decoded by an iterative probabilistic algorithm known as belief propagation [7]. The belief propagation algorithm, using the Bayes rule locally, iteratively updates the a posteriori probabilities (APPs) of each bit in the codeword. So, this algorithm can be viewed as an iterative message-passing algorithm on the associated factor graph. Moreover, for a finite codeword length, the factor graph of an LDPC code contains many cycles which lead to a suboptimal calculation of the APPs.

Each iteration of belief propagation is composed of two steps:

- (i) the data pass which updates the messages through the variable nodes;
- (ii) the check pass which updates the messages through the check nodes.

Usually, it is more convenient to use LLRs as messages. Let $v = \log(p(y|c=1)/p(y|c=-1))$ the output message of a variable node and $u = \log(p(y'|c'=1)/p(y'|c'=-1))$ the output message of a check node. During the data pass on a variable node with a connection degree equal to i , the output message v on the q th branch is as follows

$$v_q = u_0 + \sum_{\substack{n=1 \\ n \neq q}}^i u_n \quad \forall q = 1, \dots, i, \quad (3)$$

where $u_n, n = 1, \dots, i$, are the incoming messages from all the data node neighbors and u_0 is the observed LLR (or channel value). At the first decoding iteration, all the u_n are set to zero for $n = 1, \dots, i$. During a check pass, we use the following ‘‘tanh rule’’ [17] to express the output message u on the p th branch:

$$\tanh \frac{u_p}{2} = \prod_{\substack{m=1 \\ m \neq p}}^j \tanh \frac{v_m}{2} \quad \forall p = 1, \dots, j, \quad (4)$$

where $v_m, m = 1, \dots, j$, are the incoming messages from the check node neighbors.

After a few iterations of belief propagation, we can calculate the a posteriori ratio w for each data node which is equal to the sum of all messages feeding a variable node $w_k = u_0 + \sum_{n=1}^i u_n, k = 1, \dots, N$. Finally, we use w_k to estimate the information bits: $\hat{u}_k = (1 - \text{sign}(w_k))/2, k = 1, \dots, N$. Thus, after having introduced in this part the main notations and concepts which are essential to optimize the LDPC codes over OFDM frequency-selective channel, we present in the next section our proposed optimization scheme.

3. OPTIMIZATION WITH A GAUSSIAN APPROXIMATION

In order to determine the performance of LDPC codes under belief propagation, Richardson and Urbanke [18] have introduced a general method to predict the asymptotic behavior of the LDPC codes. This method called *density evolution* is based on the study of the probability density functions (pdfs) of messages being propagated in the factor graph during the decoding steps under the assumption of cycle-free graph. For memoryless binary-input continuous-output AWGN channels, Chung et al. [9] proposed a Gaussian approximation of message densities to simplify the analysis of the density evolution. For many channels, including the AWGN channel, LDPC codes (with an infinite codeword length) exhibit a threshold phenomenon. This threshold corresponds to an SNR above which the bit error probability converges to zero when the number of belief propagation iterations tends to infinity (in [9], the threshold was defined as a noise power). The criterion used by Chung to optimize the LDPC codes on the AWGN channels is to choose the code which exhibits the lowest threshold. So, the Gaussian approximation allows to calculate this threshold quickly and ensures an easier design for good LDPC codes on AWGN channels. In this section, we extend Chung’s algorithm to OFDM transmissions over frequency-selective channels.

3.1. Gaussian approximation for AWGN channels

We first introduce the notations used in the Gaussian approximation method for a stationary AWGN channel. We assume that the channel is Gaussian with zero mean and variance σ_n^2 , the constellation is BPSK, and the all-zero codeword is sent. Then the observed LLR u_0 is also Gaussian with mean $2/\sigma_n^2$ and variance $4/\sigma_n^2$. We note that the variance of u_0 is equal to twice the mean, and this property (called *consistency condition* in [13]) is preserved through the belief propagation steps. This reduces the study of the density evolution to only the mean of the pdf. It is stated in [9] that the Gaussian approximation is rather a good approximation for the variable nodes outputs v , but not so good for the check nodes outputs u . We found that Gaussian approximation has been shown sufficiently accurate to provide a good LDPC optimization.

From (3), the mean $m_{v,i}^{(l)}$ of the output message of a variable node with a degree i is given by

$$m_{v,i}^{(l)} = m_{u0} + (i-1)m_u^{(l-1)}, \quad (5)$$

where m_{u0} is the mean of observed LLR u_0 and l denotes the l th decoding iteration. At the l th iteration, an incoming message v to a check node has the following Gaussian mixture density $f_v^{(l)}$:

$$f_v^{(l)} = \sum_{i=2}^{t_c \max} \lambda_i \mathcal{N}(m_{v,i}^{(l)}, 2m_{v,i}^{(l)}). \quad (6)$$

From (4) and under the ‘‘local tree assumption’’ which states the independence between the messages v_i , the updated

mean $m_u^{(l)}$ at the l th iteration can be expressed as follows:

$$m_u^{(l)} = \sum_{j=2}^{t_r \max} \rho_j \phi^{-1} \left\{ 1 - \left[1 - \sum_{i=2}^{t_c \max} \lambda_i \phi \left(m_{v,i}^{(l)} \right) \right]^{j-1} \right\}, \quad (7)$$

where $\phi(m_v^{(l)})$ is equal to $1 - E[\tanh(v^{(l)}/2)]$ (cf. [9]). Using (5) and (7) iteratively, we can follow the evolution of $m_u^{(l)}$ along the decoding iteration. Recalling that the word error probability converges to zero if and only if $m_u^{(l)} \rightarrow +\infty$ when $l \rightarrow +\infty$. It is easy to calculate the threshold corresponding to an SNR above which $m_u^{(l)}$ tends to infinity when l tends to infinity.

