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Performance analysis of α - β - γ tracking filters using position and velocity measurements

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Abstract

This paper examines the performance of two position-velocity-measured (PVM) α - β - γ tracking filters. The first estimates the target acceleration using the measured velocity, and the second, which is proposed for the first time in this paper, estimates acceleration using the measured position. To quantify the performance of these PVM α - β - γ filters, we analytically derive steady-state errors that assume that the target is moving with constant acceleration or jerk. With these performance indices, the optimal gains of the PVM α - β - γ filters are determined using a minimum-variance filter criterion. The performance of each filter under these optimal gains is then analyzed and compared. Numerical analyses clarify the performance of the PVM α - β - γ filters and verify that their accuracy is better than that of the general position-only-measured α - β - γ filter, even when the variance in velocity measurement noise is comparatively large. We identify the conditions under which the proposed PVM α - β - γ filter outperforms the general α - β - γ filter for different ratios of noise variance in the velocity and position measurements. Finally, numerical simulations verify the effectiveness of the PVM α - β - γ filters for a realistic maneuvering target.

Keywords: α - β - γ filter; Moving target tracking; Position and velocity measurements; Steady-state error; Optimal gains; Minimum variance filter criterion

Introduction

Remote monitoring systems embedded in robots and vehicles require the capability to accurately track moving objects. Tracking filters, such as Kalman filters, extended Kalman filters (EKFs), and particle filters, are commonly used for this purpose [1-5]. These can accurately track movement based on adaptive filtering, which minimizes the error in the predicted position based on dynamical and measurement models. However, these techniques have a relatively heavy computational load, and in some cases their use is impractical. Moreover, their design is conducted empirically, because it is difficult to evaluate the validity of the design parameters (i.e., the process noise) [6,7].

One effective approach that does not suffer from these problems is known as an α - β - γ filter. These are simple tracking filters that assume constant acceleration during the sampling interval [8]. Because of their small computational load, they have been employed in various tracking

systems [9-12]. Moreover, there are only three design parameters (the α , β , and γ gains), from which the performance indices can be analytically calculated. Consequently, it is simpler to design an appropriate α - β - γ filter than to construct other tracking filters (e.g., the Kalman filter). Many researchers have studied the analytical performance and design methodology of optimal gains in the α - β - γ filter by assuming simple and practical motion models [8,13-17]. Based on these fundamental studies, recent work has investigated effective gain-setting algorithms for various maneuvering targets [18,19]. Simple gain-setting algorithms have enabled the effectiveness of α - β - γ filters to be verified in various real-world applications, such as motor position control [9] and human fall detection [10].

Traditionally, tracking filter techniques have been applied to radar, sonar, and global positioning systems that measure position only [6]. However, various sensing systems that can accurately measure velocity have recently been developed thanks to technical advances in various sensors and sensor networks, such as the micro-Doppler radar network [20,21]. Consequently, the application of tracking filters to such sensing systems has become an

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important area of research [22-25]. We can expect measured velocities to improve the accuracy of tracking compared with trackers that use position measurements alone. However, when the reliability of the velocity measurements is low, the tracking accuracy may deteriorate. Thus, the relationship between tracking accuracy and measurement noise is very important for the implementation of α - β - γ filters using both position and velocity measurements. Although position-velocity-measured (PVM) tracking filters have been investigated [23-28], the number of such studies is quite small compared with those on general tracking filters that measure only position. Additionally, most studies on PVM tracking filters use Kalman or particle filters. Several applications of PVM α - β - γ filters have been reported [29,30], but these studies do not investigate the filters' theoretical performance, meaning the tracking system parameters are designed empirically. Thus, their analytical properties have not been adequately investigated.

This paper analyzes PVM α - β - γ filters and compares their performance with that of a general α - β - γ filter. For a fair comparison with the general α - β - γ filter, the number of filter gains is fixed to three. As a result, two PVM α - β - γ filters are considered, one of which is being proposed for the first time in this paper. We analytically derive filter performance indices for PVM α - β - γ filters. The derived performance indices are then calculated using the gains determined by a minimum-variance (MV) filter criterion [31], which is the optimal gain design for the general α - β - γ filter. A performance evaluation using numerical analyses and simulations verifies the relationships between measurement noise, filter gains, and filter performance. Moreover, we show that the accuracy of the proposed PVM α - β - γ filter is better than that of the general α - β - γ filter, even when the error in velocity measurements is relatively large.

General α - β - γ filter using position-only measurements

In this section, we summarize the definition and performance of the general α - β - γ filter, which uses position measurements alone. We also review some design methods for filter gains.

The α - β - γ filter predicts the position, velocity, and acceleration of a moving target based on a constant acceleration model using three filter gains [8,13]. This filter iterates prediction and smoothing processes. The prediction process is expressed by the following equations:

$$x_{pk} = x_{sk-1} + Tv_{sk-1} + (T^2/2) a_{sk-1}, \quad (1)$$

$$v_{pk} = v_{sk-1} + Ta_{sk-1}, \quad (2)$$

$$a_{pk} = a_{sk-1}, \quad (3)$$

where x_{sk} is the smoothed target position at time kT , T is the sampling interval, x_{pk} is the predicted target position, v_{sk} is the smoothed target velocity, v_{pk} is the predicted target velocity, a_{sk} is the smoothed target acceleration, and a_{pk} is the predicted target acceleration. The smoothing process is expressed as follows:

$$x_{sk} = x_{pk} + \alpha(x_{ok} - x_{pk}), \quad (4)$$

$$v_{sk} = v_{pk} + (\beta/T)(x_{ok} - x_{pk}), \quad (5)$$

$$a_{sk} = a_{pk} + (\gamma/T^2)(x_{ok} - x_{pk}), \quad (6)$$

where x_{ok} is the measured target position, and α , β , and γ are filter gains. The definition of the α - β - γ filter does not include process noise [8,31].

Filter performance indices

To evaluate the tracking performance of the α - β - γ filters, the two steady-state error performance indices can be derived from (1) to (6) [6,8,13,14]. These indices are more effective in evaluating the steady-state tracking accuracy than the error covariance matrix in the Kalman filter equation, which is the usual performance indicator for tracking filters. This is because the error covariance matrix overrates the variance in the errors that is caused by measurement noise, as verified by Ekstrand (see Section 9.8 of [6]). In addition, the relationship between basic properties such as the filter bandwidth and the error covariance matrix is not sufficiently clarified [6,7]. Thus, the indices that are explained in the following subsections are useful when designing α - β - γ filters.

Steady-state error for a target under constant acceleration (smoothing performance index)

An important function of the tracking filter is the reduction of random errors caused by measurement noise. One index of this performance is the steady-state error of a target under constant acceleration considering sensor noise. We assume that x_{ok} contains noise with variance B_x , and that the target moves with constant acceleration. The variance of the predicted target position in the steady-state is calculated using B_x and filter gains as [8,13]:

$$\begin{aligned} \sigma_p^2 &= E \left[(x_{pk} - x_{tk})^2 \right] \\ &= \frac{8\beta^2 + \alpha(4 - 2\alpha - \beta)(2\alpha\beta - \gamma(2 - \alpha))}{(2 - \alpha)(4 - 2\alpha - \beta)(2\alpha\beta - \gamma(2 - \alpha))} B_x, \quad (7) \end{aligned}$$

where x_{tk} is the true target position and $E[\]$ indicates the mean. Note that the mean error $E[x_{pk} - x_{tk}]$ is zero, because the assumed target motion is the same as the motion model of the α - β - γ filter (constant acceleration target). We call σ_p^2 the smoothing performance index.

Steady-state error for a target with constant jerk (tracking performance index)

The tracking filter is required to track complicated motion including jerks. In the α - β - γ filter, steady-state bias error occurs when tracking a target moving with constant jerk, because the filter is based on a constant acceleration model. This error is an index of the tracking performance. When $x_{ok} = J(kT)^3/6$ (J is the constant jerk) and the measurement errors are not considered, the steady-state predicted error is expressed as [14]:

$$e_{fin} = \lim_{k \rightarrow \infty} (x_{ok} - x_{pk}) = JT^3/\gamma. \tag{8}$$

We call e_{fin} the tracking performance index. The smaller these tracking/smoothing performance indices, the better the tracking filter. However, there is a trade-off between e_{fin} and σ_p^2 , and this is a very important consideration in the design of tracking filters [6].

Here, we discuss the design of the α - β - γ filter and compare it with the Kalman filter design. Table 1 summarizes the design parameters, performance indices, and gain calculation method for these filters [7,31,32]. (Note that details of the gain design methods of the α - β - γ filter are explained in the next subsection.) As shown in (7) and (8), the above indices can be directly calculated using the filter gains that we have designed. In contrast, for the Kalman filter, we must design the covariance matrix of the process noise. However, the relationship between this and the performance index (error covariance matrix) has not been rigorously established [7,32]. Moreover, the error covariance matrix gives a misleading evaluation of tracking filter performance, as mentioned earlier in this subsection. Therefore, the design of appropriate process noise is conducted empirically and/or by Monte Carlo simulations (see Section 6 of [6]). Consequently, it is simpler to design an appropriate α - β - γ filter than to construct a Kalman filter or EKF [7,23,26-28].

Gain design methods

Various approaches can be used to determine appropriate gains for the α - β - γ filter. The main approach is to derive gains from the Kalman filter equations, because the α - β - γ filter can be considered as the steady-state Kalman filter [15-17]. However, it is difficult to select appropriate process noise for the motion model, for the same reason as the difficulties in designing a Kalman filter mentioned

in the previous subsection. In addition, the performance of an α - β - γ filter derived from the Kalman filter is not optimal when evaluated using the performance indices expressed in (7) and (8) [31].

