# Quantization error for weak RF simultaneous signal estimation 

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#### Abstract

In a congested signal environment, it is difficult to obtain estimates of weak RF signal parameters. Determining signal parameter estimates in real time is a challenge for electronic warfare receivers that aim to receive multiple simultaneous signals. Prior work provided estimates of weak signal parameters (weak signal frequency and weak signal amplitude) without taking into account any error introduced by analog-to-digital converters that are inherently part of digital signal processing systems. In order to obtain realistic estimates, we need to take error introduced by an ADC into account. The primary aim of this paper is to quantify error introduced by a single ideal ADC as a function of angle. This paper presents a method to estimate angle resolution and quantization levels in N -bit analog-to-digital converters (ADCs) for use in a weak radiofrequency (RF) simultaneous signal estimation process. The paper quantifies the error in the angle quantization of an $N$-bit ADC for an input complex signal that is the instantaneous frequency obtained for the situation in which there are two simultaneous signals (with one strong signal and one weak signal) in a weak RF simultaneous signal estimation process. The presented method describes the process to determine the angle quantization range, angle quantization uncertainty, and angle quantization error. This approach has potential applications in electronic warfare (EW) systems. The approach also has potential for assessing ADC performance for measurements that approach the quantum limit. Results are presented for 1-bit, 2-bit, 3-bit, and 10-bit ADCs.


Keywords: Analog-to-digital converter (ADC), Electronic warfare (EW), Radiofrequency (RF), Signal processing

## 1 Introduction

This work is concerned with estimation of weak signal parameters in a congested radiofrequency (RF) environment where multiple simultaneous signals exist. The motivation for this research is the need for a more accurate model of weak RF simultaneous signal estimation process that uses phase calculation approaches to obtain weak signal parameter estimates $[1-7]$. The capability to detect at least four simultaneous signals is desirable in the development of electronic warfare receivers [8]. Figure 2 shows an example of the instantaneous frequency for the situation in which there are $N=2$ simultaneous signals with a 5 MHz frequency difference between the strong signal frequency $f_{1}$ and weak signal frequency $f_{2}$. An example of this situation occurs when the strong signal frequency
$f_{1}=850 \mathrm{MHz}$, and the weak signal frequency $f_{2}=855 \mathrm{MHz}$. Estimates of the weak signal parameters as obtained with zero crossings in the instantaneous frequency (See block diagram in Fig. 1) are discussed in [5, 6].

Prior work developed a multi-tier weak RF signal estimation process to obtain weak signal parameter estimates for the situation in which there are two simultaneous signals, with one strong signal and one weak signal. In this prior work, the weak RF estimation process provided methods to obtain estimates of the weak signal parameters. Results obtained in prior work showed that estimates of the signal parameters, including the weak signal frequency estimate and weak signal amplitude estimate, can be extracted from the instantaneous frequency using phase calculation approaches (see Fig. 3a). Prior work presented a weak radio signal estimation process [1-7] for simultaneous signals using phase calculation approaches developed by Tsui. Figure 3 in [9, Fig. 3] presents a block diagram of a typical EW system showing the signal analysis and interpretation block. Results obtained in prior work [7] showed that estimates of the weak signal frequency that were obtained with this process are within $1.2 \times 10^{-4}$, and the error is within 120 parts per million. However, these estimates of the weak signal frequency and weak signal amplitude did not take into account error introduced by ADCs inherent in digital signal processing systems. Providing a method to quantify error introduced by an ideal ADC is the aim of this paper.

Tsui discusses the incorporation of ADCs in electronic warfare (EW) receivers for the desirable properties of large bandwidth and dynamic range [8]. This approach is of interest to the area of error vector magnitude (EVM) calculations [10-13]. The development of ADCs has been a focus of state-of-the-art circuit designers since at least the late 1990 s in order to develop signal processing systems with desirable properties including large dynamic range, wide bandwidth, low quantization noise, and the ability to resolve signals with complex modulation [14]. Signal processing systems incorporating ADCs include commercial devices [15], electronic warfare (EW) receivers [8, 16, 17], and electronic intelligence (ELINT) receivers [9, 18, 19].
More recently, in 2003, Feddeler et al. reviewed ADC specifications [20], and in 2008, Kester et al. reviewed performance metrics for ADCs [21]. Feddeler et al. assumes a 10 -bit, $5.12-\mathrm{V}$ ADC [20] and define quantization error as $\pm \frac{1}{2}$ of the least significant bit (LSB), which is 5.12 V per 512 bits, or 0.1 V per bit. Murmann tracks a 20 -year summary of ADC performance trends [22-26]. Potential applications exist


Fig. 1 Block diagram to determine the instantaneous frequency with [6]