3.2. Gaussian approximation for OFDM frequency-selective channels

We will now extend the Gaussian approximation approach to OFDM channels which are not stationary. This will lead us to the optimization of the irregularity profile $(\underline{\lambda}, \underline{p}, \rho)$.

Before studying the asymptotical behavior of the decoder, we first discuss the statistical properties of the observed messages. As stated in Section 2.2, the messages at the input of the decoder are

$$u_{0,2k} = \log \frac{p(R_k | C_{2k} = 1)}{p(R_k | C_{2k} = 0)} = \frac{4 \operatorname{Re} \{R_k K_k^* H_k^*\}}{|K_k|^2 \sigma_n^2},$$

$$u_{0,2k+1} = \log \frac{p(R_k | C_{2k+1} = 1)}{p(R_k | C_{2k+1} = 0)} = \frac{4 \operatorname{Im} \{R_k K_k^* H_k^*\}}{|K_k|^2 \sigma_n^2} \quad \forall k \in \mathbb{N}. \quad (8)$$

Assuming that the all-zero codeword is sent:

$$(C_{2k}, C_{2k+1}) = (0, 0) \quad \forall k, \text{ i.e., } X_k = 1 + j, \quad (9)$$

the observed LLRs become

$$U_{0,k} = u_{0,2k} = u_{0,2k+1} = \frac{4 |H_k|^2}{\sigma_n^2} + \frac{4 \operatorname{Re} \{H_k^* N_k\}}{\sigma_n^2} \quad \forall k \in \mathbb{N}. \quad (10)$$

So, the observed message $U_{0,k}$ has a *consistent* Gaussian pdf $f_{u_{0,k}}$:

$$f_{u_{0,k}} = \mathcal{N} \left(\frac{4 |H_k|^2}{\sigma_n^2}, \frac{8 |H_k|^2}{\sigma_n^2} \right) = \mathcal{N} (m_{u_{0,k}}, 2m_{u_{0,k}}) \quad \forall k \in \mathbb{N}. \quad (11)$$

First of all, remark that the ‘‘local tree’’ assumption requires an infinite codeword length, and thereby an infinite number of subcarriers. Unlike the AWGN channel case, each observed message has different statistical properties. Since we have an infinite number of observed messages (one for each subcarrier), the model of the densities (11) involves an infinite number of equations. To circumvent this problem, we build a rectangular approximation of the channel spectrum which reduces the model to a finite number of equations.

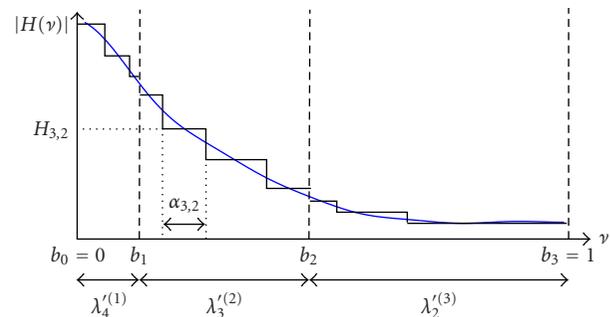


FIGURE 3: Rectangular approximation of a frequency-selective channel spectrum.

First, we split the channel spectrum into $t_{c \max} - 1$ parts according to the irregularity profile as represented in Figure 3. Each part corresponds to the spectrum bandwidth (of length $\lambda_i^{(p_i)}$), where the bits with the same connection degree i are transmitted. We sort the parts in ascending order of the positions p_i in the irregularity profile so that

$$B_i = [b_{p_i-1}; b_{p_i}] = \left[\sum_{j=k_1}^{k_{i-1}} \lambda_j^{(p_j)}; \sum_{j=k_1}^{k_i} \lambda_j^{(p_j)} \right] \quad \forall i = 2, \dots, t_{c \max}, \quad (12)$$

where the k_l are such that $p_{k_l} = l$ for all $l = 1, \dots, t_{c \max} - 1$.

In order to build the rectangular approximation of the channel spectrum $H(v)$ in each band B_i , we have chosen a staircase function such that the amplitude of the channel spectrum is divided into $N_s(i)$ equal parts. This type of approximation is called a *simple function* [19]. This means that the channel is modeled, in the band B_i , by

$$s_{N_s(i)}(v) = \sum_{k=1}^{N_s(i)} H_{i,k} \cdot \mathbf{1}_{A_{i,k}}(v) \quad \forall v \in B_i. \quad (13)$$

$\mathbf{1}_A$ represents the indicator function of set A , $N_s(i)$ is the number of stairs in each band, $H_{i,k}$ are the amplitudes of the stairs, and $A_{i,k}(v)$ are the sets defined as

$$A_{i,k} = \left\{ v; |H(v)| \in \left[m_i + \frac{k}{N_s(i)} (M_i - m_i), m_i + \frac{k+1}{N_s(i)} (M_i - m_i) \right] \right\} \quad (14)$$

with $m_i = \min_{v \in B_i} (|H(v)|)$ and $M_i = \max_{v \in B_i} (|H(v)|)$. We have drawn on Figure 3 the approximation of a typical ADSL spectrum, and its approximation with $t_{c \max} - 1 = 3$ and ($N_s(4) = 3$, $N_s(3) = 4$, $N_s(2) = 3$).