To avoid these problems, the MV filter criterion has been proposed [14,31]. This criterion determines the gains by minimizing the smoothing performance index σ_p^2 under the condition that the tracking performance index e_{fin} is constant [31]. As shown in (8), the tracking performance index depends only on γ . Thus, for the general α - β - γ filter, the optimal gains with the MV filter criterion are determined by:

$$\begin{aligned} &\arg \min_{\alpha, \beta} \sigma_p^2 \\ &\text{sub. to } \gamma = \text{const.} \end{aligned} \tag{9}$$

As shown in this equation, the MV filter criterion does not require the process noise of the motion model, unlike the Kalman filter-based approach [15-17]. In [14], it was reported that the performance (evaluated using the tracking/smoothing performance indices of (7) and (8)) is better than that of other α - β - γ filters derived from the Kalman filter equations. Thus, this paper uses the MV filter criterion to determine the optimal gains.

PVM α - β - γ filters

As described in the ‘Introduction’ section, the performance of PVM tracking filters has not been fully investigated. Hence, we focus on PVM α - β - γ filters. In this section, we derive the smoothing and tracking performance indices (σ_p^2 and e_{fin}) for PVM α - β - γ filters. As mentioned above, we ensure a fair comparison with the general α - β - γ filter by fixing the number of gains to three. We can define two types of PVM α - β - γ filter. The first has been used in several tracking systems that measure both position and velocity [29,30]. However, its performance indices have not been derived, and thus the gain determination has so far been conducted empirically. The second type is a new PVM α - β - γ filter that is being proposed for the first time in this paper. The aim of this new filter is to achieve accurate tracking, even when the noise in the velocity measurements is comparatively large. The performance indices σ_p^2 and e_{fin} are derived analytically for each PVM α - β - γ filter.

Table 1 Summary of the design and gain calculation of the α - β - γ and Kalman filters

	Design parameter	Performance index	Gain calculation
α - β - γ filter	Gain γ (or $\Gamma_{A-\gamma}$ of (29))	Tracking/smoothing performance indices	Based on relationship between gains derived using the Kalman filter equation or MV filter criterion
Kalman filter	Covariance matrix of process noise	Covariance matrix of errors	Adaptively calculated with Riccati equation

Acceleration smoothed by measured velocity (A-V)-type PVM α - β - γ filter

Using the measured velocity v_{ok} , several researchers have used a PVM α - β - γ filter with the smoothing process [29,30]:

$$x_{sk} = x_{pk} + \alpha(x_{ok} - x_{pk}), \quad (10)$$

$$v_{sk} = v_{pk} + \beta(v_{ok} - v_{pk}), \quad (11)$$

$$a_{sk} = a_{pk} + (\gamma/T)(v_{ok} - v_{pk}), \quad (12)$$

and a prediction process that is the same as that in the general α - β - γ filter (expressed in (1) to (3)). Compared with the general α - β - γ filter, the second terms of (5) and (6) have been changed to use the measured velocity. Equation (11) shows that the smoothed velocity can be estimated using the measured velocity. This is the natural expansion of the general α - β - γ filter considering the velocity measurements. Additionally, as shown in (12), the smoothed acceleration is also estimated using the measured velocity. We call this PVM α - β - γ filter the acceleration smoothed by measured velocity (A-V) filter.

The performance indices of the A-V filter are derived from (1) to (3) and (10) to (12). For simplicity, we assume that the noise in the position and velocity measurements is uncorrelated. The smoothing performance index is then derived as

$$\sigma_{p,A-V}^2 = \frac{\alpha}{2-\alpha} B_x + \frac{f_1(\alpha, \beta, \gamma)}{f_2(\alpha, \beta, \gamma)} T^2 B_v, \quad (13)$$

where B_v is the variance of the noise in v_{ok} , and

$$\begin{aligned} f_1(\alpha, \beta, \gamma) &= \alpha^2(\beta - 1)(4\beta^2 - 2\beta\gamma - \gamma^2 + 4\gamma) \\ &+ \alpha(6\beta^2\gamma - 4\beta^3 + 8\beta^2 + 3\beta\gamma^2 - 16\beta\gamma - 2\gamma^2 + 8\gamma) \\ &- 4\beta^2\gamma + 2\beta\gamma(4 - \gamma), \end{aligned} \quad (14)$$

$$\begin{aligned} f_2(\alpha, \beta, \gamma) &= 2\alpha\beta(2 - \alpha)(4 - 2\beta - \gamma) \\ &\times (\alpha^2 + \alpha\beta + \gamma - \alpha^2\beta - \alpha\beta). \end{aligned} \quad (15)$$

The derivation of (13) is given in the Appendix. Then, the tracking performance index can be derived as

$$e_{fin,A-V} = \frac{12 - 6\beta - \gamma}{12\alpha\gamma} JT^3. \quad (16)$$

Again, details of the derivation are given in the Appendix.

Acceleration smoothed by measured position (A-P)-type PVM α - β - γ filter

As it uses the measured velocity, we expect the A-V filter to realize better tracking accuracy than the general α - β - γ filter. However, the performance of the A-V filter deteriorates when the variance B_v is large. To reduce

this deterioration, we consider another PVM α - β - γ filter whose smoothing process is expressed as follows:

$$x_{sk} = x_{pk} + \alpha(x_{ok} - x_{pk}), \quad (17)$$

$$v_{sk} = v_{pk} + \beta(v_{ok} - v_{pk}), \quad (18)$$

$$a_{sk} = a_{pk} + (\gamma/T^2)(x_{ok} - x_{pk}), \quad (19)$$

and whose prediction process is the same as in the general α - β - γ filter (i.e., (1) to (3)). The difference from the A-V filter is that the smoothed acceleration is estimated using the measured position, i.e., (6) in the general α - β - γ filter. We call this new PVM α - β - γ filter the acceleration smoothed by measured position (A-P) filter. It appears that the performance of the A-P filter is better than that of the A-V filter when B_v is relatively large. In contrast, the A-V filter appears to outperform the A-P filter when B_v is relatively small. Moreover, when B_v is relatively large, it is unclear whether the performance of the A-P filter or the general α - β - γ filter is better. In the next section, these cases are investigated and clarified with theoretical analyses.

We can derive the smoothing performance index for the A-P filter as:

$$\sigma_{p,A-P}^2 = \frac{g_1(\alpha, \beta, \gamma)B_x + g_2(\alpha, \beta)T^2B_v}{g_3(\alpha, \beta, \gamma)}, \quad (20)$$

where

$$\begin{aligned} g_1(\alpha, \beta, \gamma) &= 8\alpha^3\beta(2 - \beta)(\beta - 1) \\ &+ 2\alpha^2(\beta^3\gamma + 4\beta^3 - 3\beta^2\gamma - 8\beta^2 + 6\beta\gamma - 4\gamma) \\ &+ \alpha\gamma(2\beta^3 + \beta^2\gamma + 4\beta^2 - \beta\gamma - 24\beta + 16) \\ &- 4\beta\gamma(2 - \beta)^2, \end{aligned} \quad (21)$$

$$g_2(\alpha, \beta) = 8\beta^2(\alpha + \beta - \alpha\beta - 2), \quad (22)$$

$$\begin{aligned} g_3(\alpha, \beta, \gamma) &= (16 - 8\beta - \beta\gamma - 8\alpha + 4\alpha\beta) \\ &\cdot (2\alpha^2\beta^2 - 2\alpha^2\beta - 2\alpha\beta^2 + \alpha\beta\gamma - \alpha\gamma \\ &- 2\beta\gamma + 2\gamma), \end{aligned} \quad (23)$$

and the tracking performance index is

$$e_{fin,A-P} = JT^3/\gamma. \quad (24)$$

Note that the tracking performance index is the same as in the general α - β - γ filter, as shown in (8). The derivation of these performance indices is given in the Appendix.

Performance analysis and comparison

In this section, we compare the performance of the A-V filter, A-P filter, and general MV (GMV) α - β - γ filter (which measures position only). The optimal gains are calculated with the MV filter criterion [31], and performance is analyzed using the derived tracking/smoothing performance indices and the calculated gains. The relationship

between measurement noise (B_x and B_v) and filter performance is clarified for various gain settings.

Optimal gain calculation with MV filter criterion

First, we calculate the optimal gains of the A-V filter. Under the MV filter criterion, we assume that the tracking performance index is constant. With (16), the tracking performance index depends on

$$C_{A-V} = \frac{12 - 6\beta - \gamma}{12\alpha\gamma}. \tag{25}$$

Thus, C_{A-V} is constant in the MV filter criterion. Solving this for γ , we obtain

$$\gamma = \frac{6(2 - \beta)}{12\alpha C_{A-V} + 1}. \tag{26}$$

Substituting (26) into (13) gives the smoothing performance index $\sigma_{p,A-V}^2(\alpha, \beta, C_{A-V})$, which is used to calculate the optimal gains for constant C_{A-V} . Then, we determine the optimal α and β for each C_{A-V} by:

$$\begin{aligned} &\arg \min_{\alpha, \beta} \sigma_{p,A-V}^2(\alpha, \beta, C_{A-V}) \\ &\text{sub. to } C_{A-V} = \text{const.} \end{aligned} \tag{27}$$

Next, we consider the optimal gain calculation of the A-P filter. As shown in (24), the tracking performance index of the A-P filter depends only on γ . Consequently, γ is constant when $e_{\text{fin},A-P}$ is constant. Thus, we determine the optimal α and β for each γ by:

$$\begin{aligned} &\arg \min_{\alpha, \beta} \sigma_{p,A-P}^2 \\ &\text{sub. to } \gamma = \text{const.} \end{aligned} \tag{28}$$

We now give the gain calculation results using (27) and (28) and compare these with the gains from the GMV filter. First, to simplify the discussion, we define the following two parameters.

