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Arikan and Alamouti matrices based on fast block-wise inverse Jacket transform

Moon Ho Lee[1*], Md Hashem Ali Khan[1] and Kyeong Jin Kim[2]

Abstract

Recently, Lee and Hou (IEEE Signal Process Lett 13: 461-464, 2006) proposed one-dimensional and two-dimensional fast algorithms for block-wise inverse Jacket transforms (BIJTs). Their BIJTs are not real inverse Jacket transforms from mathematical point of view because their inverses do not satisfy the usual condition, i.e., the multiplication of a matrix with its inverse matrix is not equal to the identity matrix. Therefore, we mathematically propose a fast block-wise inverse Jacket transform of orders $N = 2^k$, 3^k, 5^k, and 6^k, where k is a positive integer. Based on the Kronecker product of the successive lower order Jacket matrices and the basis matrix, the fast algorithms for realizing these transforms are obtained. Due to the simple inverse and fast algorithms of Arikan polar binary and Alamouti multiple-input multiple-output (MIMO) non-binary matrices, which are obtained from BIJTs, they can be applied in areas such as 3GPP physical layer for ultra mobile broadband permutation matrices design, first-order q-ary Reed-Muller code design, diagonal channel design, diagonal subchannel decompose for interference alignment, and 4G MIMO long-term evolution Alamouti precoding design.

Keywords: Arikan and Alamouti matrices, Fast block-wise inverse Jacket transform, Kronecker product, Sparse matrix factorization

1. Introduction

The orthogonal transforms, such as the discrete Fourier transform (DFT) and Walsh-Hadamard transform (WHT), have widely been used in images processing, data compressing and coding, spatial multiplexing, and other areas [1-11]. Using orthogonality of the WHT, the orthogonal matrices such as the element-wise or block-wise inverse matrices have been developed. It is shown that many interesting orthogonal matrices say the Hadamard matrices and the DFT matrices belong to the Jacket matrix family. Jacket matrix [8,12,13], which is motivated by the center weight Hadamard matrix [5,9,10], is a class of matrices with its inverse being determined by element-wise of matrix.

Definition 1.1: An $n \times n$ matrix $J_n = [\alpha_{ij}]_{n\times n}$ is called the element-wise inverse Jacket matrix (EIJM) of order n if its inverse matrix J_n^{-1} can be obtained by its element-wise inverse, i.e., $J_n^{-1} = 1/n[\alpha_{ij}^{-1}]_{n\times n}^T$, where T denotes the transpose, then $[J]_n[J]_n^{-1} = [I]_n$.

Since the inverse of the Jacket matrix can be calculated easily, it is very helpful to employ this kind of matrix in signal processing [1,14-16], encoding [17], mobile communication [16,18], and so on. In addition, Jacket matrices are associated with many kinds of matrices, such as unitary matrices and Hermitian matrices which are very important in signal processing [19,20] and communication [15,19,20].

Motivation of this article is to develop an efficient matrix inverse with a big size. For single-input single-output communication systems, this problem has been solved based on the EIJM [10]. However, nowadays communication environment changes to multiple-input multiple-output (MIMO)-based 4G mobile systems. For example, the coordinated multi-point transmission and reception (CoMP) technique [21] is a new promising interference cancelling scheme which has been adopted in LTE-advanced systems. For these LTE-advanced systems, CoMP is mostly required to reduce inter-cell interference. It also increases the intra cell edge user throughput and improves the coverage. Most research works based on precoding with feedback schemes mainly focus on an explicit feedback in the downlink CoMP with diagonal subchannel decompose for interference alignment. Otherwise, the diagonal block-

* Correspondence: moonho@jbnu.ac.kr
[1]Institute of Information & Communication, Chonbuk National University, Jeonju 561-756, South Korea
Full list of author information is available at the end of the article

wise precoding matrix MIMO has been studied in [16,22]. A recent study [15,19,20] is valid only for the intra cooperative reciprocal matrix with many parameters. It is shown that the inverse transform of the element-wise inverse Jacket transform can easily be obtained by taking each reciprocal entry of the forward matrix of Jacket transform and then transposing the resulting matrix. Thus, this article proposes only a simple block-wise inverse Jacket matrix (BIJM) for an orthogonal code design of inter matrices such as MIMO channel with Alamouti code.

Therefore, in this article, we mathematically propose a new fast block-wise inverse (FBI) Jacket transform. Our main contributions are summarized as follows.

1. We propose an FBI Jacket transform for the orders of $N = 2^k$, 3^k, and 5^k, where k is a positive integer. With the help of the Kronecker product of the successive lower order Jacket matrices and the identity matrix, the fast algorithms can be realized.
2. We provide Arikan polar binary and Alamouti nonbinary matrices using the proposed simple inverse and the fast inverse algorithm. It will be seen that the resulting Arikan polar binary and Alamouti nonbinary matrices become the FBI Jacket transforms.
3. We derive the space–time block code (STBC) matrix for the MIMO communications based on the Alamouti nonbinary matrices.

This remainder of this article is organized as follows. In Section 2, we present the conventional FBI Jacket transform. Section 3 presents the proposed binary block-wise inverse and Arikan polar binary basic matrices of order 2^k. Section 4 presents the binary block-wise inverse Jacket transform (BIJT) of orders $N = 3^k$ and 5^k for integer value k. Section 5 presents the two-dimensional fast algorithms for the binary BIJT. In Section 6, Alamouti MIMO nonbinary BIJT of order 2^k is presented. Finally, conclusions are drawn in Section 7.

2. The conventional fast BIJT

In this section, we present the BIJM as follows [1]. Let p be an odd prime and α denote a permutation matrix unit, $\alpha^p = I$, which is corresponding to the complex unit $\exp^{\sqrt{-1}(2\pi/p)}$ and the elements are over the same multiplicative group.

Definition 2.1: The BIJM is defined as

$$[J]_p = \left[\vec{V}_1 \vec{V} \cdots \vec{V}_{p-1}\right]^T, \tag{1}$$

Where $\vec{V}_i = \left\{\alpha^0, \alpha^{1\times i}, \alpha^{2\times i}, \ldots, \alpha^{(p-1)\times i}\right\}$, $i \in \{0, 1, \ldots, p-1\}$. The inverse matrix of $[J]_p$ is given by

$$[J]_p^{-1} = \frac{1}{p}\left[\vec{V}_0^T, \ \vec{V}_i^T \cdots \vec{V}_{p-1}^T\right] = \frac{1}{p}[J]_p^{-1} \tag{2}$$

Given a BIJM over GF (2), which is defined by $\{\alpha^0,\alpha^1\}$, the permutation matrix unit $\alpha^h = [e_{i,j}]_p$, is defined as

$$e_{i,j} = \begin{cases} 1 & for\ i=\langle j+h\rangle \\ 0 & otherwise \end{cases}, \tag{3}$$

where $\langle j + h\rangle = (j + h) \bmod p$, with $0 \le i, j, h \le p - 1$. It can easily be seen that $\{\alpha^0, \alpha^1, \ldots, \alpha^{p-1}\}$ forms one Abelian group with traditional matrix multiplication and $I = \alpha^p$ corresponding to the complex unit $\exp^{\sqrt{-1}(2\pi/p)}$. Similar to the Hadamard matrix, the BIJM $[J]_2$ can be written as

$$[J]_2 = \begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}. \tag{4}$$

From Definition 2.1, the inverse matrix of $[J]_2$ is given by

$$[J]_2^{-1} = \begin{bmatrix} (\alpha^0)^{-1} & (\alpha^0)^{-1} \\ (\alpha^0)^{-1} & (\alpha^1)^{-1} \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \tag{5}$$

with $(\alpha^h)^{-1} = (\alpha^h)^T$. Further, the higher-order BIJT can be generated from the following recursive equation

$$[J]_N = [J]_{N/2} \otimes [J]_2, \quad N \ge 4. \tag{6}$$

where $\otimes$ denotes the Kronecker product [23].