At this point, we can remark that we do not need an accurate measure of the channel shape since only a rectangular approximation is used in the optimization process. This ensures a certain robustness of our code design with respect to the channel knowledge at the transmitter. This is a clear advantage since the method is still valid for slowly varying

channels or when channel estimation errors occur. This issue is further discussed in the conclusion.

Let $\alpha_{i,k}$ denote the normalized width of the subband k located in the bandwidth B_i with

$$\alpha_{i,k} = \frac{\sup_{\nu \in A_{i,k}} |\nu| - \inf_{\nu \in A_{i,k}} |\nu|}{\lambda_i^{(p_i)}}. \quad (15)$$

So, an observed message incoming a data node with a connection degree i has a Gaussian mixture density $f_{u_{0,i}}$:

$$f_{u_{0,i}} = \sum_{k=1}^{N_s(i)} \alpha_{i,k} \mathcal{N}(m_{u_0}(H_{i,k}), 2m_{u_0}(H_{i,k})), \quad (16)$$

where $m_{u_0}(H_{i,k})$ is the mean of the LLRs located on the k th subband of B_i :

$$m_{u_0}(H_{i,k}) = \frac{4H_{i,k}^2}{\sigma_n^2} = \frac{4}{\sigma_n^2 \alpha_{i,k}} \int_{A_{i,k}} |H(\nu)|^2 d\nu. \quad (17)$$

It is important to note that the density $f_{u_{0,i}}$ is consistent for all i because it is a mixture of consistent Gaussian densities. Therefore, the variance of the messages $u_{0,i}$ is twice its mean and so, density evolution can be used for frequency-selective channels. Thus, the main difference between OFDM frequency-selective channels and AWGN channels is that the density of incoming messages to the factor graph $f_{u_{0,i}}$ is a function of the code irregularity profile. Actually, as shown in (17), the mean $m_{u_0}(H_{i,k})$ depends on the λ_i 's through the parameter $\alpha_{i,k}$ and the set $A_{i,k}$.

Now, we discuss the accuracy of the rectangular approximation. The greater $N_s(i)$ in each bandwidth B_i , the best the approximation, but the resulting density will be more computationally difficult to manage—because it is a mixture of $N_s(i)$ Gaussian densities. In order to choose the number of stairs in each B_i , we fix a maximum number $N_{s_{\max}}$ and a threshold ε . Then we evaluate the Kullback divergence iteratively between $f_{u_{0,i}}(N_{s_{\max}})$ and $f_{u_{0,i}}(n)$ for all $n \in \{1, \dots, N_{s_{\max}}\}$. We choose for $N_s(i)$ the maximum value of n such that $D_K(f_{u_{0,i}}(N_{s_{\max}}), f_{u_{0,i}}(n)) < \varepsilon$. Using a Kullback divergence [20], well suited to evaluate the distance between the tails of pdfs, we ensure that the likelihood values computed from the approximation will not be too different than the actual likelihood values—and so will be the mean $m_{u_0}(H_{i,k})$. The choice of $N_{s_{\max}}$ and of ε is a tradeoff between accuracy and computational complexity.

It is now easy to generalize the equations describing the evolution of the mean (5)–(7) to the case of OFDM frequency-selective channels. The mean $m_{v_{i,k}}^{(l)}$ of the output message of a variable node with a degree i in the k th subband B_i is given by

$$m_{v_{i,k}}^{(l)} = m_{u_0}(H_{i,k}) + (i-1)m_u^{(l-1)}. \quad (18)$$

Then, at the l th iteration of belief propagation decoding, an incoming message ν to a check node will have the follow-

ing Gaussian mixture density $f_v^{(l)}$:

$$f_v^{(l)} = \sum_{i=2}^{t_{c_{\max}}} \lambda_i^{(p_i)} \left[\sum_{k=1}^{N_s(i)} \alpha_{i,k} \mathcal{N}(m_{v_{i,k}}^{(l)}, 2m_{v_{i,k}}^{(l)}) \right] \quad (19)$$

which leads to the generalization of (7):

$$\begin{aligned} m_u^{(l)} &= \sum_{j=2}^{t_{r_{\max}}} \rho_j \phi^{-1} \left\{ 1 - \left[1 - \sum_{i=2}^{t_{c_{\max}}} \lambda_i^{(p_i)} \left(\sum_{k=1}^{N_s(i)} \alpha_{i,k} \phi(m_{v_{i,k}}^{(l)}) \right) \right]^{j-1} \right\}. \end{aligned} \quad (20)$$

Now, using (18) and (20) iteratively, we can follow the evolution of $m_u^{(l)}$ along the decoding iterations for a frequency-selective channel. We will make use of these equations in order to optimize the LDPC code profile.