- The reciprocal of C_{A-V} is defined as

$$\Gamma_{A-V} = 1/C_{A-V}. \tag{29}$$

With (8), (16), and (24), Γ_{A-V} corresponds to γ in the A-P and GMV filters.

- The ratio of the two variances of measurement noise is defined as

$$R_v = T^2 B_v / B_x. \tag{30}$$

The smoothing performance of the PVM α - β - γ filters depends on this ratio, as we can see from (13) and (20). The relationship between R_v and the performance indices is very important for the design of tracking filters that use both the measured position and velocity.

Figure 1 shows the gain calculations for the PVM and GMV filters with $R_v = 1/2$ for each value of γ or Γ_{A-V} . Here, we have used the gradient descent technique to minimize (27) and (28); the complexity (using big O notation) of this operation is $O(n^2)$ [33]. The mean calculation time to determine each (α, β, γ) is 56.3 s using an Intel CORE i7-4600U CPU@2.10 GHz 2.70 GHz. This time is acceptable, because the gain calculation is conducted in the filter design process before its application in a tracking system. As shown in Figure 1, the value of β in the A-V filter is relatively large compared with that in the GMV filter. This is because β is the gain for velocity smoothing, and the measurement accuracy of the velocity is better than that of the position. For the same reason, the value of β (α) is larger (smaller) in the A-P filter than in the other filters. These examples indicate that the gains of the PVM filters depend on the relationship between the accuracy of the position and velocity measurements.

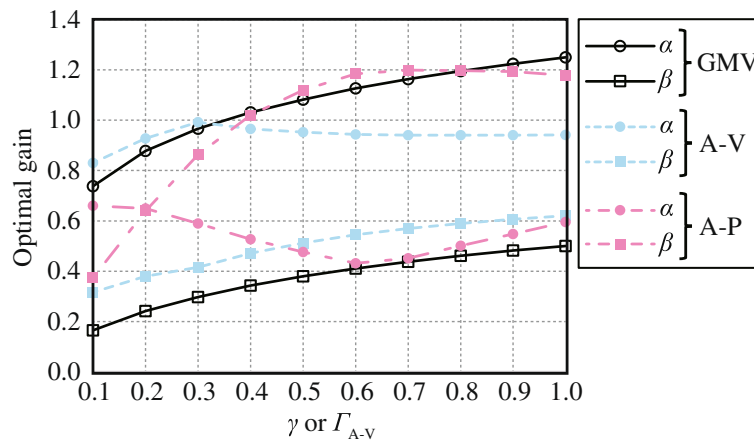


Figure 1 Calculation results for the optimal gains when $R_v = 1/2$.

Analysis results and discussion

Using the calculated optimal gains, we conduct performance analyses of the A-V, A-P, and GMV filters. The smoothing performance indices of these filters are calculated using (7), (13), and (20) under the assumption that the tracking performance indices are constant (i.e., Γ_{A-V} is constant for the A-V filter and γ is constant for the other filters). We assume that the sampling interval T and the variance of the measured position error B_x are normalized to 1.

Figure 2 shows the smoothing performance indices as a function of γ or Γ_{A-V} for $R_v = 1/2$ and 7. For relatively small R_v , shown in Figure 2a, the PVM α - β - γ filters outperform the GMV filter, especially for large values of γ or Γ_{A-V} . In this case, the A-V filter realizes the best performance. This is because accurately measured velocities improve the performance of both the smoothing and tracking. Moreover, for larger values of R_v , shown in Figure 2b, the performance deterioration in the proposed A-P filter is small compared with that in the A-V filter. This is because the smoothed acceleration in the A-P filter is calculated using the measured position. The smoothing performance of the A-P filter is better than that of the GMV filter when $\gamma \geq 0.6$ for $R_v = 7$. This result implies that the proposed A-P filter can realize better performance than the GMV filter, even when the noise in the velocity measurements is large. Figure 3 shows the smoothing performance indices as a function of R_v for γ and Γ_{A-V} values of 0.9. As shown in this figure, for $R_v = 10$ (i.e., the noise variance in the velocity measurements is ten times as large as that in the position measurements), the

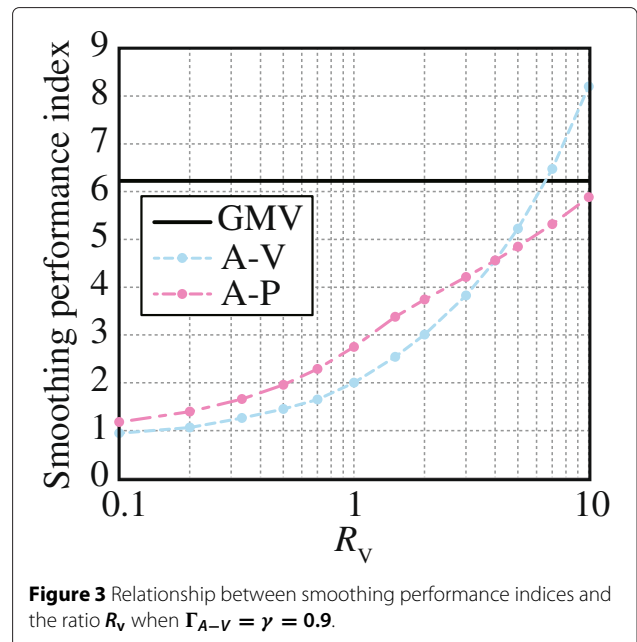


Figure 3 Relationship between smoothing performance indices and the ratio R_v when $\Gamma_{A-V} = \gamma = 0.9$.

A-P filter achieves better smoothing performance than the GMV filter. In contrast, the performance of the A-V filter deteriorates when R_v is comparatively large.

Table 2 summarizes the properties of the α - β - γ filters considered in this paper. This table indicates that the A-V filter realizes accurate tracking for small R_v , whereas the proposed A-P filter realizes better accuracy than the other filters for relatively large R_v . Additionally, when

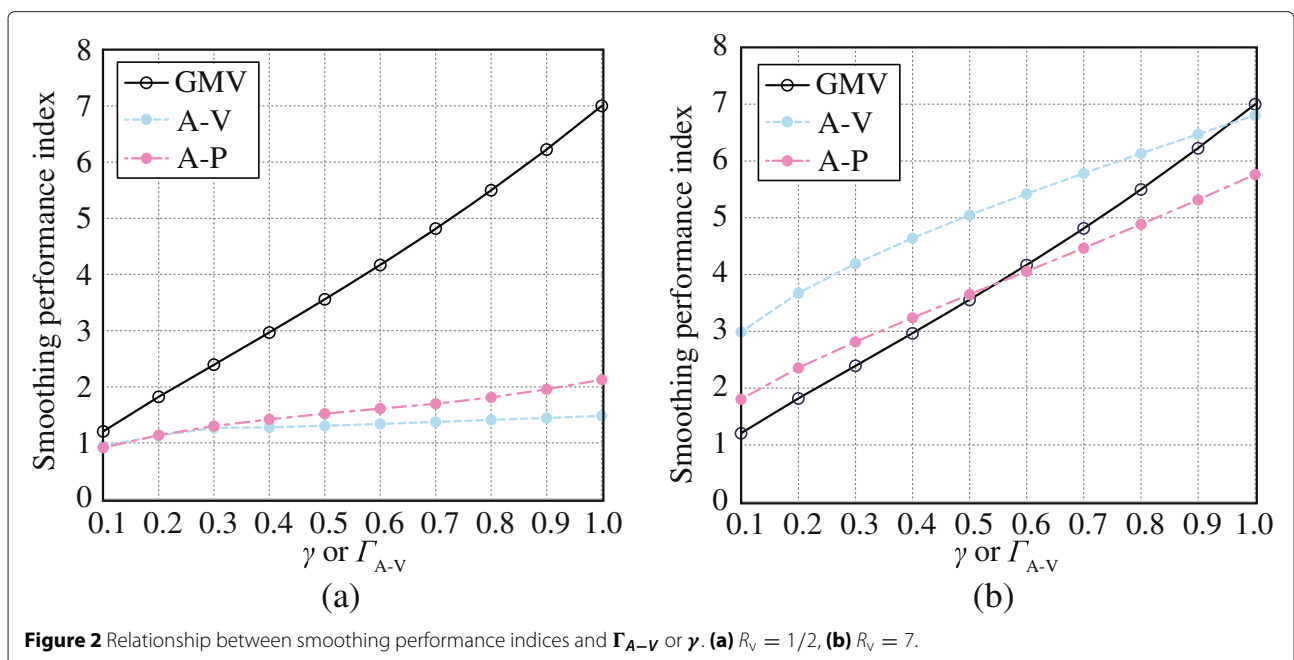


Figure 2 Relationship between smoothing performance indices and Γ_{A-V} or γ . (a) $R_v = 1/2$, (b) $R_v = 7$.