Fig. 2 Instantaneous frequency as a function of time for $\mathrm{N}=2$ simultaneous signals with a 5 MHz frequency difference between a strong signal and a weak signal. The strong signal frequency is 850 MHz
for detection of complex communication and radar signals, cognitive radio, and software defined radio. Very recently, signal generators intended for 5 G and wideband satellite applications (SatComm) are being developed for signal frequencies up to 50 GHz with bandwidths as high as 510 MHz . This hardware provides the capability to demodulate 5 G radio signals and extract signals that were not previously observable in a dense electromagnetic spectrum [27]. Keysight Technologies has introduced hardware with desirable phase noise characteristics with high dynamic range and large bandwidth (N9021B MXA X-Series Signal Analyzer) [27]. Keysight Technologies develops hardware with the capability to measure magnitude and phase error of a complex signal in a process referred to as Error Vector Magnitude (EVM) [27]. In this process, the magnitude of the error vector refers to the magnitude of the difference of the ideal vector and the test signal vector. The difference in phase between the phase of the ideal signal and the phase of the test signal is referred to as the IQ phase error, or angle quantization error. The magnitude of the difference in the test signal compared with the magnitude of the ideal signal is referred to as the magnitude error (IQ error magnitude).
In the area of photonics, in 2014, Golani et al. proposed a photonic analog-to-digital converter with 7.6 effective bits [28]. In electronics, a hybrid silicon pixel detector with a noise floor as low as 100 electrons has been manufactured and first implemented at CERN as tracking detectors. Now these Medipix and Timepix chips are widely available and have demonstrated that they can detect single X-rays one at a time [29-31].
This paper presents a method to quantify error introduced by an $N$-bit ADC [32] and focuses specifically on quantifying the error introduced in the instantaneous frequency through the quantization process for the case in which there are $N=2$ simultaneous signals. The error is inherent in the use of an ADC. Analog-to-digital converters (ADCs) take an input continuous-time signal [33] and produce an output signal in discrete-time binary coded form [33]. This paper presents the next step in the development of an electronic warfare receiver because prior work (see Figs. 2, 3a) has not yet taken into account the effect of the quantization by the ADC.


Fig. 3 a Prior work: weak radiofrequency signal estimation process with phase calculation approaches and sampled instantaneous frequency (blue boxes). $\mathbf{b}$ Contribution of this work: model of an ADC to quantize the instantaneous frequency in a weak RF signal estimation process (purple boxes). Future work (green box)

Our aim is to develop an understanding of the error introduced by an ADC when the ADC converts an analog signal (such as the instantaneous frequency) into a digital signal before any digital signal processing takes place. By taking the quantization of the signal into account in the model, more accurate estimates of the weak signal parameters will be able to be obtained in the weak RF signal estimation process (see Fig. 3b). Future work will need to incorporate this new understanding in order to obtain more accurate estimates of weak signal parameters, and after this is completed, then these more accurate estimates can be compared with estimates obtained in prior work.
As a starting point, the paper considers the representation of an $N$-bit ADC for a complex signal described by Tsui; in this representation, an $N$-bit ADC is modeled as an $N \times N$ array of squares, where each square represents one bit. For example, a 1-bit ADC is represented as a $1 \times 1$ array; a 2 -bit ADC is represented as a $2 \times 2$ array, and a 3-bit ADC is represented as a $3 \times 3$ array. A complex signal can then be represented on the ADC array as a vector having phase angle relative to the $x$-axis and magnitude that takes on a value equal to or less than $N$ in an $N$-bit ADC.
In this analysis, we choose the simplifying case of a weak signal with amplitude much lower (approximately $1 / 100$ ) compared with the amplitude of the strong signal. This paper quantifies the error in the magnitude of the signal amplitude reported by an $N$-bit ADC . The paper also quantifies the error in the angle quantization reported by an N -bit ADC . The paper presents steps to carry out the angle quantization method to determine the angle quantization range, angle quantization uncertainty, and angle quantization error as a function of angle. The angle quantization error is the difference between the angle of the sampled signal and the value of the angle reported by the ADC (phase error). The angle quantization uncertainty is the difference in angle (within the bit that captures the input signal) and the value of the angle reported by the ADC. The amplitude of the instantaneous frequency produced by the weak RF signal estimation process is assumed to take on the maximum value that happens to be the value of the upper limit of the ADC .

Results are presented for $N$-bit ADCs, where $N=\{1,2,3,10\}$. For the case in which the ADC has 10 bits, the results show that the magnitude of the angle quantization error is less than $\sim 5 \times 10^{-4} \pi$ radians. Results also show that the magnitude of the angle quantization uncertainty is less than $\sim 5 \times 10^{-4} \pi$ radians ( $0.09^{\circ}$ ).