3.3. Optimization

In [9], the optimization criterion was to minimize the LDPC decoding threshold, which seems to be the best choice for stationary channels. We think that a different optimization criterion could be used for OFDM frequency-selective channels based on their nonstationarity, leading to a codeword which is unequally fed by varying likelihoods. Indeed, it is more relevant to place the information bits in the codeword when the channel is less noisy. For this reason, as an optimization criterion, we choose to minimize the *information bit error probability* after L iterations of belief propagation at a carefully chosen E_b/N_0 . Using this optimization criterion, we get the LDPC code that asymptotically achieves the best performance after a finite number of decoding iterations for a given E_b/N_0 . We will see through simulations that the choice of this optimization criterion is relevant, especially for small block length.

The probability of error at the l th iteration for a degree i bits that is transmitted in the subband k (which is located in B_i) is given by

$$P_{e_{i,k}} = Q \left(\sqrt{\frac{m_{u_0}(H_{i,k}) + \text{im}_u^{(l)}}{2}} \right), \quad (21)$$

where $Q(\cdot)$ is the Gaussian tail function.

Let B_{info} be the band of normalized length R where the information bits are transmitted. Likewise, $B_{\text{info},i,k} = B_{\text{info}} \cap A_{i,k}$ defines the band where the information bits of degree i of the subband k are transmitted. So, the length $\alpha_{\text{info},i,k}$ of this band can be written as

$$\alpha_{\text{info},i,k} = \sup_{\nu \in B_{\text{info},i,k}} |\nu| - \inf_{\nu \in B_{\text{info},i,k}} |\nu|. \quad (22)$$

Therefore, we have the following property:

$$\sum_{i=2}^{t_{c_{\max}}} \sum_{k=1}^{N_{st}(i)} \alpha_{\text{info},i,k} = R. \quad (23)$$

With these notations, the bit error probability on the information bits is defined by

$$P_{e_{\text{info}}} = \frac{1}{R} \sum_{i=2}^{t_{c\text{max}}} \sum_{k=1}^{N_{st}(i)} \alpha_{\text{info},i,k} Q \left(\sqrt{\frac{m_{u0}(H_{i,k}) + \text{im}_u^{(i)}}{2}} \right). \quad (24)$$

We can note that this bit error probability depends on the λ_i and p_i through the parameter $\alpha_{\text{info},i,k}$ and the mean $m_{u0}(H_{i,k})$. The problem is that the dependance of (24) is *non-linear* in the required parameters λ_i , which is the main difference of our optimization scheme compared to existing work. It should be noted that the nonlinearity is not a result of the criterion chosen, but comes from the nonstationarity of the channel. It is then not possible to optimize the code profile easily with linear programming, as in [9, 18] for the erasure and the AWGN channels.

This nonlinear cost function minimization with continuous space parameters can be solved efficiently using differential evolution [21], a method that has been previously used for LDPC optimization in Rayleigh fading channels [16]. For more details about differential evolution applied, we refer the reader to [22].

4. SIMULATION RESULTS

This part presents the results obtained with our approach ($\min P_e$) for different channels including AWGN channel, ADSL channel, and a nonmonotonous spectrum channel. The results are compared with those of a regular channel coding scheme and the method which consists in the minimization of the threshold for frequency-selective channels called $\min T$. Section 4.1 presents the obtained irregular profiles while Section 4.2 gives the comparison of the performances for different codeword lengths.

4.1. Optimized irregular LDPC code profiles

In this section, we present the structure of optimized LDPC codes over the two different frequency-selective channels:

- (i) a typical ADSL channel called ch A with impulse response

$$h_A = [0.06, 0.72, 0.54, 0.36, 0.18, 0.114, 0.078, 0.054, 0.033, 0.018, 0.012]; \quad (25)$$

- (ii) a nonmonotonous spectrum channel denoted by ch B with impulse response

$$h_B = [-0.21, -0.17, 0.31, 0.68, -0.27, -0.15, 0.19, 0.13]. \quad (26)$$

In order to find the good profiles in a reasonable time, some parameters of the LDPC codes are fixed. Here we set $\rho(x) = x^7$, $t_{c\text{max}} = 10$, and we choose a coding rate equal to $R = 1/2$.

To obtain the optimized code for a given channel, we have minimized the bit error probability (using (24)) for a value of E_b/N_0 slightly lower than the threshold exhibited by the $\min T$ method. Figures 4 and 5 depict the irregularity profiles

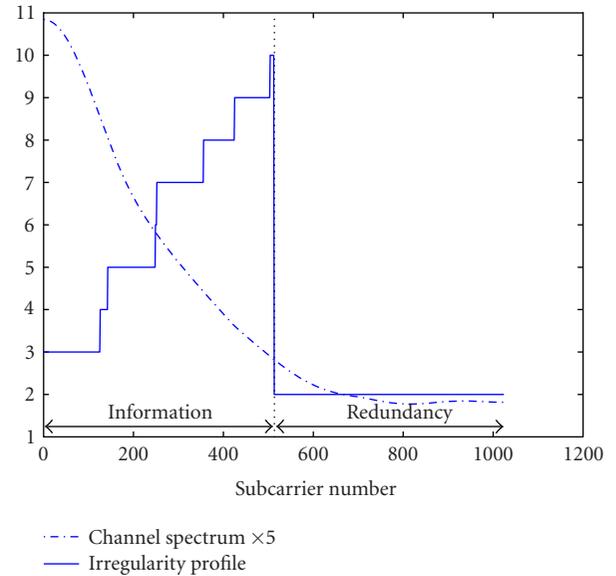


FIGURE 4: Data nodes connection degrees for the ch A with $R = 0.5$.