As an example, the order-4 matrix based on the block factorization algorithm can be written as

$$\begin{bmatrix} \alpha^0 & \alpha^0 & \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 & \alpha^0 & \alpha^1 \\ \alpha^0 & \alpha^0 & \alpha^1 & \alpha^1 \\ \alpha^0 & \alpha^1 & \alpha^1 & \alpha^0 \end{bmatrix} = \begin{bmatrix} \alpha^0 & \alpha^0 & 0 & 0 \\ \alpha^0 & \alpha^1 & 0 & 0 \\ 0 & 0 & \alpha^0 & \alpha^0 \\ 0 & 0 & \alpha^0 & \alpha^1 \end{bmatrix} = \begin{bmatrix} \alpha^0 & 0 & 0 & 0 \\ 0 & \alpha^0 & 0 & 0 \\ 0 & 0 & \alpha^0 & 0 \\ 0 & 0 & 0 & \alpha^0 \end{bmatrix}$$
$$\begin{bmatrix} \alpha^0 & 0 & \alpha^0 & 0 \\ 0 & \alpha^0 & 0 & \alpha^0 \\ \alpha^0 & 0 & \alpha^1 & 0 \\ 0 & \alpha^0 & 0 & \alpha^1 \end{bmatrix}$$
$$= \left([I]_2 \otimes \begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 \end{bmatrix}\right)(\alpha^0 [I]_N)\left([I]_{N/2} \otimes [I]_2\right), \tag{7}$$

where $[I]_N$ is the identity matrix. From Equations (6) and (7), we can derive a generalized formula for construction of order $N = 2^k$, $k \in \{1, 2, 3, 4 \ldots\}$, BIJMs as

$$[J]_N = [J]_{N/2} \otimes [J]_2 = [J]_{N/4} \otimes [J]_4 = \left([I]_{N/2} \otimes \begin{bmatrix} \alpha^0 & \alpha^0 \\ \alpha^0 & \alpha^1 \end{bmatrix}\right)\left(\alpha^0 [I]_N\right)\left([J]_{N/2} \otimes [I]_2\right). \quad (8)$$

Note that

$$[J]_N [J]_N^{-1} \neq [I]_N \quad (9)$$

which is a mathematic problem; therefore, we will perfectly solve this problem based on Arikan Polar binary and Alamouti MIMO non-binary matrices in the following sections.

3. Proposed binary block-wise inverse and Arikan polar binary basic matrices of order 2^k

3.1. Binary BIJT of order 2^k

Definition 3.1: Let B be a square binary matrix of size m, then B is called EIJM if it is invertible and $B^{-1} = B^T$.

An $m \times m$ binary matrix $[B]m$ over $GF(2)$, which has only two elements 0 and 1, is called a binary EIJM if $[B]m$ is invertible and $[B]_m^{-1} = [B]^T$, i.e., $[B]_m[B]_m^{-1} = [B]_m[B]_m^T = [I]_m$.

Example 3.1: Let $[B]_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then obviously

$$[B]_2 [B]_2^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = [I]_2 \quad (10)$$

Therefore, $[B]_2^{-1} = [B]_2^T$ and $[B]_2$ is EIJM. More generally, we give the following definition.

Definition 3.2: BIJM

Let,

$$[B]_m = \begin{pmatrix} B_{11} & B_{12} & \dots & B_{1m} \\ B_{21} & B_{22} & \dots & B_{2m} \\ \dots & \dots & \dots & \dots \\ B_{m1} & B_{m2} & \dots & B_{mm} \end{pmatrix}$$

and

$$[B]_m^T \triangleq \begin{pmatrix} B_{11} & B_{21} & \dots & B_{m1} \\ B_{12} & B_{22} & \dots & B_{m2} \\ \dots & \dots & \dots & \dots \\ B_{1m} & B_{2m} & \dots & B_{mm} \end{pmatrix}, \quad (11)$$

be an $m \times m$ block-wise matrix and the transpose of block-wise inverse matrix B, where B_{ij} is a $k \times k$ matrix for all $i, j = 1,\dots,m$. Defined $[B]_m$ is called a binary BIJM since

$$[B]_m [B]_m^T = [B]_m [B]_m^{-1} = [I]_m, \quad (12)$$

where $[I]_m$ is identity matrix of size m. In other words, $[B]_m$ is BIJM which is EIJM having the block-wise structure.

Example 3.2: BIJM of order 4 case, let

$$[\alpha]_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } [\beta]_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad (13)$$

then we have $\alpha\alpha + \beta\beta = [I]_2$ and $\alpha\beta + \beta\alpha = 0_2$, where $0_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and satisfying

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^T = \begin{pmatrix} \alpha\alpha+\beta\beta & \alpha\beta+\beta\alpha \\ \alpha\beta+\beta\alpha & \alpha\alpha+\beta\beta \end{pmatrix} = \begin{pmatrix} [I]_2 & 0 \\ 0 & [I]_2 \end{pmatrix} = [I]_4. \quad (14)$$

Therefore, we can have the followings.

$$\begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^{-1} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}^T = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \quad (15)$$

$$[J]_{2^2} \triangleq \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \quad (16)$$

and

$$[J]_{2^2}^{-1} = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}. \quad (17)$$

Clearly, $[J]_{2^2}$ is a BIJM since

$$[J]_{2^2}[J]_{2^2}^{-1} = [I]_{2^2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (18)$$

Moreover, it can be considered a circular block-wise permutation matrix as given in Example 3.3.

For our simple explanation, we consider only 2×2 case, where each block-wise matrix is a 2×2 submatrix. We introduce the binary block-wise Jacket transform (BJT) and binary BIJT. Let $[J]_m$ be a binary block-wise Jacket matrix (BJM) for the one-dimensional binary Jacket transform. We can transform a temporal or spatial vector x into a transform vector y by

$$y = [J]_m x \quad (19)$$

then the input vector can be obtained via-binary BIJTs as follows:

$$x = [J]_m^{-1}\ y = [J]_m^T\ y. \quad (20)$$

A block-wise permutation matrix $[P]_N = (p_{kl})$ is defined as for $1 \le k, l \le N$,

$$[p_{kl}] = \begin{cases} [I]_2 & \text{if } l \equiv k+1 (\bmod N) \\ 0 & \text{Otherwise} \end{cases}. \quad (21)$$

The block-wise permutation matrices P^h are referred to circulant permutation matrices. Moreover, it is easy to see that $\{I, P, \ldots, P^{N-1}\}$ forms an Abelian group with the conventional multiplication which is corresponding to the group of all complex N– roots of unity. For $N = 2$ we have

$$[P]^0_{2^2} = \begin{pmatrix} [I]_2 & 0 \\ 0 & [I]_2 \end{pmatrix} \text{ and}$$
$$[P]^1_{2^2} = \begin{pmatrix} 0 & [I]_2 \\ [I]_2 & 0 \end{pmatrix}. \quad (22)$$

While, we have the followings

$$[P]^0_{2^2} = \begin{pmatrix} [I]_2 & 0 & 0 \\ 0 & [I]_2 & 0 \\ 0 & 0 & [I]_2 \end{pmatrix},$$
$$[P]^1_{3\times 2} = \begin{pmatrix} 0 & [I]_2 & 0 \\ 0 & 0 & [I]_2 \\ [I]_2 & 0 & 0 \end{pmatrix},$$

and

$$[P]^2_{3\times 2} = \begin{pmatrix} 0 & 0 & [I]_2 \\ [I]_2 & 0 & 0 \\ 0 & [I]_2 & 0 \end{pmatrix}. \quad (23)$$

In Equation (16), $[J]_2$ is regarded to the smallest order binary BIJT. Moreover, $[J]_2$ is a circulant block-wise matrix since it can be written as

$$[J]_{2^2} = \alpha^*[I]_2 + \beta^*[P]_2 = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}. \quad (24)$$

A larger order binary BIJT can be generated by the following recursive relation, otherwise, the non-binary case is proved in Section 6.

$$[J]_N = [J]_{N/2} \otimes [J]_2, N \geq 4. \quad (25)$$

From the definition of the transpose of the block-wise transform, we have

$$[J]^T_4 = \left([J]_2 \otimes [J]_2\right)^T = [J]^T_2 \otimes [J]^T_2. \quad (26)$$

Then

$$\begin{aligned} [J]_4[J]^T_4 &= \left([J]_2 \otimes [J]_2\right)\left([J]^T_2 \otimes [J]^T_2\right) \\ &= \left([J]_2[J]^T_2\right) \otimes \left([J]_2 \otimes [J]^T_2\right) \\ &= [I]_2 \otimes [I]_2 = [I]_4. \end{aligned} \quad (27)$$