The contributions of this paper are:

- Notation for describing the quantization of a complex signal by an $N$-bit ADC;
- Method to quantify the error, angle quantization range, and angle quantization uncertainties in an $N$-bit ADC, with an application to the case in which the signal takes on a constant magnitude in an $N$-bit ADC;
- Applications to calculate the angle quantization error, angle quantization range, and angle quantization uncertainties for $N=1,2,3$, and 10-bit ADCs.
- Application to calculate the signal quantization error in a $N=10$-bit ADC.

The rest of this paper is as follows: Section 2 discusses a model of an $N$-bit ADC. Section 3 presents the proposed method for mapping the signal amplitude and angle to the amplitude and angle reported by the ADC. Section 4 discusses applications to $N=1,2,3$ and 10-bit ADCs. Section 5 presents conclusions, and Sect. 6 discusses future work.

## 2 Model of an N -bit ADC

### 2.1 Representation of a complex signal in the $x-y$ plane

In an $N$-bit ADC, the complex signal amplitude can be represented as a two-dimensional diagram with a circle in the $x-y$ plane that intersects adjacent bits on two sides (of each bit) in the ADC. For example, Fig. 4 shows a 3-bit ADC. The ADC can be represented in the $x-y$ plane as a $2^{N} \times 2^{N}$ square array. In this paper, each of the squares will be referred to as a bit. The length of the radius of the circle $A$ with the center located at the


Fig. 4 Notation for calculating $\theta_{s, n, m}$ for the sampled signal with maximum signal amplitude, $a_{p, p+1}$ and $b_{p, p+1}$ in the (darker) square with coordinates $(n, m)$ that intersects the circle representing the magnitude of the complex signal. This example shows a 3-bit ADC as an 8-bitx 8-bit array

ADC origin $(0,0)$ is taken to represent the magnitude of the amplitude of the complex signal. The reported value of the signal magnitude and reported value of the signal angle are the values of the signal magnitude and signal angle at the center $(X)$ of the bit containing the value of the signal. For the analysis that follows, the coordinates $(n, m)$ represent the lower-left corner of the bit in the two-dimensional representation; namely, for bit ( $n, m$ ), the other corners of the bit are identified with the notation $(n+1, m)$ (lower right corner), ( $n+1, m+1$ ) (upper right corner), and ( $n, m+1$ ) (upper left corner). The values in the coordinate pair $(n, m)$ take on values given by $n=1,2, \ldots, N-1$ and $m=1,2, \ldots,(N-1)-n+1=N-n$.

We introduce a term $p$ that denotes the bit(s) in an $N$-bit ADC that intersect(s) the complex signal. The value of $p$ for each bit is assigned in order of increasing angle $\theta_{\mathrm{ADC}, n, m}$. We count the $p$ bits intersected by the circle and index the bits with value $p$ in order of increasing angle from the positive $x$-axis (counter clockwise). One of each of the other two sides of the right triangle, $a_{p, p+1}$ and $b_{p, p+1}$, has integer length given by one of the following lengths: $\left\{1,2, \ldots, \frac{2^{N}}{2}\right\}$, and the other side has length obtained by the Pythagorean theorem. Note that $p$ denotes the number of the bit ( $n, m$ ) that is intersected by a circle representing a complex signal with signal magnitude equal to the radius of the circle. For simplicity, we consider the first quadrant of the ADC (that is, for angles between 0 and 90 degrees); the rest of the quantization angles for this signal can be obtained using symmetry.
The complex signal amplitude will intersect a subset ( $p \geq 1$ ) of the bits in the two-dimensional representation of the $N$-bit A DC in such a way that length of the line connecting the $\operatorname{ADC}$ origin $(0,0)$ to each bit edge forms the hypotenuse, $c$, of a right triangle having one side parallel to the $x$-axis. The length of the hypotenuse of this right triangle is equal to the magnitude $A$ of the complex signal, such that $c=A$. Each bit in the $N$-bit ADC is intersected twice by the circle except where the circle precisely intersects one corner of a bit.