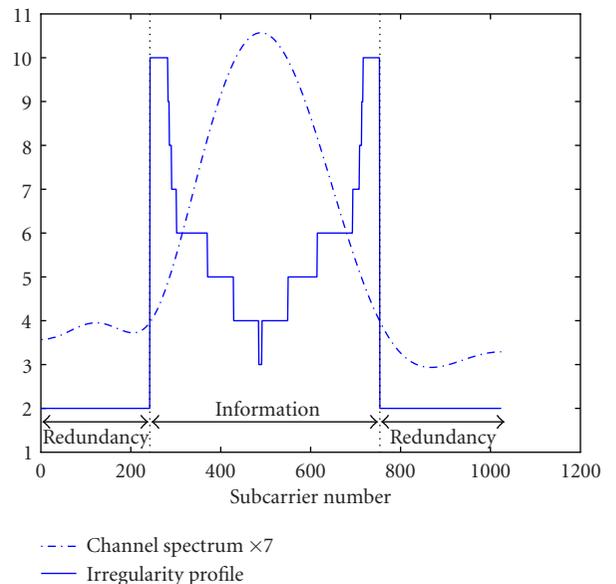


FIGURE 5: Data nodes connection degrees for the ch B with $R = 0.5$.

of LDPC codes obtained after optimization with respect to ch A and ch B. The resulting degree distributions $\underline{\lambda}'_{\text{chA}}$ and $\underline{\lambda}'_{\text{chB}}$ are

$$\begin{aligned} \underline{\lambda}'_{\text{chA}} &= [\lambda'_2, \dots, \lambda'_{10}] \\ &= [0.5, 0.1221, 0.0165, 0.1035, 0.0029, 0.1016, 0.0675, \\ &\quad 0.0772, 0.0087], \\ \underline{\lambda}'_{\text{chB}} &= [\lambda'_2, \dots, \lambda'_{10}] \\ &= [0.5, 0.0069, 0.111, 0.1202, 0.1455, 0.0263, 0.0102, \\ &\quad 0.0058, 0.0745]. \end{aligned} \quad (27)$$

Interestingly, the optimization process gives proportions of degree 2 bit nodes (λ_2) equal to $1 - R$, R being the code rate. This result is due to the minimization of the probability of error *only on the information bits*, which represent exactly a proportion of R of the codeword. This means that the optimization criterion tries to protect the information bits better than the redundancy bits by allocating more edges in the graph to the information bits. Again, this would not be an issue on stationary channels, but for OFDM frequency-selective channels, the position of the information bits in the codeword matters.

Having a look at Figures 4 and 5, we remark that putting the node degree locations as parameters in the cost function leads to an irregularity profile that has two behaviors. First, the redundancy bits are connected to exactly 2 edges and are placed in the codeword where the channel has the lowest gain. Secondly, the connections on the information bits are inversely proportional to the channel shape. This kind of profile could be interpreted as a compensation for the channel selectivity. We can explain this phenomenon by the following way: a bit which is connected to a large number of check nodes is well protected against the additive noise because it gets lots of information coming from the other bits during the decoding process. So, an information bit transmitted on a subcarrier with a small SNR must have a large connection degree in order to be well protected. We once more recall that the obtained profiles result from the optimization process thanks to the introduction of location parameters in the code profile, and do not require any a priori or heuristic.

4.2. Performance study

One of LDPC drawbacks is their high encoding complexity. Several authors have proposed LDPC encoding methods whose complexity scales as $\mathcal{O}(N)$ [12, 23, 24], but these methods need to permute the codeword bits, breaking down the desired irregularity profile of the code. In [25], MacKay et al. propose to encode the information bits directly with the parity matrix, which can be done in linear time when the parity matrix is upper triangular. This approach allows to keep the irregularity profile while ensuring a simple encoding. For this purpose, we have derived a bit filling algorithm to build an upper triangular parity matrix H with the profiles of Figures 4 and 5 and no cycles of length 4.

In our simulations, the number of subcarriers N_c is set to 1024, the length of the cyclic prefix is set to 12, and the LDPC codeword length $N = k \times N_c$. Two codeword block sizes are used: $N = 16384$ (which is decoding with 200 iterations of belief propagation) in order to obtain the performances close to the asymptotic behavior and $N = 1024$ (decoding with 40 iterations of belief propagation) in order to consider more practical cases. The results are plotted in Figures 6–10.