Note that since $[J]_4[J]^{-1}_4 = [J]_4[J]^T_4 = [I]_4$, $[J]_4$ becomes a 4×4 binary BIJM. In general, we can prove that $[J]_{2^k}$ is a $2^k \times 2^k$ binary BJM for a positive integer k. In fact

$$\begin{aligned} [J]_{2^k}[J]^T_{2k} &= \left([J]_{2^{k-1}} \otimes [J]_2\right)\left([J]_{2^{k-1}} \otimes [J]_2\right)^T \\ &= \left([J]_{2^{k-1}} \otimes [J]_2\right)\left([J]^T_{2^{k-1}} \otimes [J]^T_2\right) \\ &= \left([J]_{2^{k-1}}[J]^T_{2^{k-1}}\right) \otimes \left([J]_2[J]^T_2\right) \\ &= [I]_{2^{k-1}} \otimes [I]_2 = [I]_{2^k}. \end{aligned} \quad (28)$$

Thus, $[J]^{-1}_{2^k} = [J]^T_{2^k}$ and $[J]_{2^k}[J]^{-1}_{2^k} = [J]_{2^k}[J]^{-T}_{2^k} = [I]_{2^k}$. For the 4×4 binary BIJM, $[J]_{4^2}$ can be decomposed to the product of two sparse matrices,

$$\begin{aligned} [J]_{4^2} &= [J]_{2^2} \otimes [J]_{2^2} = \left([J]_{2^2} \otimes [I]_{2^2}\right)\left([I]_{2^2} \otimes [J]_{2^2}\right) \\ &= \begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \\ \beta & 0 & \alpha & 0 \\ 0 & \beta & 0 & \alpha \end{pmatrix}\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{pmatrix} \end{aligned} \quad (29)$$

From Equation (25), we can derive a general formula for construction of order $N = 2^k$, $k = 1, 2, \ldots$, binary block-wise inverse as

$$\begin{aligned} [J]_{2^k} &= \left([J]_{2^{k-1}} \otimes [J]_2\right) = \left([J]_{2^{k-1}} \otimes [I]_2\right)\left([I]_{2^{k-1}} \otimes [J]_2\right) \\ &= \left(\left([J]_{2^{k-2}} \otimes [J]_2\right) \otimes [I]_2\right)\left([I]_{2^{k-1}} \otimes [J]_2\right) \\ &= \left([J]_{2^{k-2}} \otimes \left([J]_2 \otimes [I]_2\right)\right)\left([I]_{2^{k-1}} \otimes [J]_2\right) \\ &= \left([J]_{2^{k-2}} \otimes [I]_{2^2}\right)\left([I]_{2^{k-2}} \otimes \left([J]_2 \otimes [I]_2\right)\right)\left([I]_{2^{k-1}} \otimes [J]_2\right) \\ &= \prod_{i=0}^{k-1}\left([I]_{2^{k-i-1}} \otimes [J]_2 \otimes [I]_{2^i}\right). \end{aligned} \quad (30)$$

This proof is given in Appendix. The fast signal flow graph corresponding to Equation (30) is similar to the graph in [1].

3.2. Arikan polar binary basic matrix of order 2^k

Polar coding introduced recently by Arikan [24] is a structured coding technique which approaches capacity for every output-symmetric discrete memoryless channels. Polarization has been applied later also in multi-terminal information-theoretic settings, such as the multiple access channel. We present here two classes of Arikan polar code matrix of order2^k. For binary Jacket matrices, the definition has a special form.

We first consider the following Arikan Polar binary basic matrices. Equation (31) discussed in Appendix.

$$[\alpha]_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } [\beta]_2 \triangleq \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (31)$$

It can easily be showed that

$$[\alpha]_2^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]_2$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = [\beta]_2^2, \quad (32)$$

$$[\alpha]_2[\beta]_2 + [\beta]_2[\alpha]_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = [I]_2, \quad (33)$$

and $[\alpha]_2[\beta]_2 + [\beta]_2[\alpha]_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]_2$.

From Equations (32)–(33), we can readily find

$$\alpha = \alpha^{-1}, \beta = \beta^{-1}, \alpha^2 + \beta^2 = I, \alpha\beta = \beta\alpha = I, (\alpha\beta)^{-1} = \beta\alpha,$$

and

$$(\beta\alpha)^{-1} = \alpha\beta. \quad (34)$$

Furthermore, we can verify that

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}\begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} \alpha\beta + \beta\alpha & \alpha^2 + \beta^2 \\ \beta^2 + \alpha^2 & \beta\alpha + \alpha\beta \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, (\text{i.e.},)$$

$$\begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}^{-1} = \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix}. \quad (35)$$

Now we present a BIJM based on submatrices α and β defined above

$$[J]_{2^2} \triangleq \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} (\text{and})$$

$$[J]_{2^2}^{-1} \triangleq \begin{bmatrix} \beta & \alpha \\ \alpha & \beta \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}^T = [J]_2^T \quad (36)$$

From Definition 3.1, it is certain that $[J]_2$ is a binary BIJM since it satisfies

$$[J]_{2^2}[J]_{2^2}^{-1} = [I]_{2^2}. \quad (37)$$

These Jacket block matrices are useful in coding theory and orthogonal code design. Let p be a positive number, then from Equation (3) yields

$$[E]^h = [ei, j]_{p\times p} \text{ and } e_{i,j} = \begin{cases} 1 & \text{for } i = \langle j+h \rangle \\ 0 & \text{otherwise} \end{cases}, \quad (38)$$

where $\langle j + h \rangle = j + h \bmod p$ and $0 \le i, j, h \le p - 1$. The matrices $[E]^h$, for $0 \le h \le p - 1$, are refer to circulant permutation matrices. It can be seen that $\{I, E, \ldots, Ep^{-1}\}$ form an Abelian group with conventional matrix multiplication and $I = E^p$.

Example 3.3: For 3GPP ultra mobile broadband, permutation matrices based on order-4 Abelian group are defined as follows.

$$[I]_{2^2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, [E]_{2^2}^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$
$$[E]_{2^2}^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \text{and } [E]_{2^2}^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (39)$$

Similar fashion as Equation (31), we define two additional matrices:

$$[\Lambda]_{2^3} \triangleq \begin{bmatrix} I & 0 \\ E^h & I \end{bmatrix} \text{ and } [\Omega]_{2^3} \triangleq \begin{bmatrix} I & E^{-h} \\ 0 & I \end{bmatrix}, \quad (40)$$

where

$$[\Lambda]_{2^3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and}$$

$$[\Omega]_{2^3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, we have

$$[\Lambda]_{2^3}[\Omega]_{2^3}+[\Omega]_{2^3}[\Lambda]_{2^3}=\begin{bmatrix} I & 0\\ E^h & I\end{bmatrix}\begin{bmatrix} I & E^{-h}\\ 0 & I\end{bmatrix}+\begin{bmatrix} I & E^{-h}\\ 0 & I\end{bmatrix}\begin{bmatrix} I & 0\\ E^h & I\end{bmatrix}=\begin{bmatrix} I & I\\ I & 0\end{bmatrix}+\begin{bmatrix} 0 & I\\ I & I\end{bmatrix}=[I]_{2^3}. \quad (41)$$

Furthermore,

$$\begin{bmatrix} \Lambda & \Omega\\ \Omega & \Lambda\end{bmatrix}_{16}\begin{bmatrix} \Omega & \Lambda\\ \Lambda & \Omega\end{bmatrix}_{16}=\begin{bmatrix} \Lambda\Omega+\Omega\Lambda & \Lambda^2+\Omega^2\\ \Omega^2+\Lambda^2 & \Omega\Lambda+\Lambda\Omega\end{bmatrix}=\begin{bmatrix} I & 0\\ 0 & I\end{bmatrix} \quad (42)$$

Now we can evaluate two matrices

$$[J]_{2^4}\triangleq\begin{bmatrix} \Lambda & \Omega\\ \Omega & \Lambda\end{bmatrix}=\begin{bmatrix} I & 0 & I & E^{-h}\\ E^h & I & 0 & I\\ I & E^{-h} & I & 0\\ 0 & I & E^h & I\end{bmatrix} \quad \text{and}$$

$$[J]^{-1}_{2^4}\triangleq\begin{bmatrix} \Omega & \Lambda\\ \Lambda & \Omega\end{bmatrix}=\begin{bmatrix} I & E^{-h} & I & 0\\ 0 & I & E^h & I\\ I & 0 & I & E^{-h}\\ E^h & I & 0 & I\end{bmatrix},$$

which satisfies

$$[J]_{2^4}[J]^{-1}_{2^4}=[I]_{16}. \quad (43)$$

Thus, the proposed Arikan Polar binary matrix becomes BIJM.