Table 2 Summary of the properties of the α - β - γ filters considered in this paper

	GMV filter	A-V filter	A-P filter ^a
Measurement parameter	Position	Position and velocity	
Prediction process		(1) to (3)	
Smoothing process	(4) to (6)	(10) to (12)	(17) to (19) ^a
Smoothing performance index	(7)	(13) ^a	(20) ^a
Tracking performance index	(8)	(16) ^a	(8) ^a
Suitable when R_v is	very large ^a	small ^a	large ^a

^aIndicates novel results in this paper.

R_v becomes large, the GMV filter realizes the best performance, which suggests that we should not use the measured velocity in this case.

Cramér–Rao bound evaluation

This section calculates the fundamental performance limitation of the PVM and conventional tracking problems using a Cramér–Rao bound (CRB) evaluation. Moreover, we evaluate the tracking accuracy of the PVM filters using Monte Carlo simulations and compare this with the CRBs.

The CRBs in the position estimation are calculated by the Riccati-like recursion used in [34]. The CRB is the lower bound of the covariance of the state estimation, which is expressed as

$$E[(\hat{\mathbf{x}}_k - \mathbf{x}_{tk})(\hat{\mathbf{x}}_k - \mathbf{x}_{tk})^T] \geq \mathbf{J}_k^{-1} = \mathbf{P}_k, \tag{31}$$

where $\hat{\mathbf{x}}_k$ is the target state estimate based on all measurements collected up to and including time kT , the target state is composed of the position, velocity, and acceleration in the form $(x_k, v_k, a_k)^T$, \mathbf{J}_k is the filtering information

matrix defined in [35], and \mathbf{P}_k is the CRB. When we do not use the process noise, the recursive formula for \mathbf{J}_k can be expressed as [36]:

$$\mathbf{J}_{k+1} = (\mathbf{F}^{-1})^T \mathbf{J}_k \mathbf{F}^{-1} + \mathbf{H}^T \mathbf{R}^{-1} \mathbf{H}, \tag{32}$$

where \mathbf{F} is the state transition matrix, \mathbf{H} is the observation matrix, and \mathbf{R} is the covariance matrix of measurement noise. In the PVM tracking problem, these are expressed as [26]:

$$\mathbf{F} = \begin{pmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{pmatrix}, \tag{33}$$

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{34}$$

$$\mathbf{R} = \begin{pmatrix} B_x & 0 \\ 0 & B_v \end{pmatrix}. \tag{35}$$

A detailed explanation is provided in [34].

First, we calculate and compare the CRBs of the general position-only-measured and PVM tracking problems. In the position-only-measured tracking problem, \mathbf{H} and \mathbf{R} are expressed as:

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \tag{36}$$

$$\mathbf{R} = \begin{pmatrix} B_x \end{pmatrix}, \tag{37}$$

and \mathbf{F} is same as (33). We set $B_x = 1$ and $T = 1$. Figure 4a shows the calculated CRBs of the position estimation. As shown in this figure, the performance limitation of the PVM tracking problem is less than that of the position-only-measured tracking problem. The limitation of the PVM tracking problem with $R_v = 0.1$ is small even for relatively large k . When $R_v = 10$, the CRB is almost the same as that for the position-only-measured tracking problem

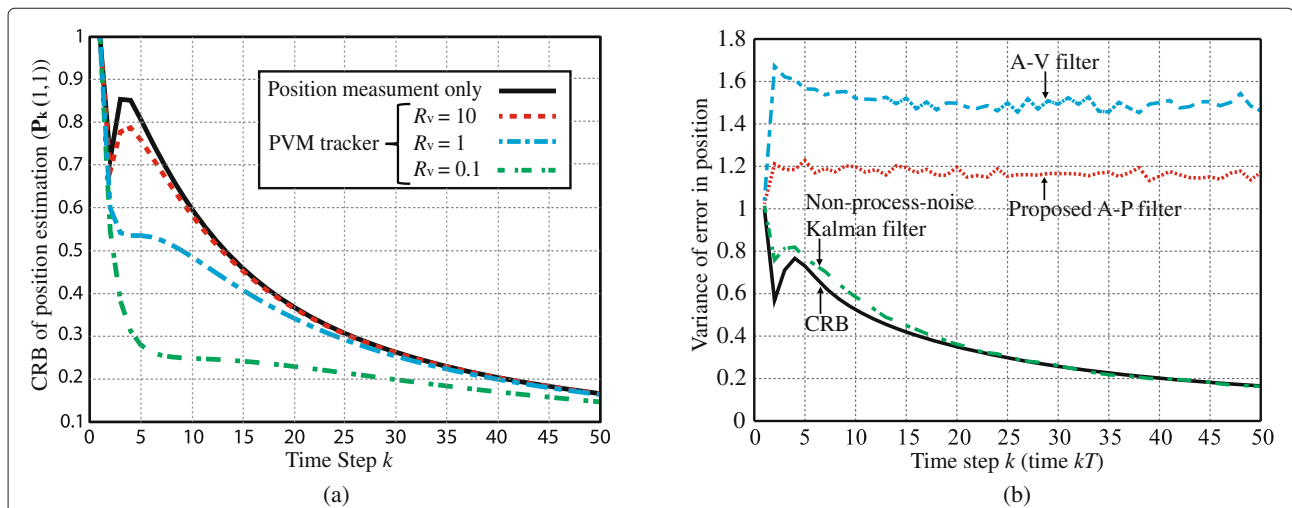


Figure 4 CRB evaluation results. **(a)** CRBs of PVM and position-only-measured tracking problems, **(b)** Monte Carlo simulation results of the PVM tracking filters when $R_v = 10$.

at approximately $k > 10$. However, no deterioration in the CRBs has occurred.

Next, we compare the CRBs with the performance of the PVM α - β - γ filters calculated using Monte Carlo simulations. We set the number of Monte Carlo simulations to 10,000, the initial state to $(0, 0.5, 0.005)^T$, $B_x = 1$, and $T = 1$. For reference, the simulation results for the non-process-noise Kalman filter [32] are also presented. Figure 4b shows the CRB and the Monte Carlo simulation results for the A-V and A-P filters for γ (Γ_{A-V}) = 0.1 and the non-process-noise Kalman filters where $R_v = 10$. As shown in this figure, the accuracy of the PVM α - β - γ filters is worse than that of the non-process-noise Kalman filter whose performance is close to the CRB. This is because the α - β - γ filter uses fixed gains, unlike the Kalman filter. However, the proposed A-P filter produces a smaller difference between the CRBs and error variances than the A-V filter. Additionally, the computational load of the proposed filter is smaller than that of the Kalman filter, as we shall discuss later.

Simulation assuming radar tracking of a maneuvering target

Finally, we use numerical simulations to investigate the performance of each filter for a realistic maneuvering target. In this subsection, we simulate the Doppler radar tracking [20,21,29] of a maneuvering target and compare the tracking errors given by the three filters considered in this paper and an EKF [1,2]. Figure 5 shows the simulation scenario. Figure 5a,b shows the true target motion and the radar position, respectively. Two-dimensional (2D) tracking of the point target is assumed, and the received radar signals are calculated using ray-tracing, as in [29]. We

assume there are two Doppler radars located at $(x, y) = (0, 0)$ and $(0.5, 0)$. The sampling interval T is 1 ms, and the transmitting signal is an ultrawide-band pulse with a center frequency of 26.4 GHz and bandwidth of 2 GHz. The radars measure the position using ranging results and the velocity using the Doppler shift [29]. White Gaussian noise is added to the ranging and Doppler shift estimations to control R_v . Figure 5c shows the true target position at each time.

We now describe the composition of the tracking filters. For 2D tracking, the α - β - γ filter is composed as follows along each axis:

$$\begin{pmatrix} x_{sk} \\ v_{xsk} \\ a_{xsk} \\ y_{sk} \\ v_{ysk} \\ a_{ysk} \end{pmatrix} = \begin{pmatrix} x_{pk} \\ v_{xpk} \\ a_{xpk} \\ y_{pk} \\ v_{ypk} \\ a_{ypk} \end{pmatrix} + \mathbf{K} \begin{pmatrix} x_{ok} - x_{pk} \\ v_{xok} - v_{xpk} \\ y_{ok} - y_{pk} \\ v_{yok} - v_{ypk} \end{pmatrix}, \quad (38)$$

$$\begin{pmatrix} x_{pk} \\ v_{xpk} \\ a_{xpk} \\ y_{pk} \\ v_{ypk} \\ a_{ypk} \end{pmatrix} = \begin{pmatrix} 1 & T & T^2/2 & 0 & 0 & 0 \\ 0 & 1 & T & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & T & T^2/2 \\ 0 & 0 & 0 & 0 & 1 & T \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_{sk-1} \\ v_{xsk-1} \\ a_{xsk-1} \\ y_{sk-1} \\ v_{ysk-1} \\ a_{ysk-1} \end{pmatrix}, \quad (39)$$

where \mathbf{K} is the gain matrix, x_k , v_{xk} , and a_{xk} denote position, velocity, and acceleration along the x -axis, y_k , v_{yk} , and a_{yk} denote position, velocity, and acceleration along the y -axis, and subscripts 's', 'p', and 'o' denote 'smoothed', 'predicted', and 'observed (measured)', respectively. In the