Figures 6 and 7 illustrate the performance between regular and optimized irregular coding schemes for a transmission over the AWGN, ch A, and ch B channels. The regular code is defined by $\lambda(x) = x^3$ and $\rho(x) = x^7$. For an error probability equal to 10^{-5} , we can note an improvement of 1.5 dB for the optimized code compared to the regular one

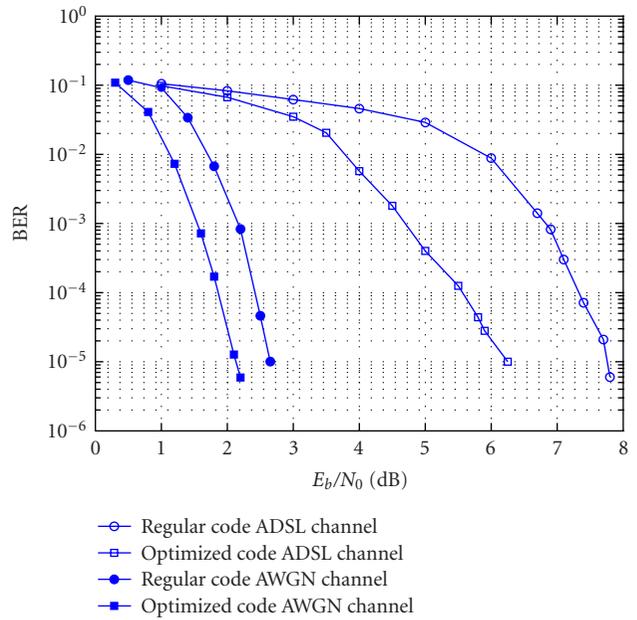


FIGURE 6: Performance comparison between our method and the regular coding scheme for a rate $R = 1/2$ over ch A and over the AWGN channel. $N = 1024$.

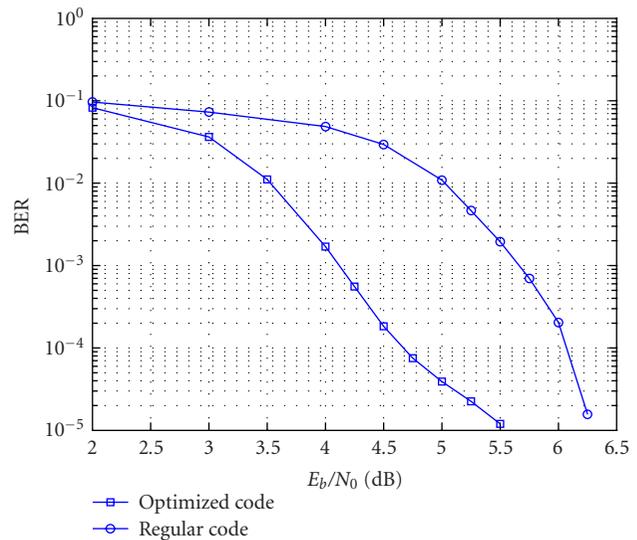


FIGURE 7: Performance comparison between our method and the regular coding scheme for a rate $R = 1/2$ over ch B. $N = 1024$.

for ch A with roughly the same decoding complexity. The same remarks hold true for ch B with an improvement of 0.8 dB.

We have also performed the simulations with the optimized code (obtained for ch A) and the regular one on the AWGN channel. These results show that the gap is greater in the case of frequency-selective channel than for the AWGN channel. So, we can conclude that optimized irregular coding strategy is well suited to transmissions over frequency-selective channels.

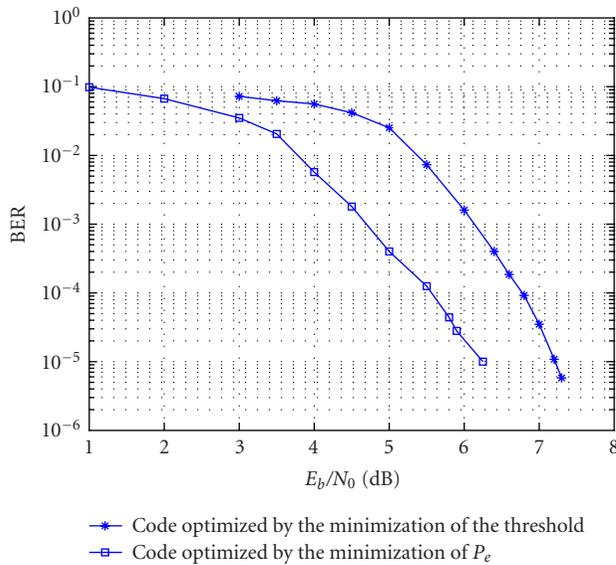


FIGURE 8: Performance comparison over chA between the code obtained using the $\min P_e$ and the $\min T$ methods. $R = 1/2$ and $N = 1024$.

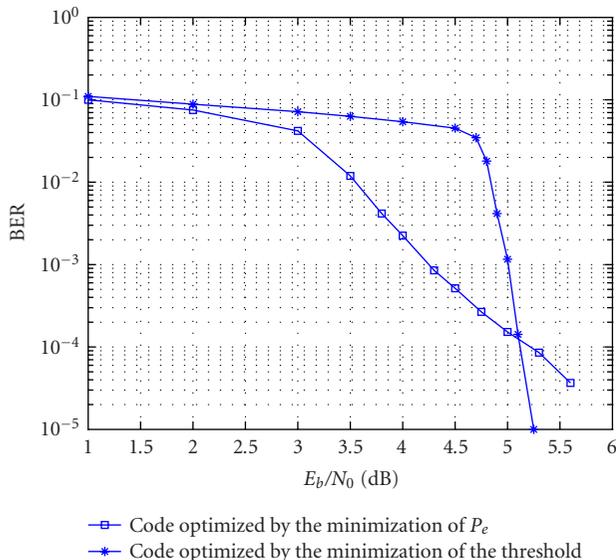


FIGURE 9: Performance comparison over chA between the code obtained using the $\min P_e$ and the $\min T$ methods. $R = 1/2$ and $N = 16384$.