4. Binary BIJT of order 3^k and 5^k

In this section, a binary BJT and binary BIJT with orders 3^k and 5^k are proposed. From Equation (23), the smallest order 3 × 3 binary BJT can be written as

$$[J]_3=\alpha_0[P]^0_3+\alpha_1[P]^1_3+\alpha_2[P]^2_3. \quad (44)$$

From the definition of the BJM, we can show that

$$\begin{aligned}[J]_3[J]^T_3&=\left(\alpha_0[P]^0_3+\alpha_1[P]^1_3+\alpha_2[P]^2_3\right)\times\left(\alpha_0[P]^0_3+\alpha_1[P]^1_3+\alpha_2[P]^2_3\right)^T\\&=\left(\alpha_0\alpha_0^T+\alpha_1\alpha_1^T+\alpha_2\alpha_2^T\right)[P]^0_3+\left(\alpha_0\alpha_2^T+\alpha_1\alpha_0^T+\alpha_2\alpha_1^T\right)[P]^1_3+\left(\alpha_0\alpha_1^T+\alpha_1\alpha_2^T+\alpha_2\alpha_0^T\right)[P]^2_3\\&=[I]_3\end{aligned} \quad (45)$$

if the following conditions are satisfied,

$$\begin{aligned}&\left(\alpha_0\alpha_0^T+\alpha_1\alpha_1^T+\alpha_2\alpha_2^T\right)=I_2,\left(\alpha_0\alpha_2^T+\alpha_1\alpha_0^T+\alpha_2\alpha_1^T\right)\\&\quad=0,\ \textit{and}\ \left(\alpha_0\alpha_1^T+\alpha_1\alpha_2^T+\alpha_2\alpha_0^T\right)\\&\quad=0.\end{aligned} \quad (46)$$

Example 4.1: Let

$$\alpha_0=\begin{pmatrix} 0 & 0\\ 0 & 1\end{pmatrix},\alpha_1=\begin{pmatrix} 1 & 0\\ 0 & 0\end{pmatrix}\textit{ and }\alpha_2=\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \quad (47)$$

be three 2 × 2 binary matrices over *GF*(2) which satisfy the above conditions in Equation (46). Similarly, the smallest order 3 binary BIJM can be written as follows

$$[J]_{3\times 2}=\begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2\\ \alpha_2 & \alpha_0 & \alpha_1\\ \alpha_1 & \alpha_2 & \alpha_0\end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}. \quad (48)$$

Then we can show that

$$[J]_3[J]^{-1}_3=[J]_3[J]^T_3=[I]_3$$

Clearly, $[J]_3$ is also circulant binary BIJM. Using the Kronecker product of BJMs, the larger order 3^k binary BJT can be determined by the following recursive relation, i.e.,

$$[J]_N=[J]_{NB}\otimes[J]_3, N\geq 9. \quad (49)$$

By a similar method, it can be proved that $[J]_{3^k}$ is a binary block-wise inverse Jacket of order 3^k according to Equation (50). We can also derive a fast algorithm based on the proposed factorization of the binary BIJT.

$$i=0,[J]_{3^k}=\prod_{i=0}^{k-1}\left([I]_{3^{k-1-i}}\otimes[J]_3\otimes[I]_{3^i}\right). \quad (50)$$

Example 4.2: For DFT 3 × 3 matrix, $N=3^2$, $p=3$, $\omega=e^{2\pi i/3}$, and $N=p^n$, we can get the fast transform of Jacket matrix J_{3^2} as follows:

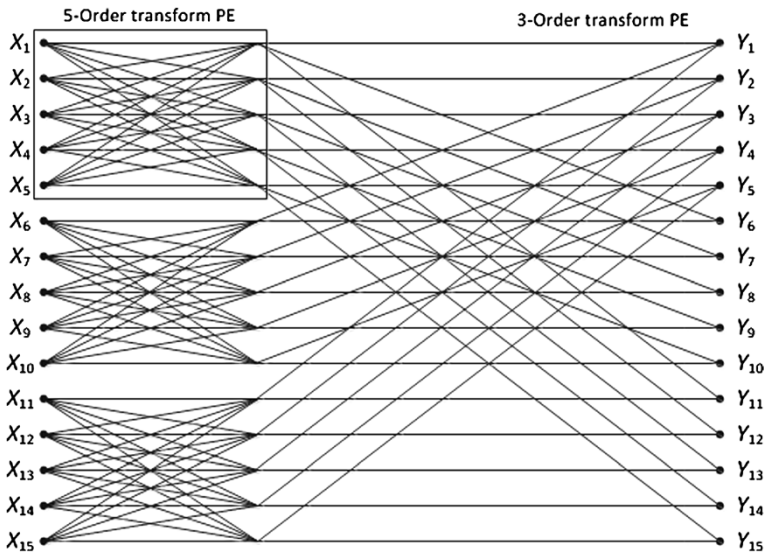

Figure 1 Fast signal flow graph of the proposed 1-dimensional $[J]_{15}$ Jacket transform.

$$[J]_{3^2} = [DFT]_3 \otimes [DFT]_3 = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^1 \end{bmatrix} \otimes \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^1 \end{bmatrix} = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^0 & \omega^1 & \omega^2 & \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^2 & \omega^1 & \omega^0 & \omega^2 & \omega^1 & \omega^0 & \omega^2 & \omega^1 \\ \omega^0 & \omega^0 & \omega^0 & \omega^1 & \omega^1 & \omega^1 & \omega^2 & \omega^2 & \omega^2 \\ \omega^0 & \omega^1 & \omega^2 & \omega^1 & \omega^2 & \omega^0 & \omega^2 & \omega^0 & \omega^1 \\ \omega^0 & \omega^2 & \omega^1 & \omega^1 & \omega^0 & \omega^2 & \omega^2 & \omega^1 & \omega^0 \\ \omega^0 & \omega^0 & \omega^0 & \omega^2 & \omega^2 & \omega^2 & \omega^1 & \omega^1 & \omega^1 \\ \omega^0 & \omega^1 & \omega^2 & \omega^2 & \omega^0 & \omega^1 & \omega^1 & \omega^2 & \omega^0 \\ \omega^0 & \omega^2 & \omega^1 & \omega^2 & \omega^1 & \omega^0 & \omega^1 & \omega^0 & \omega^2 \end{bmatrix} \tag{51}$$

Similarly, we have

$$[J]_{3^2}^{-1} = \begin{bmatrix} \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 & \omega^0 \\ \omega^0 & \omega^2 & \omega^1 & \omega^0 & \omega^2 & \omega^1 & \omega^0 & \omega^2 & \omega^1 \\ \omega^0 & \omega^1 & \omega^2 & \omega^0 & \omega^1 & \omega^2 & \omega^0 & \omega^1 & \omega^2 \\ \omega^0 & \omega^0 & \omega^0 & \omega^1 & \omega^1 & \omega^1 & \omega^2 & \omega^2 & \omega^2 \\ \omega^0 & \omega^2 & \omega^1 & \omega^1 & \omega^0 & \omega^2 & \omega^2 & \omega^1 & \omega^0 \\ \omega^0 & \omega^1 & \omega^2 & \omega^1 & \omega^2 & \omega^0 & \omega^2 & \omega^0 & \omega^1 \\ \omega^0 & \omega^0 & \omega^0 & \omega^2 & \omega^2 & \omega^2 & \omega^1 & \omega^1 & \omega^1 \\ \omega^0 & \omega^2 & \omega^1 & \omega^2 & \omega^1 & \omega^0 & \omega^1 & \omega^0 & \omega^2 \\ \omega^0 & \omega^1 & \omega^2 & \omega^2 & \omega^0 & \omega^1 & \omega^1 & \omega^2 & \omega^0 \end{bmatrix}. \tag{52}$$

Certainly, we get a fast algorithm Equation (50) for the binary BIJT according to the following equation:

$$[J]_{3^2}[J]_{3^2}^{-1} = [I]_{3^2}. \tag{53}$$

We can take the left-hand side Equation (51) of coefficient

$$RM(1,3^2) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 & 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}. \tag{54}$$

This is the first-order ternary generator matrix of the Reed-Muller (RM) matrix [25].