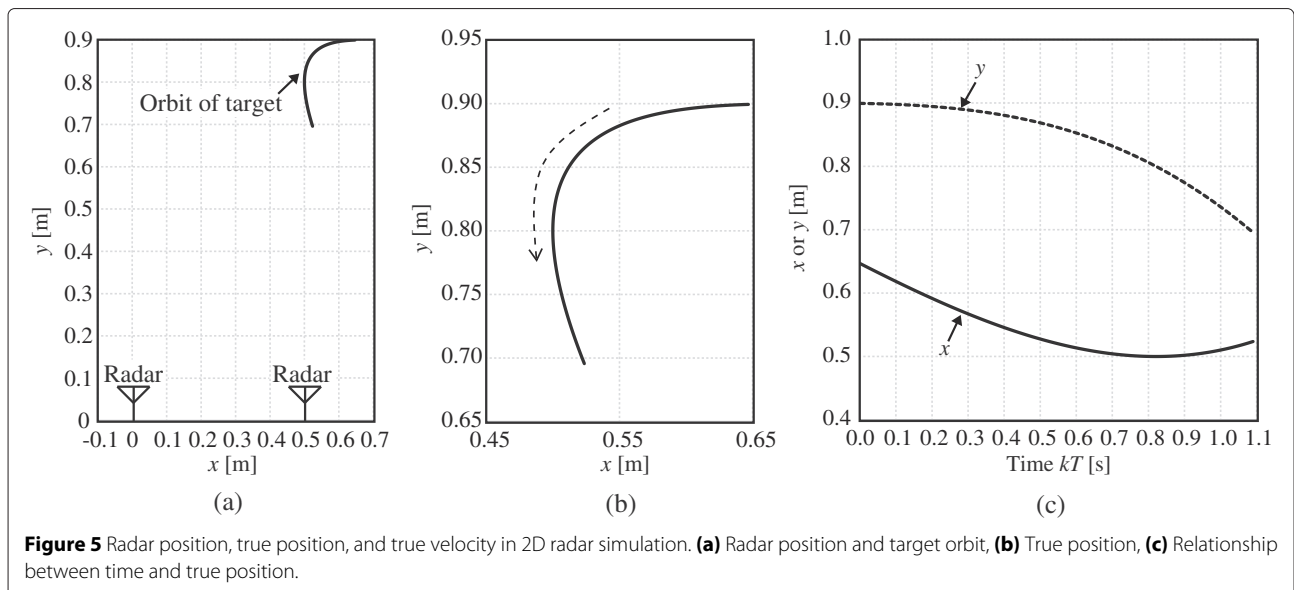


Figure 5 Radar position, true position, and true velocity in 2D radar simulation. **(a)** Radar position and target orbit, **(b)** True position, **(c)** Relationship between time and true position.

A-V filter tracking, \mathbf{K} is expressed as:

$$\mathbf{K}_{\text{A-V}} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & \gamma/T & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & 0 & \gamma/T \end{pmatrix}. \quad (40)$$

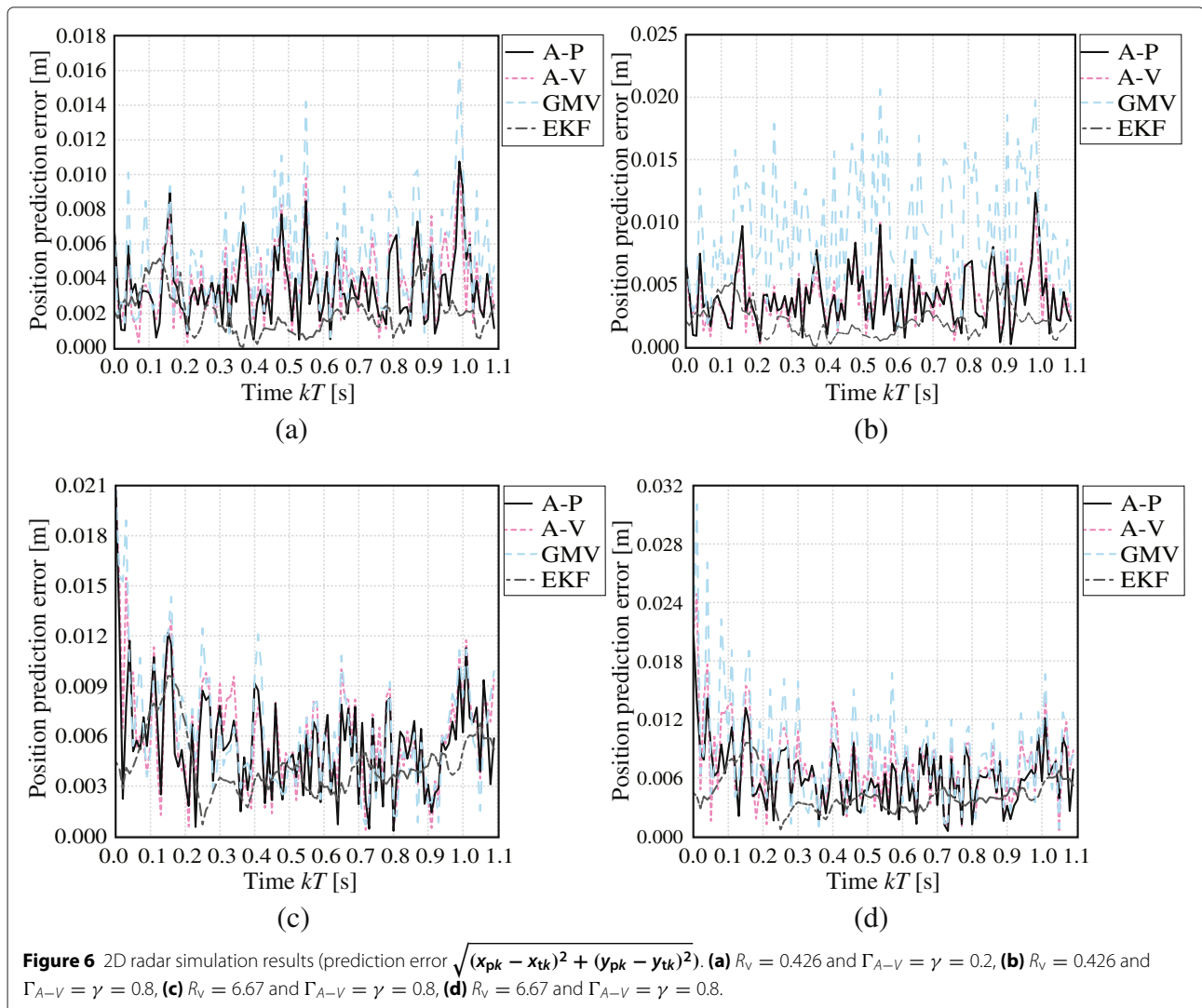
In the A-P filter tracking, \mathbf{K} is expressed as:

$$\mathbf{K}_{\text{A-P}} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ \gamma/T^2 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & \gamma/T^2 & 0 \end{pmatrix}. \quad (41)$$

Additionally, in the EKF tracking, \mathbf{K} is the Kalman gain matrix calculated by the Kalman filter equations, and a

nonlinear measurement model is used [1]. The EKF considers the correlation between the $x - y$ axes, unlike the α - β - γ filters. The process noise is taken to be the zero-mean random-acceleration noise given in [2], and we empirically set this to realize errors that are as small as possible.

Figure 6 shows the results of the 2D radar tracking. Figure 6a,b shows the position prediction error $\sqrt{(x_{pk} - x_{tk})^2 + (y_{pk} - y_{tk})^2}$ (where y_{tk} is the true y) for γ and $\Gamma_{\text{A-V}}$ values of 0.2 and 0.8 with the GMV and PVM α - β - γ filters when the mean R_v is 0.426. The gains of the GMV and PVM α - β - γ filters are calculated according to this mean R_v . As for the previous analyses and simulations, the accuracy of the PVM α - β - γ filters is somewhat better than that of the GMV filters when the gains are relatively large. In both cases, the EKF realizes the best performance. This is because it considers the correlated noise of the axes and has four times as many gains as



the α - β - γ filters. Moreover, these gains change adaptively. However, the PVM α - β - γ filters realize relatively good accuracy with fixed gains and a small computational load. Table 3 shows the required number of addition, multiplication, and inversion operations of matrices for each time step k for the EKF and PVM α - β - γ filter. As shown in this table, the computational load of the PVM α - β - γ filter is smaller than that of the EKF.

Next, we present results for when the velocity measurement noise is large compared with the position measurement noise. Figure 6c,d shows the position prediction error for γ and Γ_{A-V} values of 0.2 and 0.8 with the GMV and PVM α - β - γ filters when the mean R_v is 6.67. The EKF realizes the best performance in both cases, as for the previous scenario. When $\gamma = \Gamma_{A-V} = 0.2$, the difference between the three α - β - γ filters is slight, as suggested by our theoretical analysis. When $\gamma = \Gamma_{A-V} = 0.8$, the proposed A-P filter realizes slightly better accuracy than the other two α - β - γ filters. Table 4 lists the mean error of the results shown in Figure 6. From this table, we can see that the mean accuracy of the EKF is better than that of the PVM α - β - γ filters in all cases. However, the accuracy of the PVM α - β - γ filters is sufficiently high. As shown in Figure 6, the positioning errors of the PVM α - β - γ filters are almost smaller than the wavelength of the radar signal (which is approximately 0.0114 m, corresponding to 26.4 GHz). For various remote sensing applications using radar, sonar, and laser, an accuracy of better than the wavelength is often expected. Thus, the above results indicate that the accuracy of the PVM filters is sufficient for various remote sensing applications. In contrast, the errors in the GMV filter are often larger than the wavelength. The mean error of the A-V filter is 0.418 times that of the GMV filter when $R_v = 0.426$ and $\gamma = \Gamma_{A-V} = 0.8$. Moreover, the A-P filter even realizes better accuracy when $R_v = 6.67$. These results indicate that the PVM filters enable accurate tracking with simple calculations and few gains when the velocity measurement noise is relatively small, even for 2D radar tracking applications.