The magnitude of the signal reported by the ADC is the length of the sampled signal, $s$, and the reported value for the magnitude of the sampled signal $s_{\mathrm{ADC}}$ is the length of the vector pointing to the center of the bit $(n=0, m=0)$. For a 1 -bit ADC, $s_{\mathrm{ADC}}=\sqrt{\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{2}}=\frac{\sqrt{2}}{2}$. For a 2-bit ADC, $s_{\mathrm{ADC}}=\sqrt{\left(\frac{3}{4}\right)^{2}+\left(\frac{3}{4}\right)^{2}}=\frac{3 \sqrt{2}}{4}$. For a 3-bit ADC, $s_{\mathrm{ADC}}=\sqrt{\left(\frac{7}{8}\right)^{2}+\left(\frac{5}{8}\right)^{2}}=\frac{\sqrt{74}}{8}$. We also observe that:
i. For a 1-bit ADC , each bit is intersected twice by the circle with amplitude $A=1$;
ii. For a 2-bit ADC, each bit is intersected twice by the circle except for the single case in which the signal amplitude $A$ takes on a value that is equal to $\frac{1}{2}$ and $\frac{\sqrt{2}}{2}$, where the circle intersects the lower-left corner of the bit;
iii. For a 3-bit ADC, each bit is intersected twice by the circle except for signals with amplitude $A$ that takes on any of the following values: $\frac{1}{4}, \frac{1}{2}$, and $\frac{3}{4}, \sqrt{\left(\frac{1}{4}\right)^{2}+\left(\frac{1}{4}\right)^{2}}$; $\sqrt{\left(\frac{1}{4}\right)^{2}+\left(\frac{1}{2}\right)^{2}} ; \sqrt{\left(\frac{1}{2}\right)^{2}+\left(\frac{1}{2}\right)^{2}} ; \sqrt{\left(\frac{3}{4}\right)^{2}+\left(\frac{1}{2}\right)^{2}}$.
iv. Each bit in the $N$-bit ADC is intersected twice by the circle except for the cases in which the signal amplitude $A$ takes on values that are given by $\sqrt{\left(\frac{n}{2^{N-1}}\right)^{2}+\left(\frac{m}{2^{N-1}}\right)^{2}}$, where $n=0,1, \ldots, N$ and $m=0,1, \ldots, N$.

As an example, the visualization of a complex signal $s=A e^{i 2 \pi f_{s} t}$ with constant amplitude $A$ and constant frequency $\omega_{s}=2 \pi f_{s}$ is shown in Fig. 4 as a circle with center at the origin $(0,0)$ with radius $A$. For the case in which the signal takes on the maximum value of the ADC, then $A=2^{N-1}$.

### 2.2 Notation

The sampled signal of interest is shown as the red arrow taking on the maximum signal amplitude shown in Fig. 4. For simplicity, the discussion that follows will identify each bit with coordinates $(n, m)$. In the figure, the angle of the maximum signal $s$ in bit ( $n, m$ ) is $\theta_{s, n, m}$; the angle reported by the ADC is $\theta_{\mathrm{ADC}, n, m}$; the corners of the bit are located at the coordinates $\{(n, m),(n+1, m),(n, m+1),(n+1, m+1)\}$; and the numbers of the bit ( $n, m$ ) that intersects the signal with maximum amplitude, where $-2^{N-1} \leq n \leq 2^{N-1}$ and $-2^{N-1} \leq m \leq 2^{N-1}$. In the first quadrant, $0 \leq n \leq 2^{N-1}$ and $0 \leq m \leq 2^{N-1}$, respectively.
Figure 5 shows three diagrams of a (a) 1-bit, (b) 2-bit, and (c) 3-bit ADC. In each figure, the red arrow indicates the sampled signal used to illustrate the notation for the angle of the sampled signal at the coordinates $(n, m)$. The angle reported by each ADC, $\theta_{\mathrm{ADC}, n, m}$, is the angle at the center of the bit $(n, m)$ that captures the sampled signal. For the 1 -bit ADC (Fig. 5a), the sampled signal is captured in the bit with coordinates ( $n=0, m=0$ ). The sampled signal has angle $\theta_{s, n=0, m=0}$, and the reported angle by the ADC using the center of the bit is $\theta_{\mathrm{ADC}, n=0, m=0}$. For the 2-bit ADC (Fig. 5b), the sampled signal with the same magnitude as in Fig. 5 a is captured in the bit with coordinates ( $n=1, m=1$ ). The sampled signal has angle $\theta_{s, n=1, m=1}$, and the reported angle by the ADC using the center of the bit is $\theta_{\mathrm{ADC}, n=1, m=1}$. For the 3-bit ADC (Fig. 5c), the sampled signal with the same magnitude as in Fig. 5a is captured in the bit with coordinates ( $n=3, m=2$ ). The sampled signal has angle $\theta_{s, n=3, m=2}$, and the reported angle by the ADC using the center of the bit is $\theta_{\mathrm{ADC}, n=3, m=2}$.

Figure 6 shows the magnitude and quantized angles of (a) 1-bit, (b) 2-bit, and (c) 3-bit ADCs in the first quadrant, for the case in which the complex signal takes on the maximum value of the ADC. The magnitude and quantized angles in the other quadrants are obtained using symmetry. The values are shown in Table 2.