Now, we consider an order 5^k binary BJT. The smallest order 5 binary BJT can be defined as follows.

$$[J]_5 = \left(\beta_0[P]_5^0 + \beta_1[P]_5^1 + \beta_2[P]_5^2 + \beta_3[P]_5^3 + \beta_4[P]_5^4\right), \tag{55}$$

Table 1 Computational complexity of the proposed fast algorithms for the block Jacket transforms compared with its direct computation (DC)

DC	Proposed					
	$N = 2^k$	$N = 3^k$	$N = 5^k$	$N = p^k$	$N = p^m q^n$	Two-dimensional: $p = q$, $m = n$
ADD						
$N(N-1)$	$N\log_2 N$	$2N\log_3 N$	$4N\log_5 N$	$(p-1)N\log_p N$	$n(p-1)p^n + m(q-1)q^m$	$2m(p-1)p^m$
MUL						
$N \times N$	$1/2N\log_2 N$	$4/3N\log_3 N$	$16/5N\log_5 N$	$\frac{(p-1)^2}{p} N \log_p N$	$n(p-1)^2 p^{n-1} + 2\, m(q-1)q^{m-1}$	$m(p^2-1)p^{m-1}$

In Table 1, ADD and MUL, respectively, abbreviate real additions and real multiplications.

where $[P]_5^0$ is the 5 × 5 block-wise identity matrix and $[P]_5^1$, $[P]_5^2$, $[P]_5^3$, and $[P]_5^4$ are the 5 × 5 block-wise permutation matrices. Thus, we can get

$$\begin{aligned}
[J]_5[J]_5^T &= \left(\beta_0[P]_5^0 + \beta_1[P]_5^1 + \beta_2[P]_5^2 + \beta_3[P]_5^3 + \beta_4[P]_5^4\right) \\
&\quad \times \left(\beta_0[P]_5^0 + \beta_1[P]_5^1 + \beta_2[P]_5^2 + \beta_3[P]_5^3 + \beta_4[P]_5^4\right)^T \\
&= \left(\beta_0\beta_0^T + \beta_1\beta_1^T + \beta_2\beta_2^T + \beta_3\beta_3^T + \beta_4\beta_4^T\right)[P]_5^0 \\
&\quad + \left(\beta_0\beta_4^T + \beta_1\beta_0^T + \beta_2\beta_1^T + \beta_3\beta_2^T + \beta_4\beta_3^T\right)[P]_5^1 \\
&\quad + \left(\beta_0\beta_3^T + \beta_1\beta_4^T + \beta_2\beta_0^T + \beta_3\beta_1^T + \beta_4\beta_2^T\right)[P]_5^2 \\
&\quad + \left(\beta_0\beta_2^T + \beta_1\beta_3^T + \beta_2\beta_4^T + \beta_3\beta_0^T + \beta_4\beta_1^T\right)[P]_5^3 \\
&\quad + \left(\beta_0\beta_1^T + \beta_1\beta_2^T + \beta_2\beta_3^T + \beta_3\beta_4^T + \beta_4\beta_0^T\right)[P]_5^4 \\
&= [I]_5
\end{aligned} \tag{56}$$

if the following conditions are satisfied

$$\begin{aligned}
&\left(\beta_0\beta_0^T + \beta_1\beta_1^T + \beta_2\beta_2^T + \beta_3\beta_3^T + \beta_4\beta_4^T\right) = [I]_2, \\
&\left(\beta_0\beta_4^T + \beta_1\beta_0^T + \beta_2\beta_1^T + \beta_3\beta_2^T + \beta_4\beta_3^T\right) = 0, \\
&\left(\beta_0\beta_3^T + \beta_1\beta_4^T + \beta_2\beta_0^T + \beta_3\beta_1^T + \beta_4\beta_2^T\right) = 0, \\
&\left(\beta_0\beta_2^T + \beta_1\beta_3^T + \beta_2\beta_4^T + \beta_3\beta_0^T + \beta_4\beta_1^T\right) = 0, \quad \textit{and} \\
&\left(\beta_0\beta_1^T + \beta_1\beta_2^T + \beta_2\beta_3^T + \beta_3\beta_4^T + \beta_4\beta_0^T\right) = 0.
\end{aligned} \tag{57}$$

The signal flow graph corresponding Equation (56) is shown in Figure 1 and its factorization is defined by Equation (57). The general formula is given in Appendix.

Example 4.3: Let

$$\beta_0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \beta_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \textit{ and } \beta_4 \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{58}$$

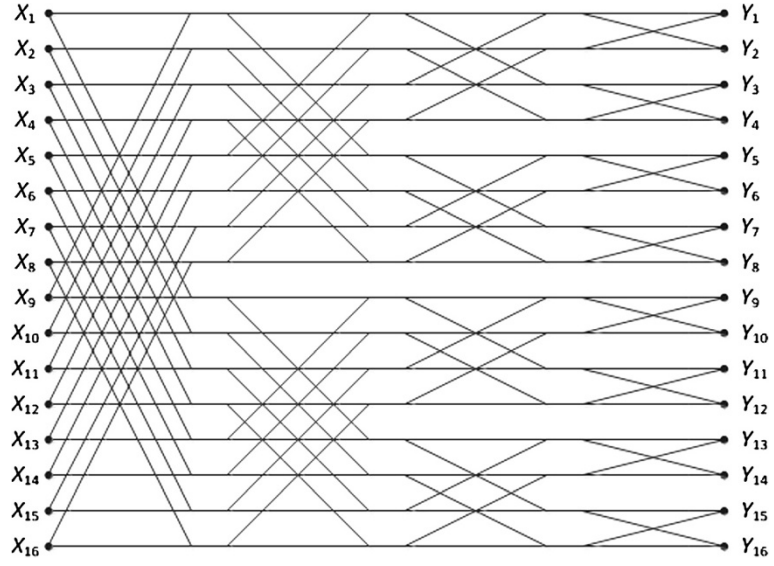

Figure 2 Two-dimensional 4 × 4 fast signal flow graph of the proposed Jacket transform.

be five 2 × 2 binary matrices over $GF(2)$ which satisfy the condition defined in Equation (57). From this knowledge, the order 5 binary BJT can be written as

$$[J]_5 = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_4 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_3 & \beta_4 & \beta_0 & \beta_1 & \beta_2 \\ \beta_2 & \beta_3 & \beta_4 & \beta_0 & \beta_1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (59)$$

Then, we can show that

$$[J]_5[J]_5^{-1} = [J]_5[J]_5^T = [I]_5.$$

Clearly, $[J]_5$ is also circulant binary BIJM. Using the Kronecker product of two matrices, we can generate a higher order binary BJT according to the following recursive form:

$$\begin{aligned} [J]_{5^k} &= [J]_{5^{k-1}} \otimes [J]_5 \\ &= \prod_{i=0}^{k-1} \left([I]_{5-1-i} \otimes [J]_5 \otimes [I]_5^i \right), \end{aligned} \quad (60)$$

where $[I]_{5^0} = 1$. Further, it is easy to construct binary BJMs with orders 6, 10, 15, 25, and so on.

Similarly as in Equations (30) and (60), we can derive an order p^k fast binary BJT, where p is the prime number. For the matrix of order 6, the binary BJT can be written as $[J]_6 = ([J]_2 \otimes [J]_3)$. We can further express $[J]_6$ in following block-wise form

$$[J]_6 = \left([J]_2 \otimes [I]_3 \right) \left([I]_2 \otimes [J]_3 \right). \quad (61)$$

Now, with the aid of recursive relation, the matrix $[J]_{6^k} = [J]_{6^{k-1}} \otimes [J]_6$ becomes an order of 6^k binary BJT. The fast algorithm depends on the following sparse factorization:

$$i = 0, [J]_{6^k} = \prod_{i=0}^{k-1} \left([I]_{6^{k-1-i}} \otimes [J]_6 \otimes [I]_6^i \right), \quad (62)$$

If the matrix is of order 15, we can decompose $[J]_{15}$ as

$$[J]_{15} = [J]_3 \otimes [J]_5. \quad (63)$$

A corresponding block-wise form of Equation (63) is given by

$$\begin{aligned} [J]_{15} &= \left([I]_3 \otimes [J]_5 \right) \left([J]_3 \otimes [I]_5 \right) \\ &= \left([I]_3 \otimes \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \beta_4 & \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \beta_3 & \beta_4 & \beta_0 & \beta_1 & \beta_2 \\ \beta_2 & \beta_3 & \beta_4 & \beta_0 & \beta_1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_0 \end{pmatrix} \right) \times \left([J]_3 \otimes [I]_5 \right). \end{aligned} \quad (64)$$

It can be seen that the computation of order-15 matrix is the combination of three times of order 5 and five times of order 3 of two sparse matrices as shown in Figure 1. In general, the computational complexity of the proposed fast algorithm and higher order binary BJT implementations of inverse are similar to those in [1]. For example, a binary BJT of order $N = 2^k$ requires $N\log_2 N$ additions and $1/2N\log_2 N$ multiplications. Another BJT of order $N = 3^k$ requires $2N\log_3 N$ additions and $4/3N\log_3 N$ multiplications. Also, a binary BJT of order $N = 5^k$ requires $4N\log_5 N$ additions and $16/5N\log_5 N$ multiplications. These results are summarized in Table 1.