Conclusions

In this paper, we have examined the performance of two PVM α - β - γ filters: the A-V filter and the newly proposed A-P filter. The A-V filter estimates smoothed acceleration using the measured velocity, whereas the proposed A-P filter uses the measured position. We analytically derived the tracking and smoothing performance indices of each

Table 3 Required number of matrix operations for the tracking filtering in each time step

	Addition	Multiplication	Inversion
PVM α - β - γ filter	1	2	0
EKF	5	11	1

Table 4 Mean of the predicted errors in 2D radar simulations (units: mm)

R_v	Γ_{A-V} or γ	A-P filter	A-V filter	GMV filter	EKF
0.426	0.2	3.49	3.82	5.29	2.07
	0.8	3.93	3.91	9.35	2.07
6.67	0.2	5.48	5.91	6.21	4.28
	0.8	6.07	7.09	8.40	4.28

filter. Based on these performance indices, we calculated the optimal gains of the PVM α - β - γ filters with the MV filter criterion. The performance of the A-V and A-P filters was investigated in terms of the calculated gains, and we compared the output with that from the GMV filter. Numerical analyses verified that the A-V filter realizes better accuracy than the GMV filter when the ratio R_v is relatively small. Moreover, the proposed A-P filter achieved better performance when both R_v and the gain γ were comparatively large. The proposed A-P filter achieved the best performance for $R_v = 7$ and $\gamma \geq 0.6$. In particular, even for $R_v = 10$, which means that the variance of noise in the velocity measurements is ten times that in the position measurements, the A-P filter was more accurate than the GMV filter when $\gamma = 0.9$. Finally, numerical simulations verified the effectiveness of the A-V and A-P filters for a realistic 2D radar application. Moreover, the simulation results matched those from numerical analyses using the derived performance indices. Thus, the performance analyses presented in this paper will be useful for the design of actual tracking systems using position and velocity measurements. One limitation of the current study is our assumption that there are three filter gains. The relaxation of this assumption is an important area of future work that will enable the realization of more accurate tracking filters.

Appendix

Derivation of (13)

The true position of a target under constant acceleration is expressed as

$$x_{tk} = x_{tk-1} + Tv_{tk-1} + (T^2/2) a_{tk-1}, \quad (42)$$

where v_t and a_t are the true velocity and acceleration. With (1) and (42), the variance of the predicted position errors is

$$\begin{aligned} \sigma_p^2 &= E[(x_{pk} - x_{tk})^2] \\ &= E[(x_{sk-1} - x_{tk-1})^2] + T^2 E[(v_{sk-1} - v_{tk-1})^2] \\ &\quad + (T^4/4) E[(a_{sk-1} - a_{tk-1})^2] \\ &\quad + 2TE[(x_{sk-1} - x_{tk-1})(v_{sk-1} - v_{tk-1})] \\ &\quad + T^2 E[(x_{sk-1} - x_{tk-1})(a_{sk-1} - a_{tk-1})] \\ &\quad + T^3 E[(v_{sk-1} - v_{tk-1})(a_{sk-1} - a_{tk-1})]. \end{aligned} \quad (43)$$

Because we assume a steady state, the variances and covariances in (43) do not depend on k . Consequently, we can define these variances and covariances as:

$$\sigma_{sx}^2 = E[(x_{sk} - x_{tk})^2] = E[(x_{sk-1} - x_{tk-1})^2], \quad (44)$$

$$\sigma_{sv}^2 = E[(v_{sk} - v_{tk})^2] = E[(v_{sk-1} - v_{tk-1})^2], \quad (45)$$

$$\sigma_{sa}^2 = E[(a_{sk} - a_{tk})^2] = E[(a_{sk-1} - a_{tk-1})^2], \quad (46)$$

$$\sigma_{sxv}^2 = E[(x_{sk} - x_{tk})(v_{sk} - v_{tk})] = E[(x_{sk-1} - x_{tk-1})(v_{sk-1} - v_{tk-1})], \quad (47)$$

$$\sigma_{sxa}^2 = E[(x_{sk} - x_{tk})(a_{sk} - a_{tk})] = E[(x_{sk-1} - x_{tk-1})(a_{sk-1} - a_{tk-1})], \quad (48)$$

$$\sigma_{sva}^2 = E[(v_{sk} - v_{tk})(a_{sk} - a_{tk})] = E[(v_{sk-1} - v_{tk-1})(a_{sk-1} - a_{tk-1})]. \quad (49)$$

Substituting (44) to (49) into (43), we have

$$\sigma_p^2 = \sigma_{sx}^2 + T^2\sigma_{sv}^2 + (T^4/4)\sigma_{sa}^2 + 2T\sigma_{sxv}^2 + T^2\sigma_{sxa}^2 + T^3\sigma_{sva}^2. \quad (50)$$

The variances and covariances in this equation are derived as functions of the filter gains and the variances of measurement noise. With (1) and (10), we have

$$x_{sk} = (1 - \alpha)(x_{sk-1} + Tv_{sk-1} + (T^2/2)a_{sk-1}) + \alpha x_{ok}. \quad (51)$$

We can rewrite (42) as

$$x_{tk} = (1 - \alpha)(x_{tk-1} + Tv_{tk-1} + (T^2/2)a_{tk-1}) + \alpha x_{tk}. \quad (52)$$

Using (51) and (52), the smoothing error is expressed as

$$x_{sk} - x_{tk} = (1 - \alpha)\{x_{sk-1} - x_{tk-1} + T(v_{sk-1} - v_{tk-1}) + (T^2/2)(a_{sk-1} - a_{tk-1})\} + \alpha(x_{ok} - x_{tk}). \quad (53)$$

Thus, the variance of this error is calculated as

$$\begin{aligned} \sigma_{sx}^2 &= E[(x_{sk} - x_{tk})^2] \\ &= (1 - \alpha)^2 \{E[(x_{sk-1} - x_{tk-1})^2] + T^2E[(v_{sk-1} - v_{tk-1})^2] \\ &\quad + (T^4/4)E[(a_{sk-1} - a_{tk-1})^2] \\ &\quad + 2TE[(x_{sk-1} - x_{tk-1})(v_{sk-1} - v_{tk-1})] \\ &\quad + T^2E[(x_{sk-1} - x_{tk-1})(a_{sk-1} - a_{tk-1})] \\ &\quad + T^3E[(v_{sk-1} - v_{tk-1})(a_{sk-1} - a_{tk-1})]\} \\ &\quad + \alpha^2E[(x_{ok} - x_{tk})^2] \\ &\quad + 2\alpha(1 - \alpha)\{E[(x_{sk-1} - x_{tk-1})(x_{ok} - x_{tk})] \\ &\quad + TE[(v_{sk-1} - v_{tk-1})(x_{ok} - x_{tk})] \\ &\quad + (T^2/2)E[(a_{sk-1} - a_{tk-1})(x_{ok} - x_{tk})]\}. \end{aligned} \quad (54)$$

Here,

$$E[(x_{ok} - x_{tk})^2] = B_x. \quad (55)$$

The following relations are satisfied because of the steady-state assumption and because the smoothed parameters are a linear combination of the measured parameters:

$$E[(x_{sk-1} - x_{tk-1})(x_{ok} - x_{tk})] = 0, \quad (56)$$

$$E[(v_{sk-1} - v_{tk-1})(x_{ok} - x_{tk})] = 0, \quad (57)$$

$$E[(a_{sk-1} - a_{tk-1})(x_{ok} - x_{tk})] = 0. \quad (58)$$

Substituting (44) to (49) and (55) to (58) into (54), we obtain

$$\begin{aligned} \sigma_{sx}^2 &= (1 - \alpha)^2(\sigma_{sx}^2 + T^2\sigma_{sv}^2 + (T^4/4)\sigma_{sa}^2 + 2T\sigma_{sxv}^2 \\ &\quad + T^2\sigma_{sxa}^2 + T^3\sigma_{sva}^2) + \alpha^2B_x. \end{aligned} \quad (59)$$

This can be simplified to

$$\begin{aligned} \alpha(2 - \alpha)\sigma_{sx}^2 - (1 - \alpha)^2(T^2\sigma_{sv}^2 + (T^4/4)\sigma_{sa}^2 + 2T\sigma_{sxv}^2 \\ + T^2\sigma_{sxa}^2 + T^3\sigma_{sva}^2) = \alpha^2B_x. \end{aligned} \quad (60)$$