Figures 8 and 9 compare the performances of LDPC codes obtained with two different optimization criteria, namely, the minimization of the bit error probability $\min P_e$ and the threshold minimization $\min T$. In the case where $N = 1024$ (Figure 8), the LDPC code obtained by $\min P_e$ outperforms the one obtained by the $\min T$ method. For example, for a bit error rate (BER) equal to 10^{-5} , we observe an improvement equal to 1.1 dB. For $N = 16384$ depicted in Figure 9, two regions can be distinguished, namely, a region in which the performances obtained by $\min P_e$ are bet-

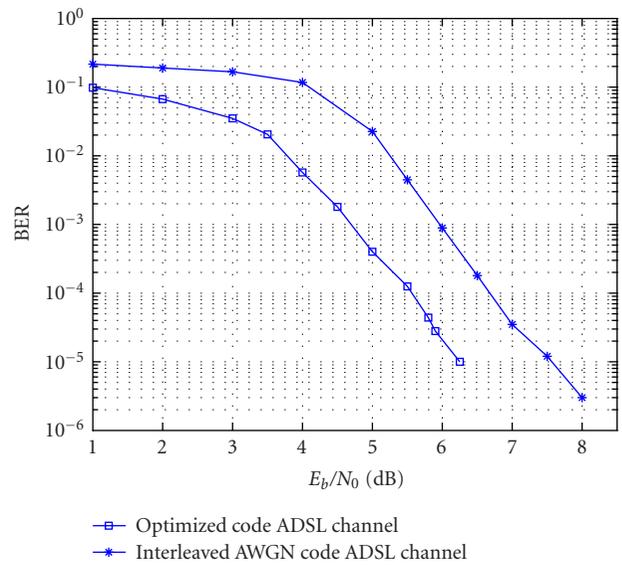


FIGURE 10: Performance comparison over chA between our method and the LDPC code obtained by optimization on AWGN channel for a rate $R = 1/2$. $N = 1024$.

ter than ones obtained with $\min T$. This region is defined for the SNRs lower than 5.15 dB. For SNR higher than 5.15 dB, the $\min T$ code presents a threshold phenomenon and outperforms the $\min P_e$ code. So, for applications where N is constrained to a relatively small value, the use of $\min P_e$ criterion to design an optimized channel coding scheme will be preferable.

In order to be sure that the optimization of LDPC codes for a specific channel is of great importance, we have also compared our codes to irregular LDPC codes presented in [18], which have been optimized for the AWGN channel (with the same parameters, i.e., $\rho(x) = x^7$, $t_{c\max} = 10$, and $R = 1/2$). Because the positions of the codeword bits are a result of our optimization process, we have added an interleaver in the case of the code optimized on the AWGN channel. The comparison is done in Figure 10 for a codeword of length 1024 and with the frequency-selective channel chA. We can notice that adapting the LDPC code irregularity to the channel shape leads to a high improvement of the coding performances.

5. DISCUSSION AND CONCLUSION

In this paper, we have optimized the structure of LDPC codes for transmissions over OFDM frequency-selective channel. The optimization is based on a new and general parameterization of the LDPC code irregularity: the *irregularity profile*. We have optimized the irregularity profile using a Gaussian approximation technique. We have shown that it is relevant to minimize the bit error probability instead of trying to get a vanishing block error probability LDPC code. This has been shown by simulations on several channels and for small and large codeword lengths. We obtain optimized LDPC codes that exhibit a performance improvement of several dBs

compared to a regular LDPC code. This improvement is greater than the one observed on stationary Gaussian channel, which means that using irregular LDPC codes is even more important on OFDM frequency-selective channels than more *simple* channels.

We must emphasize that the proposed method does not require a perfect a priori knowledge on the CSI for the optimization of the channel code and therefore is well suited to practical case. Actually, the a priori knowledge on the CSI is quite often incomplete due to the use of a short training sequence to estimate it. Moreover, we think that our method is still interesting for a slowly varying channel and/or when channel estimation mismatches occurs. However, we are currently working on the robustness of the optimized codes with respect to partial or wrong CSI. This work will be reported in future publication. Another discussion can be engaged to compare our approach and power allocation approach as in DMT standard. When perfect CSI knowledge is available at the transmitter, the power allocation transforms a frequency-selective channel into an AWGN channel. So, in theory, it is possible to find the *optimal* channel code leading to the frequency channel capacity. We can notice that this assumes a *perfect power allocation with respect to waterfilling principle*. Once again, in some practical case, perfect power allocation is not a realistic assumption due to the CSI estimation but also because power allocation implies the use of a bit loading algorithm which authorizes to use just a few number of constellations which assume to use integer bits. For these reasons, we believe that power allocation and irregularity profile optimization are not necessarily competitive and could be used jointly in order to achieve performance close to the capacity.