### 2.3 Angle quantization error

This section focuses on developing an understanding of the angle quantization error, presents a method to calculate the angle quantization error, and quantifies the performance of an N -bit ADC. This section also presents an analysis of bounds on the angle quantization error, angle quantization uncertainty, and angle quantization range. The angle quantization error for a sampled signal is taken to be the difference between the angle of the sampled signal and the angle reported by the ADC. The number of samples for a signal with constant magnitude at the maximum value of the ADC produces a wide variation in the angle quantization range and angle quantization uncertainty, depending on the angle of the sampled signal. This situation occurs specifically because there exist some (rare) cases where the error is very small. These cases occur when the signal magnitude intersects a corner of each bit of the ADC.


Fig. 5 Diagrams of a 1-bit, b 2-bit, and c 3-bit ADC, with the sampled signal (red arrow) with maximum signal amplitude, angle $\theta_{s, n, m}$ of the maximum signal s in bit $(n, m)$, angle reported by the ADC $\theta_{\text {ADC,n,m }}$


Fig. 6 Diagrams of the quantized angles that can be reported $\theta_{A D C, n, m}$ by a a 1-bit, b 2-bit, c 3-bit ADC (first quadrant) as tabulated in Table 2

The steps to determine the angle quantization error are shown in the block diagram in Fig. 7 and are listed below:

1. Assume that the signal $s$ (see the red vector in Figs. 4 and 5 ) has magnitude $A$ that takes on a value up to the maximum of the $N$-bit ADC such that the signal magnitude is less than $N$.
i. For the case in which the amplitude $A$ takes on a constant value, the signal defines a circle in the ADC.
ii. The signal projection on the ADC will take on a non-circular shape in the case that the signal is time-varying, such as for a linear FM pulsed waveform with time-varying amplitude.
2. Count the $p$ bits intersected by the signal and index the bits with value $p$ in order of increasing angle from the positive $x$-axis.
i. For example, $p=1$ indicates the first bit intersected by the signal, and $p=2$ indicates the next bit intersected by the signal.
ii. Each bit in the $N$-bit ADC is intersected twice by the circle, except in the rare cases where the circle is precisely intersecting one corner of a bit.
3. The rest of these steps consider the case in which the signal magnitude $A$ takes on a constant value. For this case, we observe that the circle defined by the signal intersects each bit at two locations specified by pairs of coordinates ( $x_{A 1}, y_{A 1}$ ) and $\left(x_{A 2}, y_{A 2}\right)$ according to one of the following three scenarios:
i. Two vertical sides are intersected by the circle;
ii. Two horizontal sides are intersected by the circle;
iii. One horizontal size and 1 vertical side are intersected by the circle.


Fig. 7 Block diagram of steps to determine the angle quantization error
4. Construct a right triangle with the hypotenuse defined by the signal vector with signal magnitude $A$. One side of the triangle is parallel to the $x$-axis, and the other side is parallel to the $y$-axis. For the case of a signal with constant amplitude $A$, we consider the symmetry of the circle intersecting the bits and focus our analysis on the first half quadrant, $0 \leq \theta \leq \frac{\pi}{4}$, without loss of generality:
i. One corner of each triangle is located at $(0,0)$.
ii. The distance from the origin to point of intersection defines the hypotenuse with magnitude $A$.
iii. The right triangle has at least one side with integer length $a=1,2, \ldots, N$.
iv. The third side has length $b$ given by $b=\sqrt{A^{2}-a^{2}}$.
5. Identify all triangles with integer sides along the vertical, parallel to the $y$-axis, in the first $\frac{1}{8}$ of the circle, $0 \leq \theta \leq \frac{\pi}{4}$, and calculate the acute angle $\theta_{i}$ relative to the $x$-axis for each triangle; create a list of the values $\left\{\theta_{i}\right\}$, where $i=0, \ldots H-1$, and $H$ is the number of triangles with integer sides along the vertical.
6. Identify all triangles with integer sides along the horizontal, parallel to the $x$-axis, in the first $\frac{1}{8}$ of the circle and calculate the acute angle $\theta_{j}$ relative to the $y$-axis for each triangle; create a list of the values $\left\{\theta_{j}\right\}$, where $j=0, \ldots, V-1$, and $V$ is the number of triangles with integer sides along the horizontal.
7. Create a list of all angles $\left\{\theta_{k}\right\}$ by concatenating the lists $\left\{\theta_{i}\right\}$ and $\left\{\theta_{j}\right\}$, where $k$ takes on values $k=0,1,2, \ldots, H+V-1$.
8. Sort the combined list obtained in the previous step in order of increasing angle, starting from angle with the least value, 0 radians, to the angle with the largest value, $\frac{\pi}{4}$, for $k=0,1, \ldots, H+V-1$.
9. For each bit, form pairs of angles $\left(\theta_{k}, \theta_{k+1}\right)$ that indicates the pair of intersections at each bit, where the angle with lowest value is listed first, and the angle with the larger value is listed second in the pair, where $k=0,1, \ldots, H+V-1$.
10. Calculate the coordinates of a point within the bit by calculating the average of the $x$-value and $y$-values of each intersection such that

$$
\begin{align*}
& x_{\mathrm{ave}}=\frac{1}{2}\left(x_{A 1}+x_{A 2}\right)  \tag{1}\\
& y_{\mathrm{ave}}=\frac{1}{2}\left(y_{A 1}+y_{A 2}\right) \tag{2}
\end{align*}
$$