5. Two-dimensional fast algorithm for binary BJT

The two-dimensional matrix transforms a temporal/spatial matrix X into a transformed matrix Y as

$$Y = [J]_N X \left([J]_N \right)^T. \quad (65)$$

In general, the linear transform of matrix X verified as $AXB = Y$ can be expressed by the transformation of the column-wise stacking vector of X as in [1,5,18].

$$\left([J]_N \otimes [J]_N \right) vec(X) = vec(Y). \quad (66)$$

Thus, the two-dimensional binary BJM in Equation (56) can be expressed by

$$vec(Y) = \left([J]_N \otimes [J]_N \right) vec(X). \quad (67)$$

Based on this one-dimensional fast algorithm, the two-dimensional fast algorithm for the binary BJT decomposition can be described as follows:

$$\begin{aligned} [J]_N &= [J]_{N_2} \otimes [J]_{N_1} \\ &= \left([I]_{N_2} \otimes [J]_{N_1} \right) \left([J]_{N_2} \otimes [I]_{N_1} \right) \end{aligned} \quad (68)$$

For example, $N_1 = N_2 = 4 = 2^2$, then we have

$$\begin{aligned} [J]_4 \otimes [J]_4 &= \left([I]_{2^2} \otimes [J]_{2^2} \right) \left([J]_{2^2} \otimes [I]_{2^2} \right) \\ &= \left([I]_{2^2} \otimes \left([J]_2 \otimes [J]_2 \right) \right) \left(\left([J]_2 \otimes [J]_2 \right) \otimes [I]_{2^2} \right) \\ &= \underbrace{\left([I]_{2^2} \otimes [I]_2 \otimes [J]_2 \right)}_{\left([I]_2 \otimes [J]_2 \otimes [I]_{2^2} \right)} \underbrace{\left([I]_{2^2} \otimes [J]_2 \otimes [I]_2 \right)}_{\left([J]_2 \otimes [I]_2 \otimes [I]_{2^2} \right)}. \end{aligned} \quad (69)$$

It is shown that block matrix $[J]_{2^2}$ is a 2-order BJT that can be constructed in the recursive fashion on the basis of $[J]_2$ with the fast algorithm. The two-dimensional BJT can be designed by the fast algorithm described by

Equation (69) and Figure 2. There are four sparse matrices stages in Figure 2, i.e., $\log_2 16 = \log_2 2^4 = 4$. For the two-dimensional 4 × 4 case, the fast algorithm requires 64 additions and 32 multiplications as shown in Table 1.

6. Alamouti MIMO non-binary BIJT of order 2^k

Similar to the binary case of Equations (18), (23), and (44), a non-binary BIJT can be developed by the Kronecker product of the successive lower order identity matrix and the basis $[J]_2$ matrix. The mobile communication diagonal channel matrix is given by [17], Equation (32)

$$[H]_N = [I]_{N/2} \otimes [J]_2, \tag{70}$$

where $u = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} e^{it} & 0 \\ 0 & e^{it} \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$

then

$$[J]_2 = \begin{pmatrix} \cos 45^\circ & i \sin 45^\circ \\ \sin 45^\circ & i \cos 45^\circ \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$
$$= \begin{bmatrix} 0.8881 & -.3251 + 0.3251i \\ 0.3251 + 0.3251i & 0.8881 \end{bmatrix}$$
$$\begin{bmatrix} 0.9659 - 0.2588i & 0 \\ 0 & -0.2588 + 0.9659i \end{bmatrix}$$
$$\begin{bmatrix} 0.8881 & 0.3251 - 0.3251i \\ -0.3251 - 0.3251i & 0.8881 \end{bmatrix} = U \Lambda U^H, \tag{71}$$

where U is eigenvector matrix and Λ is eigenvalue matrix.

From Equations (70) and (71), the MIMO channel matrix H is decomposed by the singular value decomposition (SVD) [26], that is, we have

$$H = U \Sigma V^H, \tag{72}$$

where U and V are unitary matrices, and Σ is a rectangular diagonal matrix with non-negative real elements which means the EIJM. Figure 3 shows the block diagram of the considered MIMO channel. The diagonal elements of Σ are the singular values of the channel matrix H, denoting by $\sigma_1, \sigma_2, \ldots, \sigma_{N_{\min}}$, where $N_{\min} = \min(N_T, N_R)$. In case of $N_{\min} = N_T$, SVD in Equation (72) can be expressed as

$$H = \mathrm{U} \Sigma \mathrm{V}^H = \underbrace{[\mathrm{U}_{N_{\min}} \mathrm{U}_{N_R - N_{\min}}]}_{U} \underbrace{\begin{bmatrix} \Sigma_{N_{\min}} \\ 0_{N_R - N_{\min}} \end{bmatrix}}_{\Sigma}$$
$$\times \mathrm{V}^H = \mathrm{U}_{N_{\min}} \Sigma_{N_{\min}} \mathrm{V}^H, \tag{73}$$

where $U_{N_{\min}}$ is composed of $N_{\min}$ left-singular vectors and $\Sigma_{N_{\min}}$ is a square matrix. In case of $N_{\min} = N_R$, SVD in Equation (72) can be expressed as

$$H = U \underbrace{[\Sigma_{N_{\min}} 0_{N_T - N_{\min}}]}_{\Sigma} \underbrace{\begin{bmatrix} V^H_{N_{\min}} \\ V^H_{N_T - N_{\min}} \end{bmatrix}}_{V^H}$$
$$= U \Sigma_{N_{\min}} V^H_{N_{\min}}. \tag{74}$$

where $V_{N_{\min}}$ is composed of $N_{\min}$ right-singular vectors. Then we get eigenvalue decomposition,

$$HH^H = U \Sigma \Sigma^H U^H = U \Lambda U^H \tag{75}$$

where $U^H U = I_{N_R}$ and $\Lambda \in \mathbb{C}^{N_R \times N_R}$ is a diagonal matrix. Equation (75) is same as Equation (71). Based on Equation (71), the 2 × 2 Jacket matrix, the 4 × 4 BJM can be evaluated as

$$[H]_2 = \begin{pmatrix} [J]_2 & 0 \\ 0 & [J]_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = [I]_2 \otimes [J]_2. \tag{76}$$

Certainly, the inverse matrix of $[H]_2$ is given by

$$[H]_2^{-1} = \sqrt{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ i & -i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & i & -i \end{pmatrix} = [I]_2 \otimes [J]_2^{-1}. \tag{77}$$

From Equations (76) and (77), we can show that

$$[H]_2 [H]_2^{-1} = 2[I]_4, \tag{78}$$

which satisfies the property of the Jacket matrix.

Similar fashion as Equation (70), the Alamouti encoder [22,27] encodes two consecutive symbols x_1 and x_2 by

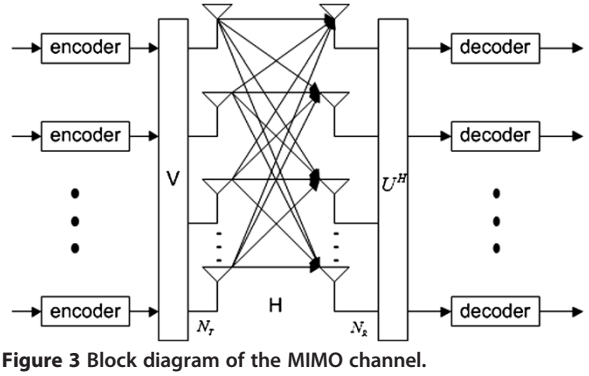

Figure 3 Block diagram of the MIMO channel.

the following space–time codeword matrix,

$$[A]_2 = \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} \text{ and } [A]_2^H = \begin{pmatrix} x_1^* & -x_2 \\ x_2^* & x_1 \end{pmatrix}. \quad (79)$$

Encoded signals are transmitted from two transmit antennas over two symbols intervals. In the first symbol interval, two symbols x_1 and x_2 are transmitted from the two transmit antennas, whereas in the second symbol interval, $-x_2^*$ is transmitted from the first transmit antenna and x_1^* transmitted from the second transmit antenna. From this employment, the Alamouti codeword becomes a complex-valued orthogonal matrix as follows

$$\begin{aligned}[A]_2[A]_2^H &= \begin{bmatrix} |x_1|^2 + |x_2|^2 & 0 \\ 0 & |x_1|^2 + |x_2|^2 \end{bmatrix} \\ &= \left(|x_1|^2 + |x_2|^2\right) I_2, \end{aligned} \quad (80)$$

where I_2 denotes the 2 × 2 identity matrix. Also, Equation (79) becomes the EIJM.