In the same way, other variances and covariances are calculated using (1) to (3) and (10) to (12) as follows:

$$\begin{aligned} \sigma_{sv}^2 &= E[(v_{sk} - v_{tk})^2] \\ &= (1 - \beta)^2(\sigma_{sv}^2 + T^2\sigma_{sa}^2 + 2T\sigma_{sva}^2) + \beta^2B_v, \end{aligned} \quad (61)$$

$$\begin{aligned} \sigma_{sa}^2 &= E[(a_{sk} - a_{tk})^2] \\ &= \sigma_{sa}^2 + (\gamma^2/T^2)B_v + (\gamma^2/T^2)(\sigma_{sv}^2 + T^2\sigma_{sa}^2 + 2T\sigma_{sva}^2) \\ &\quad - (2\gamma/T)(\sigma_{sva}^2 + \sigma_{sa}^2) \end{aligned} \quad (62)$$

$$\begin{aligned} \sigma_{sxv}^2 &= E[(x_{sk} - x_{tk})(v_{sk} - v_{tk})] \\ &= (1 - \alpha)(1 - \beta)(\sigma_{sxv}^2 + T\sigma_{sxa}^2 + T\sigma_{sv}^2 + T^2\sigma_{sva}^2 \\ &\quad + (T^2/2)\sigma_{sva}^2 + (T^3/2)\sigma_{sa}^2) \end{aligned} \quad (63)$$

$$\begin{aligned} \sigma_{sxa}^2 &= E[(x_{sk} - x_{tk})(a_{sk} - a_{tk})] \\ &= (1 - \alpha)(\sigma_{sxa}^2 + T\sigma_{sva}^2 + (T^2/2)\sigma_{sa}^2) \\ &\quad - (g(1 - a)/T)(\sigma_{sxv}^2 + T\sigma_{sxa}^2 + T\sigma_{sv}^2 \\ &\quad + T^2\sigma_{sva}^2 + (T^2/2)\sigma_{sva}^2 + (T^3/2)\sigma_{sa}^2), \end{aligned} \quad (64)$$

$$\begin{aligned} \sigma_{sva}^2 &= E[(v_{sk} - v_{tk})(a_{sk} - a_{tk})] \\ &= (1 - \beta)(1 - \gamma)(\sigma_{sva}^2 + T\sigma_{sa}^2) - (\gamma(1 - \beta)/T) \\ &\quad \times (\sigma_{sv}^2 + T\sigma_{sva}^2) + (\beta\gamma/T)B_v, \end{aligned} \quad (65)$$

where $E[(v_{ok} - v_{tk})^2] = B_v,$ (66)

and the following is satisfied because we assume that the measurement position and velocity noise are uncorrelated:

$$E[(x_{ok} - x_{tk})(v_{ok} - v_{tk})] = 0. \quad (67)$$

Equations (61) to (65) can be simplified to:

$$\beta(2 - \beta)\sigma_{sv}^2 - (1 - \beta)^2(T^2\sigma_{sa}^2 + 2T\sigma_{sva}^2) = \beta^2B_v, \quad (68)$$

$$\gamma(2 - \gamma)\sigma_{sa}^2 - (\gamma^2/T^2)\sigma_{sv}^2 - (2\gamma(1 - \gamma)/T)\sigma_{sva}^2 = (\gamma^2/T^2)B_v, \quad (69)$$

$$(\alpha + \beta - \alpha\beta)\sigma_{sv}^2 - (1 - \alpha)(1 - \beta)(T\sigma_{sv}^2 + (T^3/2)\sigma_{sa}^2 + T\sigma_{sva}^2 + (3T^2/2)\sigma_{sva}^2) = 0, \quad (70)$$

$$\begin{aligned} &(\alpha + \gamma - \alpha\gamma)\sigma_{sv}^2 + (\gamma(1 - \alpha)/T)(T\sigma_{sv}^2 + \sigma_{sv}^2) \\ &- ((\alpha - 1)(\gamma - 1)T^2/2)\sigma_{sa}^2 \\ &- ((\alpha - 1)(3\gamma - 2)T/2)\sigma_{sva}^2 = 0, \end{aligned} \quad (71)$$

$$\begin{aligned} &(\beta + 2\gamma - 2\beta\gamma)\sigma_{sva}^2 + \gamma(1 - \beta)\sigma_{sv}^2 \\ &- (1 - \beta)(1 - \gamma)T\sigma_{sa}^2 = (\beta\gamma/T)B_v. \end{aligned} \quad (72)$$

Solving the linear system involving (59) and (68) to (72), we obtain:

$$\sigma_{sx}^2 = \frac{\alpha}{2 - \alpha}B_x + \frac{f_1(\alpha, \beta, \gamma)(1 - \alpha)^2}{f_2(\alpha, \beta, \gamma)}T^2B_v, \quad (73)$$

$$\sigma_{sv}^2 = \frac{2\beta^2 + 2\gamma - 3\beta\gamma}{\beta(4 - 2\beta - \gamma)}B_v, \quad (74)$$

$$\sigma_{sa}^2 = \frac{2\gamma^2}{\beta(4 - 2\beta - \gamma)}\frac{B_v}{T^2}, \quad (75)$$

$$\sigma_{svv}^2 = \frac{(1 - \alpha)(1 - \beta)(4\alpha\beta^2 - \alpha\gamma^2 - 4\alpha\beta\gamma + 4\alpha\gamma + 4\beta\gamma)}{f_2(\alpha, \beta, \gamma)/\alpha}TB_v, \quad (76)$$

$$\sigma_{sva}^2 = \frac{\gamma(\alpha - 1)(4\alpha\beta^2 - 4\alpha\beta - \gamma^2 + 4\gamma)}{f_2(\alpha, \beta, \gamma)/\alpha}B_v, \quad (77)$$

$$\sigma_{sva}^2 = \frac{\gamma(2\beta - \gamma)}{\beta(4 - 2\beta - \gamma)}\frac{B_v}{T}, \quad (78)$$

where $f_1(\alpha, \beta, \gamma)$ and $f_2(\alpha, \beta, \gamma)$ are expressed as (14) and (15). Substituting (73) to (78) into (50), we arrive at (13).

Derivation of (16)

We first derive the relationship between the measured signals (x_{ok} and v_{ok}) and the predicted position x_{pk} in the z -domain and then obtain the tracking performance index using the final value theorem. Applying a z -transform to (1) to (3) and (10) to (12), we obtain:

$$X_p(z) = X_s(z)/z + TV_s(z)/z + (T^2/2)A_s(z)/z, \quad (79)$$

$$V_p(z) = V_s(z)/z + TA_s(z)/z, \quad (80)$$

$$A_p(z) = A_s(z)/z, \quad (81)$$

$$X_s(z) = X_p(z) + \alpha(X_o(z) - X_p(z)), \quad (82)$$

$$V_s(z) = V_p(z) + \beta(V_o(z) - V_p(z)), \quad (83)$$

$$A_s(z) = A_p(z) + (\gamma/T)(V_o(z) - V_p(z)). \quad (84)$$

Substituting (84) into (81), we have

$$A_p(z) = \frac{\gamma}{z - 1} \cdot \frac{V_o(z) - V_p(z)}{T}. \quad (85)$$

Substituting (83) into (80) gives

$$(z + \beta - 1)V_p(z) = \beta V_o(z) + zTA_p(z). \quad (86)$$

Substituting (85) into (86), the relationship between the predicted and measured velocities is calculated as

$$V_p(z) = \frac{(\beta + \gamma)z - \beta}{z^2 + (\beta + \gamma - 2)z - \beta + 1}V_o(z). \quad (87)$$

Substituting (87) into (85), the relationship between the predicted acceleration and the measured velocities is calculated as

$$A_p(z) = \frac{\gamma(z - 1)}{z^2 + (\beta + \gamma - 2)z - \beta + 1} \frac{V_o(z)}{T}. \quad (88)$$

Substituting (82) to (84), (87), and (88) into (79), the relationship between the predicted position and the measured position and velocity is written as

$$\begin{aligned} X_p(z) &= \frac{\alpha}{z + \alpha - 1}X_o(z) \\ &+ \frac{z(2\beta + \gamma)z - 2\beta + \gamma}{2(z + \alpha - 1)(z^2 + (\beta + \gamma - 2)z - \beta + 1)}TV_o(z). \end{aligned} \quad (89)$$

Thus, the z -transform of the error $x_{ok} - x_{pk}$ is expressed as

$$\begin{aligned} E_p(z) &= \frac{z - 1}{z + \alpha - 1}X_o(z) \\ &- \frac{z(2\beta + \gamma)z - 2\beta + \gamma}{2(z + \alpha - 1)(z^2 + (\beta + \gamma - 2)z - \beta + 1)}TV_o(z). \end{aligned} \quad (90)$$

Here, the measured position and velocity of a target with constant jerk J are:

$$x_{ok} = J(kT)^3/6, \quad (91)$$

$$v_{ok} = J(kT)^2/2, \quad (92)$$

and their z -transforms are:

$$X_o(z) = \frac{z(z^2 + 4z + 1)}{6(z - 1)^4}JT^3, \quad (93)$$

$$V_o(z) = \frac{z(z + 1)}{2(z - 1)^3}JT^2. \quad (94)$$

Substituting (93) and (94) into (90), we have

$$E_p(z) = \frac{z(2z^2 + (8 - 4\beta - \gamma)z + 2 - 2\beta)}{12(z - 1)(z + \alpha - 1)(z^2 + (\beta + \gamma - 2)z - \beta + 1)}JT^3. \quad (95)$$

With the final value theorem $\lim_{z \rightarrow 1}(z - 1)E_p(z)$, we have (16).