11. Identify the bit index $(n, m)$ of the bit containing the point identified in the previous step by calculating the integer values of the coordinates of the lower left coordinate of the bit where $n=\left\lfloor x_{\text {ave }}\right\rfloor$ and $m=\left\lfloor y_{\text {ave }}\right\rfloor$.
12. Calculate the coordinates $\left(x_{n}, y_{n}\right)$ of the center of the bit corresponding to the value reported by the ADC,

$$
\begin{equation*}
\left(x_{n}=n+\frac{1}{2}, y_{m}=m+\frac{1}{2}\right)=\left(\left\lfloor x_{\mathrm{ave}}\right\rfloor+\frac{1}{2},\left\lfloor y_{\mathrm{ave}}\right\rfloor+\frac{1}{2}\right) \tag{3}
\end{equation*}
$$

where

$$
\begin{align*}
& x_{n}=\left\lfloor x_{\mathrm{ave}}\right\rfloor+\frac{1}{2},  \tag{4}\\
& y_{n}=\left\lfloor y_{\mathrm{ave}}\right\rfloor+\frac{1}{2} . \tag{5}
\end{align*}
$$

13. Calculate the angle reported by the ADC such that

$$
\begin{equation*}
\theta_{\mathrm{ADC}_{n, m}}=\arctan \left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right) \tag{6}
\end{equation*}
$$

14. Calculate the angle quantization range for each bit $p$ having coordinates $(n, m)$.
i. The angle quantization range is taken to be the difference between adjacent pairs of acute angles of triangles formed at the intersections of the circle and each bit, such that

$$
\begin{align*}
& \Delta \theta_{p}=\theta_{i+1}-\theta_{i}  \tag{7}\\
& =\theta_{p+1, p+2}-\theta_{p, p+1} \tag{8}
\end{align*}
$$

ii. Note that $\theta_{i}$ and $\theta_{i+1}$ represent the angles of the two intersected sides of each bit $\left(\theta_{i}<\theta_{i+1}\right)$.
iii. Note that $\theta_{i}=\theta_{p, p+1}$ and $\theta_{i+1}=\theta_{p+1, p+2}$.
15. Calculate the angle quantization uncertainties ("high" and "low") for each bit ( $n, m$ ).
i. These are calculated such that they taken on positive values.
ii. The "low" angle quantization uncertainty is taken to be the difference between the angle reported by the ADC and the lower angle $\theta_{i}$, such that

$$
\begin{align*}
& \delta \theta_{\mathrm{ADC}, n, m, \mathrm{low}}=\theta_{\mathrm{ADC}, n, m}-\theta_{i},  \tag{9}\\
& =\theta_{\mathrm{ADC}, n, m}-\theta_{p, p+1} . \tag{10}
\end{align*}
$$

iii. The "high" angle quantization uncertainty is taken to be the difference between the upper angle $\theta_{i+1}$ and the angle reported by the ADC , such that

$$
\begin{align*}
& \delta \theta_{\mathrm{ADC}, n, m, \text { high }}=\theta_{i+1}-\theta_{\mathrm{ADC}, n, m}  \tag{11}\\
& =\theta_{p+1, p+2}-\theta_{\mathrm{ADC}, n, m} \tag{12}
\end{align*}
$$

## 3 Mapping the signal amplitude and angle to the amplitude and angle reported by the ADC

The proposed method is obtained in a straightforward manner using geometry as discussed in the preceding figures. The complexity arises when the actual sample obtained by the ADC is taken into account. The reason that the actual sample matters
is that, for a signal with a given amplitude, depending on the phase angle of the sample, the angle value reported by the ADC may have more or less error (that is, the angle quantization error) because of the bit that detects the sample. As will be shown in figures that follow, the angle quantization error is a function of angle, and for some values of the angle, the error can be minimal whereas for other values of the angle, the error can be much larger. From the understanding of the angular dependence of the angle quantization error provided in this paper, future work might take this effect into account in future ADCs and digital signal processing systems.
Table 1 shows the mapping of bit locations $(n, m)$ to bit numbers $p$ for a complex signal with maximum signal magnitude in a $1-, 2-$, 3 -, and $N$-bit ADC. For each bit ( $n, m$ ) intersected by the complex signal, the ADC reports the value of the center of the bit that is intersected by the complex signal. For each bit ( $n, m$ ), the bit center is located at coordinates $\left(n+\frac{1}{2}, m+\frac{1}{2}\right)$.