For the STBC with two antennas in the transmitter and receiver in the LTE [21,28,29], we can define the Alamouti matrix as

$$\begin{aligned}[A]_2 &= \begin{pmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{pmatrix} \\ &= \begin{pmatrix} a & a \\ b & -b \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} c & d \\ c & -d \end{pmatrix} \\ &= [J]_1[\Lambda][J]_2. \end{aligned} \quad (81)$$

We now show that the Alamouti coding matrix can be the BIJM.

$$[J]_1 = \begin{pmatrix} a & a \\ b & -b \end{pmatrix}, [\Lambda] = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } [J]_2 = \begin{pmatrix} c & d \\ c & -d \end{pmatrix},$$

then yields

$$[J]_1[\Lambda][J]_2 = \begin{pmatrix} ac(\lambda_1 + \lambda_2) & ad(\lambda_1 - \lambda_2) \\ bcac(\lambda_1 - \lambda_2) & bdac(\lambda_1 + \lambda_2) \end{pmatrix}.$$

By comparing with both sides of $[A]_2 = [J_1][\Lambda][J_2]$, it is easy to get the following equations

$$\begin{aligned} ac(\lambda_1 + \lambda_2) &= x_1 \quad , \quad ad(\lambda_1 - \lambda_2) = x_2 \\ bc(\lambda_1 - \lambda_2) &= -x_2^* \quad , \quad bd(\lambda_1 + \lambda_2) = x_1^* \end{aligned}$$

By solving the above equations, we have

$$\begin{aligned} \frac{ac(\lambda_1 + \lambda_2)}{ad(\lambda_1 - \lambda_2)} &= \frac{x_1}{x_2}, \quad \frac{c(\lambda_1 - \lambda_2)}{d(\lambda_1 + \lambda_2)} = \frac{-x_2^*}{x_1^*} \\ \left(\frac{c}{d}\right)^2 &= -\frac{x_1 x_2^*}{x_2 x_1^*} \qquad = -\frac{x_1^2 \left(x_2^*\right)^2}{x_1 x_2^* x_2 x_2^*} \end{aligned}$$

We can put $c = \frac{ix_1}{|x_1|}$, $d = \frac{x_2^*}{|x_2|}$. Similarly, we have

$$\frac{a(\lambda_1 + \lambda_2)}{b(\lambda_1 - \lambda_2)} = -\frac{x_1}{-x_2^*}, \quad \frac{a(\lambda_1 - \lambda_2)}{b(\lambda_1 + \lambda_2)} = \frac{x_2}{x_1^*}$$

From these computations, we can obtain the following two forms for transmitted symbols

$$-\frac{x_1^2}{|x_1|^2}(\lambda_1 + \lambda_2) = x_1 \text{ and } \frac{ix_1 x_2^*}{|x_1||x_2|}(\lambda_1 - \lambda_2) = x_2.$$

Having obtained these two symbols, we are able to get the half rate Alamouti matrix $[A]_4$ as follows

$$\begin{aligned}[A]_4 &= \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ -x_2^* & x_1^* & 0 & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & -x_2^* & x_1^* \end{pmatrix} = \begin{pmatrix} a & a & 0 & 0 \\ b & -b & 0 & 0 \\ 0 & 0 & a & a \\ 0 & 0 & b & -b \end{pmatrix} \\ &\quad \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & b & \lambda_4 \end{pmatrix} \begin{pmatrix} c & d & 0 & 0 \\ c & -d & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & c & -d \end{pmatrix} \\ &= [I]_2 \otimes \left([J]_1[\Lambda][J]_2\right) = \left([I]_2 \otimes [J]_1\right) diag(\Lambda) \left([I]_2 \otimes [J]_2\right) \end{aligned} \quad (82)$$

From Equation (82), we can rewrite as

$$\begin{aligned}[A]_4 &= [I]_2 \otimes [J^1]_2 \, \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) [J^2]_2 \\ &= \left([I]_2 \otimes [J^1]_2\right) \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \left([I]_2 \otimes [J^2]_2\right) \end{aligned} \quad (83)$$

where $[I]_2$ is identity matrix, $[J^1]_2$ is a unitary matrix, $\lambda_1,\lambda_2,\ldots,\lambda_4$ eigenvalue, and $[J^2]_2$ is another unitary matrix. We know that the binary fast block-wise Jacket transform (BFBJT) of 2^k, 3^k, 5^k, and 6^k is general Equations (30), (50), (60), and (62). The non-binary Alamouti fast BJT based on the BFBJT, then the general equation is given

$$[A]_k = \left([I]_{k/2} \otimes [J^1]_k\right) \mathrm{diag}(\lambda_1, \lambda_2, \cdots \lambda_k) \left([I]_{k/2} \otimes [J^2]_k\right) \quad (84)$$

From Equation (82), if $|x_1|^2 = |x_2|^2 = 1$, we can show that

$$
\begin{aligned}
[A]_4[A]_4^{-1} &= \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ -x_2^* & x_1^* & 0 & 0 \\ 0 & 0 & x_1 & x_2 \\ 0 & 0 & -x_2^* & x_1^* \end{pmatrix} \begin{pmatrix} x_1^* & -x_2 & 0 & 0 \\ x_2^* & x_1 & 0 & 0 \\ 0 & 0 & x_1^* & -x_2 \\ 0 & 0 & x_2^* & x_1 \end{pmatrix} \\
&= \begin{pmatrix} |x_1|^2 + |x_2|^2 & 0 & 0 & 0 \\ 0 & |x_1|^2 + |x_2|^2 & 0 & 0 \\ 0 & 0 & |x_1|^2 + |x_2|^2 & 0 \\ 0 & 0 & 0 & |x_1|^2 + |x_2|^2 \end{pmatrix} = 2[I]_4.
\end{aligned} \tag{85}
$$

Also, a full rate 4 × 4 Alamouti matrix can be decomposed as

$$
\begin{aligned}
[A]_4[A]_4^{-1} &= \begin{pmatrix} x_1 & x_2 & 0 & 0 \\ -x_2^* & x_1^* & 0 & 0 \\ 0 & 0 & x_3 & x_4 \\ 0 & 0 & -x_4^* & x_3^* \end{pmatrix} \begin{pmatrix} x_1^* & -x_2 & 0 & 0 \\ x_2^* & x_1 & 0 & 0 \\ 0 & 0 & x_3^* & -x_4 \\ 0 & 0 & x_4^* & x_3 \end{pmatrix} \\
&= \begin{pmatrix} |x_1|^2 + |x_2|^2 & 0 & 0 & 0 \\ 0 & |x_1|^2 + |x_2|^2 & 0 & 0 \\ 0 & 0 & |x_3|^2 + |x_4|^2 & 0 \\ 0 & 0 & 0 & |x_3|^2 + |x_4|^2 \end{pmatrix} = 2[I]_4.
\end{aligned} \tag{86}
$$

Note that if $|x_1|^2 = |x_2|^2 = |x_3|^2 = |x_4|^2 = 1$, then $[A]_4[A]_4^{-1} = 2[I]_4$, so that $[H]_2$ and $[A]_4$, respectively, in Equations (76) and (82) become BIJTs.

7. Conclusion

The fast Arikan Polar binary BIJTs and Alamouti MIMO non-binary block-wise inverse transforms are proposed, which overcome the mathematical problem raised in [1] and also satisfy the relation $[J]_N[J]_N^{-1} = [I]_N$. The orders 2^k, 3^k, 5^k, and 6^k binary BJTs are constructed and their binary block-wise inverse transforms are easily obtained by the transpose of binary block-wise transforms. The one- and two-dimensional binary block-wise fast transforms are proposed based on their recursive forms. The Kronecker product of the successive lower order matrix and binary block-wise basis matrix are used in recursive forms. Furthermore, the Alamouti MIMO non-binary BJT is verified that it is the Kronecker product of lower order identity matrix and the basis $[J]_2$ matrix. From our verification, the proposed BIJTs can be applied in areas such as 3GPP ultra mobile broadband permutation matrices design, RM code design, diagonal channel, MIMO 4G LTE Alamouti code, and precoding design. The diagonal block-wise basis identity matrices are well suitable to without inter-symbol interference orthogonal frequency division multiplexing, diagonal block zero forcing precoding, and SVD for multiuser MIMO.