Derivation of (20)

Using the same procedure as for the A-V filter, the linear system with respect to the variances and covariances of the smoothing parameters is calculated using (1) to (3) and (17) to (19) as:

$$\alpha(2-\alpha)\sigma_{sx}^2 - (1-\alpha)^2(T^2\sigma_{sv}^2 + (T^4/4)\sigma_{sa}^2 + 2T\sigma_{sxx}^2 + T^2\sigma_{sxa}^2 + T^3\sigma_{sva}^2) = \alpha^2 B_x, \tag{96}$$

$$\beta(2-\beta)\sigma_{sv}^2 - (1-\beta)^2(T^2\sigma_{sa}^2 + 2T\sigma_{sva}^2) = \beta^2 B_v, \tag{97}$$

$$(\gamma(4-\gamma)/4)\sigma_{sa}^2 - (\gamma-2/T^4)(\sigma_{sx}^2 + T^2\sigma_{sv}^2 + 2T\sigma_{sxx}^2) + \gamma(2-\gamma)(\sigma_{sxa}^2/T^2 + \sigma_{sva}^2/T) = (\gamma^2/T^4) B_x, \tag{98}$$

$$(\alpha + \beta - \alpha\beta)\sigma_{sxx}^2 - (1-\alpha)(1-\beta)(T\sigma_{sv}^2 + (T^3/2)\sigma_{sa}^2 + T\sigma_{sxa}^2 + (3T^2/2)\sigma_{sva}^2)\sigma_{sxx}^2 = 0, \tag{99}$$

$$(\alpha + \gamma - \alpha\gamma)\sigma_{sxa}^2 - (\alpha-1)(\gamma-1)T\sigma_{sva}^2 - ((\alpha-1)(\gamma-2)/4)T^2\sigma_{sa}^2 + \gamma(1-\alpha)(\sigma_{sx}^2/T^2 + \sigma_{sv}^2 + 2\sigma_{sxx}^2/T) = (\alpha\gamma/T^2) B_x, \tag{100}$$

$$(2\beta + 3\gamma - 3\beta\gamma)\sigma_{sva}^2/2 + \gamma(1-\beta) \times (\sigma_{sxx}^2/T^2 + \sigma_{sv}^2/T + \sigma_{sxa}^2/T^2) - (\beta-1)(\gamma-2)T\sigma_{sa}^2/2 = 0. \tag{101}$$

Solving the linear system involving (96) to (101), we obtain:

$$\sigma_{sx}^2 = \frac{g_{1x}(\alpha, \beta, \gamma)B_x + (1-\alpha)^2g_2(\alpha, \beta)T^2B_v}{g_3(\alpha, \beta, \gamma)}, \tag{102}$$

$$\sigma_{sv}^2 = \frac{g_{1v}(\alpha, \beta, \gamma)B_x/T^2 + g_{2v}(\alpha, \beta, \gamma)B_v}{(2-\beta)g_3(\alpha, \beta, \gamma)}, \tag{103}$$

$$\sigma_{sa}^2 = \frac{g_{1a}(\alpha, \beta, \gamma)B_x/T^4 + g_{2a}(\alpha, \beta, \gamma)B_v/T^2}{(2-\beta)g_3(\alpha, \beta, \gamma)}, \tag{104}$$

$$\sigma_{sxx}^2 = \frac{g_{1xx}(\alpha, \beta, \gamma)B_x/T + g_{2xx}(\alpha, \beta, \gamma)TB_v}{g_3(\alpha, \beta, \gamma)}, \tag{105}$$

$$\sigma_{sxa}^2 = \frac{g_{1xa}(\alpha, \beta, \gamma)B_x/T^2 + 2(1-\alpha)(2-\beta)\gamma^2g_2(\alpha, \beta)B_v}{\gamma(2-\beta)g_3(\alpha, \beta, \gamma)}, \tag{106}$$

$$\sigma_{sva}^2 = \frac{g_{1va}(\alpha, \beta, \gamma)B_x/T^3 + g_{2va}(\alpha, \beta, \gamma)B_v/T}{(2-\beta)g_3(\alpha, \beta, \gamma)}, \tag{107}$$

where $g_2(\alpha, \beta)$ and $g_3(\alpha, \beta, \gamma)$ are expressed as (22) and (23), and

$$g_{1x}(\alpha, \beta, \gamma) = 8\alpha^3\beta(2-\beta)(\beta-1) - 2\alpha^2(3\beta^3\gamma - 4\beta^3 - 9\beta^2\gamma + 8\beta^2 + 2\beta\gamma + 4\gamma) + \alpha\gamma(10\beta^3 + \beta^2\gamma - 28\beta^2 - \beta\gamma + 8\beta + 16) - 4\beta\gamma(2-\beta)^2, \tag{108}$$

$$g_{1v}(\alpha, \beta, \gamma) = 8\gamma^2(1-\beta)^2(2-\beta)(\alpha + \beta - \alpha\beta - 2), \tag{109}$$

$$g_{2v}(\alpha, \beta, \gamma) = 8\alpha^3\beta^2(2-\beta)(1-\beta) + 2\alpha^2\beta^2(\beta-2) \times (3\beta\gamma - 12\beta - 3\gamma + 8) - \alpha\beta^2(22\beta^2\gamma - 16\beta^2 + \beta\gamma^2 - 64\beta\gamma + 32\beta - \gamma^2 + 40\gamma) + 2\beta^2\gamma(\beta-1)(8\beta + \gamma - 16), \tag{110}$$

$$g_{1a}(\alpha, \beta, \gamma) = 4\beta\gamma^2(\beta-2)(2\alpha\beta^2 - 6\alpha\beta - 2\beta^2 - \beta\gamma + 4\alpha + 4\beta + \gamma), \tag{111}$$

$$g_{2a}(\alpha, \beta, \gamma) = 4\beta^2\gamma(\alpha-2)(2\alpha\beta^2 - 6\alpha\beta - 2\beta^2 - \beta\gamma + 4\alpha + 4\beta + \gamma), \tag{112}$$

$$g_{1xx}(\alpha, \beta, \gamma) = 2\beta\gamma(\alpha-1)(\beta-2)(\beta-1)(4\alpha + \gamma), \tag{113}$$

$$g_{2xx}(\alpha, \beta, \gamma) = 2\beta^2(\alpha-1)(\alpha-2)(\beta-1)(4\alpha + \gamma), \tag{114}$$

$$g_{1xa}(\alpha, \beta, \gamma) = \gamma^2(2-\beta)(8\alpha^2\beta^3 - 24\alpha^2\beta^2 + 16\alpha^2\beta + 2\alpha\beta^3\gamma - 8\alpha\beta^3 - 6\alpha\beta^2\gamma + 16\alpha\beta^2 - 4\alpha\beta\gamma + 8\alpha\gamma - 2\beta^3\gamma - \beta^2\gamma^2 + \beta\gamma^2 + 16\beta\gamma - 16\gamma), \tag{115}$$

$$g_{1va}(\alpha, \beta, \gamma) = 2\beta\gamma^2(2-\beta)(\beta-1)(4\alpha\beta - 8\alpha - 4\beta - \gamma + 8), \tag{116}$$

$$g_{2va}(\alpha, \beta, \gamma) = 2\beta^2\gamma(2-\alpha)(\beta-1)(4\alpha\beta - 8\alpha - 4\beta - \gamma + 8). \tag{117}$$

Substituting (102) to (107) into (50), we arrive at (20).

Derivation of (24)

The derivation process is the same as for the A-V filter. By applying a z-transform to (1) to (3) and (17) to (19)

and their simplified forms, the predicted parameters in the z -domain are derived as:

$$A_p(z) = \frac{\gamma z}{z-1} \cdot \frac{X_o(z) - X_p(z)}{T^2}, \quad (118)$$

$$V_p(z) = \frac{\gamma z^2}{(z-1)(z+\beta-1)} \cdot \frac{X_o(z) - X_p(z)}{T} + \frac{\beta}{z+\beta-1} V_o(z), \quad (119)$$

$$X_p(z) = \frac{(2\alpha + \gamma)X_o(z) + 2\beta TV_o(z) + 2(1 - \beta)TV_p(z) + T^2 A_p(z)}{2z + 2\alpha + \gamma - 2}. \quad (120)$$

Substituting (118) and (119) into (120), the relationship between the predicted position and the measured position and velocity is expressed as

$$X_p(z) = \frac{h_1(z)}{h_2(z)} X_o(z) + \frac{2\beta z(z-1)}{h_2(z)} TV_o(z), \quad (121)$$

where

$$h_1(z) = \gamma z^3 + (2\alpha - \beta\gamma + 2\gamma)z^2 + (2\alpha\beta - 4\alpha + \beta\gamma - 2\gamma)z - 2\alpha\beta - \beta\gamma + 2\alpha + \gamma, \quad (122)$$

$$h_2(z) = (\gamma + 2)z^3 + (2\alpha + 2\beta + 2\gamma - \beta\gamma - 6)z^2 + (2\alpha\beta + \beta\gamma - 4\alpha - 4\beta - 2\gamma + 6)z - 2\alpha\beta - \beta\gamma + 2\alpha + 2\beta + \gamma - 2. \quad (123)$$

From (121), the z -transform of the predicted error is

$$E_p(z) = X_o(z) - X_p(z) = \frac{2(z-1)^2(z+\beta-1)}{h_2(z)} X_o(z) - \frac{2\beta z(z-1)}{h_2(z)} TV_o(z). \quad (124)$$

Substituting (93) and (94) into (124), the error for a target with constant jerk is given by

$$E_p(z) = \frac{z(z^2 - 2\beta z + 4z - \beta + 1)}{3(z-1)h_2(z)} JT^3. \quad (125)$$

Applying the final value theorem to (125), we have (24).

Abbreviations

PVM: position-velocity-measured; MV: minimum-variance; A-V filter: acceleration smoothed by measured velocity-type PVM α - β - γ filter; A-P filter: acceleration smoothed by measured position-type PVM α - β - γ filter; GMV filter: general minimum-variance α - β - γ filter; CRB: Cramér–Rao bound; EKF: extended Kalman filter.

Competing interests

The authors declare that they have no competing interests.

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