For a signal sampled with magnitude (radius) $A$ and phase angle that is contained within bit $(n, m)$, the $\operatorname{ADC}$ reports the angle $\theta_{\mathrm{ADC}, n, m}$ given by the expression,

$$
\begin{equation*}
\tan \theta_{\mathrm{ADC}, n, m}=\left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right) \tag{13}
\end{equation*}
$$

such that

$$
\begin{equation*}
\theta_{\mathrm{ADC}, n, m}=\arctan \left(\frac{m+\frac{1}{2}}{n+\frac{1}{2}}\right) \tag{14}
\end{equation*}
$$

Table 1 Mapping of bit locations ( $n, m$ ) to bit numbers $p$ for a complex signal with maximum signal magnitude in N -bit ADCs

| $\boldsymbol{N}$-bit ADC | $\boldsymbol{p}^{\text {th }}$ bit (in order of increasing angle) | $(\boldsymbol{n}, \boldsymbol{m})$ |
| :--- | :--- | ---: |
| $N=1$ | 1 | $(0,0)$ |
| $N=2$ | 1 | $(1,0)$ |
|  | 2 | $(1,1)$ |
| $N=3$ | 3 | $(0,1)$ |
|  | 1 | $(3,0)$ |
|  | 2 | $(3,1)$ |
|  | 3 | $(3,2)$ |
|  | 4 | $(2,2)$ |
|  | 5 | $(2,3)$ |
|  | 6 | $(1,3)$ |
|  | 7 | $(0,3)$ |
|  | 1 | $(N, 0)$ |
|  | 2 | $(N, 1)$ |
|  | $\vdots$ | $\vdots$ |
|  |  | $(0, N)$ |

For bit ( $n, m$ ), the ADC reports the amplitude $s_{\mathrm{ADC}, n, m}$ for the complex signal according to the expression,

$$
\begin{equation*}
s_{\mathrm{ADC}, n, m}=\sqrt{\left(n+\frac{1}{2}\right)^{2}+\left(m+\frac{1}{2}\right)^{2}} \tag{15}
\end{equation*}
$$

As a result of the quantization, the reported amplitude is not generally equal to the complex signal amplitude.

The values of the angle obtained $\left\{\theta_{\mathrm{ADC}, \mathrm{n}, \mathrm{m}}\right\}$ are then sorted in increasing value for those angles contained within bits $(n, m)$ with distance to the signal magnitude less than or equal to the distance between the center of the bit and the closest point in the bit to the origin of the $\operatorname{ADC}\left(\frac{\sqrt{2}}{2}\right)$. The sorted values represent the angle quantization values of the $N$-bit ADC.

### 3.1 Angle quantization range

Figure 8a shows the angle quantization range for sampled signal $s$ in an $N$-bit ADC. In the figure, the sampled bit $(n, m)$ is in the $(p+1)^{t h}$ bit in order of increasing sampled angle in the first quadrant of the ADC. Each bit is identified by its coordinates ( $n, m$ ) in the lower left corner with corners that are adjacent to neighboring bits $\{(n, m+1),(n+1, m+1),(n+1, m)\}$. The lower and upper angles at which the sampled signal intersects the bit ( $\mathrm{n}, \mathrm{m}$ ) are denoted by the terms $\theta_{p+1, p+2}$ and $\theta_{p, p+1}$, respectively, where coordinates $(n, m)$ denote the $p^{t h}$ bit sampled by the signal with maximum amplitude. The coordinates $(p, p+1)$ indicate that the maximum signal intersects the $p^{\text {th }}$ bit $(n, m)$ at the boundary between bit $p$ and bit $p+1$, and the coordinates $(p+1, p+2)$ indicate that the maximum signal intersects the $p^{t h}$ bit at the boundary between bit $p+1$ and bit $p+2$. We let $\theta_{p+1, p+2}$ represent the angle at which the complex signal amplitude intersects the $p^{t h}$ bit with coordinates $(n, m)$ at the boundary between the $(p+1)^{t h}$ bit and the $(p+2)^{\text {th }}$ bit. We let $\theta_{p, p+1}$ represent the angle at which the complex signal amplitude intersects the $p^{t h}$ bit $(n, m)$ at the boundary between the $p^{t h}$ bit and the $(p+1)^{t h}$ bit.