The methods that we have developed in this paper could also be successfully applied to multiuser multicarrier modulations such as MC-CDMA.

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REFERENCES

- [1] J. A. C. Bingham, "Multicarrier modulation for data transmission: an idea whose time has come," *IEEE Communications Magazine*, vol. 28, no. 5, pp. 5–14, 1990.
- [2] H. Sari, G. Karam, and I. Jeanclaude, "Transmission techniques for digital terrestrial TV broadcasting," *IEEE Communications Magazine*, vol. 33, no. 2, pp. 100–109, 1995.
- [3] W. Y. Zou and Y. Wu, "COFDM: an overview," *IEEE Transactions on Broadcasting*, vol. 41, no. 1, pp. 1–8, 1995.
- [4] C. E. Shannon, "A mathematical theory of communication," *Bell System Technical Journal*, vol. 27, pp. 379–423, 623–656, July–October 1948.
- [5] C. Berrou, A. Glavieux, and P. Thitimajshima, "Near Shannon limit error-correcting coding and decoding: Turbo-codes," in *Proc. IEEE International Conference on Communications (ICC '93)*, vol. 2, pp. 1064–1070, Geneva, Switzerland, May 1993.
- [6] M. Sipser and D. A. Spielman, "Expander codes," *IEEE Transactions on Information Theory*, vol. 42, no. 6, pp. 1710–1722, 1996.
- [7] D. J. C. MacKay, "Good error-correcting codes based on very sparse matrices," *IEEE Transactions on Information Theory*, vol. 45, no. 2, pp. 399–431, 1999.
- [8] A. Shokrollahi and R. Storn, *Design of Efficient Erasure Codes with Differential Evolution*, Bell Laboratories, Murry Hill, NJ, USA, 1999.
- [9] S.-Y. Chung, T. J. Richardson, and R. L. Urbanke, "Analysis of sum-product decoding of low-density parity-check codes using a Gaussian approximation," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 657–670, 2001.
- [10] D. Doan and K. Narayanan, "Design of good low rate codes for ISI channels based on spectral shaping," in *Proc. 3rd International Symposium on Turbo-Codes & Related Topics*, Brest, France, September 2003.
- [11] H. Pishro-Nik, N. Rahnvard, J. Ha, F. Fekri, and A. Adibi, "Low-density parity-check codes for volume holographic memory systems," *Applied Optics-IP*, vol. 42, no. 5, pp. 861–870, 2003.
- [12] D. Declercq, G. Gelle, and V. Mannoni, "Irregular channel coding for OFDM transmission," in *Proc. 6th International Symposium on Communication Theory and Applications (ISCTA '01)*, Ambleside, UK, July 2001.
- [13] T. J. Richardson, M. A. Shokrollahi, and R. L. Urbanke, "Design of capacity-approaching irregular low-density parity-check codes," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 619–637, 2001.
- [14] F. Verdier and D. Declercq, "A LDPC parity check matrix construction for parallel hardware decoding," in *Proc. 3rd International Symposium on Turbo-Codes & Related Topics*, Brest, France, September 2003.
- [15] F. R. Kschischang, B. J. Frey, and H.-A. Loeliger, "Factor graphs and the sum-product algorithm," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 498–519, 2001.
- [16] J. Hou, P. H. Siegel, and L. B. Milstein, "Performance analysis and code optimization of low density parity-check codes on Rayleigh fading channels," *IEEE Journal on Selected Areas in Communications*, vol. 19, no. 5, pp. 924–934, 2001.
- [17] C. R. Hartmann and L. D. Rudolph, "An optimum symbol-by-symbol decoding rule for linear codes," *IEEE Transactions on Information Theory*, vol. 22, no. 5, pp. 514–517, 1976.
- [18] T. J. Richardson and R. L. Urbanke, "The capacity of low-density parity-check codes under message-passing decoding," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 599–618, 2001.
- [19] M. Capinski and E. Kopp, *Measure, Integral and Probability*, Springer-Verlag, New York, NY, USA, 1999.
- [20] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics*, C. Griffin, London, UK, 2nd edition, 1963.
- [21] V. Mannoni, D. Declercq, and G. Gelle, "Optimized irregular Gallager codes for OFDM transmission," in *Proc. 13th IEEE International Symposium on Personal, Indoor and Mobile Radio Communications (PIMRC '02)*, vol. 1, pp. 222–226, Lisboa, Portugal, September 2002.
- [22] R. Storn and K. Price, "Differential evolution—a simple and efficient heuristic for global optimization over continuous spaces," *Journal of Global Optimization*, vol. 11, no. 4, pp. 341–359, 1997.
- [23] D. A. Spielman, "Linear-time encodable and decodable error-correcting codes," *IEEE Transactions on Information Theory*, vol. 42, no. 6, pp. 1723–1731, 1996.

- [24] T. J. Richardson and R. L. Urbanke, "Efficient encoding of low-density parity-check codes," *IEEE Transactions on Information Theory*, vol. 47, no. 2, pp. 638–656, 2001.
- [25] D. J. C. MacKay, S. T. Wilson, and M. C. Davey, "Comparison of constructions of irregular Gallager codes," *IEEE Trans. Communications*, vol. 47, no. 10, pp. 1449–1454, 1999.

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