Appendix

Proof of Table 1 with $N = p^m q^n$

Suppose $[J]_p$ and $[J]_q$ are constructed using the above approaches specified in Equation (30), where p, q are prime numbers and m and n are non-negative integer numbers. Then a larger size BIJM $[J]_{N=p^m q^n}$ can be constructed in the following way similar to those in [30],

$$
\begin{aligned}
[J]_N &= \left\{ I_{q^n} \otimes \left(\prod_{i=0}^{m-1} I_{q^{m-i-1}} \otimes J_p \otimes I_{p^i} \right) \right\} \\
&\quad \times \left\{ \left(\prod_{i=0}^{n-1} I_{q^{n-i-1}} \otimes J_q \otimes I_{q^i} \right) \otimes I_{p^m} \right\} \\
&= \left\{ I_{q^n} \otimes \left(\prod_{i=1}^{m} I_{p^{m-i}} \otimes J_p \otimes I_{p^{i-1}} \right) \right\} \\
&\quad \times \left\{ \left(\prod_{i=1}^{n} I_{q^{n-i}} \otimes J_q \otimes I_{p^{i-1}} \right) \otimes I_{p^m} \right\}
\end{aligned} \tag{87}
$$

Proof: Block-wise Jacket matrices $[J]_p$ and $[J]_q$ are given by

$$
[J]_p^i = I_{p^{m-i-1}} \otimes J_p \otimes I_{p^i} \quad \textit{and} \quad [J]_q^i = I_{q^{m-i-1}} \otimes J_q \otimes I_{q^i}. \tag{88}
$$

We can prove left side of Equation (88),

$$
[J]_p = \left(J_p \otimes I_{p^{m-i-1}} \right) \left(I_p \otimes J_{m-i-1} \right),
$$

where I_p is an identity matrix of order p^2 and $[J]_{p-1} = \left(I_1 \otimes J_p \otimes I_{p^{m-i-2}}\right)\left(I_p \otimes J_{m-i-2}\right)$. Thus, we obtain

$$\left(J_p \otimes I_{p^{m-i-1}}\right)\left(I_p \otimes J_{m-i-1}\right) = J_p \otimes J_{m-i-1}. \tag{89}$$

For a general form, we rewrite Equation (88) as

$$[J]_p = \left(I_1 \otimes J_p \otimes I_{p^{m-i-1}}\right)\left(I_p \otimes J_{m-i-1}\right).$$

The second term can be written as

$$\begin{aligned}[J]_p \otimes [J]_{m-i-1} &= \begin{pmatrix} I_1 \otimes J_p \otimes I_{p^{m-i-2}} & 0 \\ 0 & I_1 \otimes J_p \otimes I_{p^{m-i-2}} \end{pmatrix} \\ &\quad \begin{pmatrix} I_p \otimes J_{m-i-2} & 0 \\ 0 & I_p \otimes J_{m-i-2} \end{pmatrix} \\ &= \left(I_p \otimes J_p \otimes I_{p^{m-i-1}}\right)\left(I_{p^2} \otimes J_{m-i-2}\right).\end{aligned} \tag{90}$$

From Equations (88)–(90), we have

$$[J]_p = \prod_{i=0}^{m-1} \left(I_{p^{i-1}} \otimes J_p \otimes I_{p^{m-1}}\right).$$

Since $J_p^T = J_p$, we have

$$[J]_p = \prod_{i=0}^{m-1} \left(I_{p^{m-i-1}} \otimes J_p \otimes I_{p^i}\right). \tag{91}$$

Similarly, we prove the right side of Equation (88).

$$[J]_q = \prod_{i=0}^{m-1} \left(I_{q^{m-i-1}} \otimes J_q \otimes I_{q^i}\right), \tag{92}$$

where $k \in \{p, q\}$ and $l \in \{m, n\}$. Subsequently, a general $[J]_N$ can be constructed as

$$[J]_N = \left(I_{p^m} \otimes J_{q^n}\right)\left(J_{p^m} \otimes I_{q^n}\right). \tag{93}$$

In [31], Equation (15), similar fashion as Equations (6) and (93) are Kronecker product of diagonal subchannel and decomposed of interference alignment for cellular networks. The idea is to align interferences into multidimensional decompose subspace to one dimension.

Discussed of Equation (31) over *GF*(2)

We consider here matrices only over *GF*(2). Let 0 be the all-zero matrix of size *N*.

For an arbitrary matrix $[M]_N$, form the matrix

$$[\Lambda]_{2N} = \begin{pmatrix} [I]_N & 0 \\ [M]_N & [I]_N \end{pmatrix}. \tag{94}$$

For the matrix $[M]_N$, we can check easily

$$\begin{aligned}[\Lambda]_{2N}^2 = [\Lambda]_{2N}[\Lambda]_{2N} &= \begin{pmatrix} [I]_N & 0 \\ 0 & [I]_N \end{pmatrix}, \left[\Lambda^T\right]_{2N}^2 \\ &= \begin{pmatrix} [I]_N & 0 \\ 0 & [I]_N \end{pmatrix}, \quad \text{and}\end{aligned}$$

$$[\Lambda]_{2N} + \left[\Lambda^T\right]_{2N}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{95}$$

Suppose $[M]_N = [U]_N$ is an orthogonal matrix then

$$\begin{aligned}[\Lambda]_{2N}[\Lambda]_{2N}^T &= \begin{pmatrix} [I]_N & 0 \\ [U]_N & [I]_N \end{pmatrix}\begin{pmatrix} [I]_N & [U]_N^T \\ 0 & [I]_N \end{pmatrix} \\ &= \begin{pmatrix} [I]_N & [U]_N^T \\ [U]_N & 0 \end{pmatrix} \quad \text{and}\end{aligned}$$

$$\begin{aligned}[\Lambda]_{2N}^T[\Lambda]_{2N} &= \begin{pmatrix} [I]_N & [U]_N^T \\ 0 & [I]_N \end{pmatrix}\begin{pmatrix} [I]_N & 0 \\ [U]_N & [I]_N \end{pmatrix} \\ &= \begin{pmatrix} 0 & [U]_N^T \\ [U]_N & [I]_N \end{pmatrix}.\end{aligned} \tag{96}$$

That is,

$$[\Lambda]_{2N}[\Lambda]_{2N}^T = [\Lambda]_{2N}^T[\Lambda]_{2N} = \begin{pmatrix} [I]_N & 0 \\ 0 & [I]_N \end{pmatrix}. \tag{97}$$

Now we form the matrix $[\Omega]_{4N}$.

$$[\Omega]_{4N} = \begin{pmatrix} [\Lambda]_{2N} & [\Lambda]_{2N}^T \\ [\Lambda]_{2N}^T & [\Lambda]_{2N} \end{pmatrix}, \tag{98}$$

this satisfies the orthogonality as

$$\begin{aligned}[\Omega]_{4N}[\Omega]_{4N}^T &= \begin{pmatrix} [\Lambda]_{2N} & [\Lambda]_{2N}^T \\ [\Lambda]_{2N}^T & [\Lambda]_{2N} \end{pmatrix}\begin{pmatrix} [\Lambda]_{2N}^T & [\Lambda]_{2N} \\ [\Lambda]_{2N} & [\Lambda]_{2N}^T \end{pmatrix} \\ &= \begin{pmatrix} [\Lambda]_{2N}[\Lambda]_{2N}^T + [\Lambda]_{2N}^T[\Lambda]_{2N} & [\Lambda]_{2N}^2 + \left[\Lambda^T\right]_{2N}^2 \\ \left[\Lambda^T\right]_{2N}^2 + [\Lambda]_{2N}^2 & [\Lambda]_{2N}^T[\Lambda]_{2N} + [\Lambda]_{2N}[\Lambda]_{2N}^T \end{pmatrix} \\ &= \begin{pmatrix} [I]_N & 0 \\ 0 & [I]_N \end{pmatrix}.\end{aligned} \tag{99}$$

Therefore, $[\Omega]$ is an orthogonal of size 4 *N*. If $[U]$ is an orthogonal of size *N* then

$$[\Omega]_{4N} = \begin{pmatrix} [I]_N & 0 & [I]_N & [U]_N^T \\ [U]_N & [I]_N & 0 & [I]_N \\ [I]_N & [U]_N^T & [I]_N & 0 \\ 0 & [I]_N & [U]_N & [I]_N \end{pmatrix}. \quad (100)$$

Competing interests
The authors do not have competing interests.

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Author details
[1]Institute of Information & Communication, Chonbuk National University, Jeonju 561-756, South Korea. [2]Mitsubishi Electric Research Laboratories, Cambridge, MA 02139, USA.

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