We introduce the angular quantization range $\theta_{\mathrm{ADC}, n, m}$ of the ADC as the angular range between the angles at which the maximum signal amplitude intersects the $p^{\text {th }}$ bit $(n, m)$ according to

$$
\begin{equation*}
\theta_{\mathrm{ADC}, n, m}=\theta_{p+1, p+2}-\theta_{p, p+1} \tag{16}
\end{equation*}
$$

The ADC reports the value of the angle $\theta_{\mathrm{ADC}, n, m}$ in the $p^{t h}$ bit with coordinates $(n, m)$ at the bit center (' $X^{\prime}$ ), namely ( $n+\frac{1}{2}, m+\frac{1}{2}$ ) (see Fig. 8a).

Consider the case in which the complex signal magnitude takes on a constant value of the maximum signal amplitude $c=\frac{2^{N}}{2}$. The length of the hypotenuse of each right triangle then has length $\frac{N}{2}$ (that is, the maximum signal amplitude). One of the other sides of the triangle has lengths $\left\{j=1,2, \ldots, \frac{2^{N}}{2}\right\}$, as discussed above. The length of the third side of each triangle, $b_{p, p+1}$, can then be obtained from $c$ and $a_{p, p+1}$ according to the Pythagorean Theorem, such that

$$
\begin{equation*}
b_{p, p+1}=\sqrt{c^{2}-a_{p, p+1}^{2}} . \tag{17}
\end{equation*}
$$



Fig. 8 a Angle quantization range (green dashed semicircle), $\mathbf{b}$ angle quantization uncertainty, both "low" and "high" (green dashed arcs), c angle quantization error (green dashed arc) in an N -bit ADC

We consider the $j^{t h}$ right triangle for which the side parallel to the abscissa is equal to an integer number of bits. In this case, the angle of the $j^{\text {th }}$ triangle relative to the $x$-axis is given by the expression,

$$
\begin{align*}
& \tan \theta_{j}=\frac{j}{\sqrt{c^{2}-j^{2}}}  \tag{18}\\
& =\frac{j}{\sqrt{\left(\frac{2^{N}}{2}\right)^{2}-j^{2}}} \tag{19}
\end{align*}
$$

where $j=\left\{1,2, \ldots, \frac{2^{N}}{2}-1\right\}$. Next, we consider the $k^{t h}$ right triangle for which the side parallel to the ordinate is equal to an integer number of bits. In this case, the angle of the $k^{t h}$ triangle relative to the $x$-axis is given by,

$$
\begin{align*}
& \tan \theta_{k}=\frac{\sqrt{c^{2}-k^{2}}}{k}  \tag{20}\\
& =\frac{\sqrt{\left(\frac{2^{N}}{2}\right)^{2}-k^{2}}}{k} \tag{21}
\end{align*}
$$

where $k=\left\{1,2, \ldots, \frac{2^{N}}{2}-1\right\}$.
These angles are ordered according to increasing angle $\left\{\theta_{j}, \theta_{k}\right\}$. The angles in the sorted list are then grouped into a sequential list of angle pairs. Each pair of angles is an arc segment and an angle quantization range for the ADC. The series of the angle pairs forms a series of arc segments and the set of angle quantization ranges for the ADC. We let $M$ denote the number of bits $p$ that intersect the complex signal represented by a circle taking on a radius equal to the magnitude of the complex signal in the $x-y$ plane. Then, the sequence of arc segments between the pairs of points along adjacent bits that intersect the maximum signal amplitude can be written in order of increasing angle as

$$
\widehat{\theta_{k, k+1} \theta_{k+1, k+2}}, \text { where } \theta_{k=0,1}=0 \text { radians for } k=0,1, \ldots, M
$$

### 3.2 Angle quantization uncertainty

The angle quantization uncertainty $\delta \theta_{\mathrm{ADC}, n, m, \text { low }}$ for bit $(n, m)$ is defined as the difference between the angle $\theta_{\mathrm{ADC}, n, m}$ and the lower angle corresponding to the point of intersection at the side of the bit, such that

$$
\begin{equation*}
\delta \theta_{\mathrm{ADC}, n, m, \mathrm{low}}=\theta_{\mathrm{ADC}, n, m}-\theta_{p, p+1} \tag{22}
\end{equation*}
$$

Similarly, the angle quantization uncertainty $\delta \theta_{\mathrm{ADC}, n, m, \text { high }}$ for bit $(n, m)$ is defined as the difference between the angle $\theta_{\mathrm{ADC}, n, m}$ and the upper angle corresponding to the point of intersection at the upper side of the bit, such that

$$
\begin{equation*}
\delta \theta_{\mathrm{ADC}, n, m, \text { high }}=\theta_{p+1, p+2}-\theta_{\mathrm{ADC}, n, m} \tag{23}
\end{equation*}
$$

Figure 8b shows the "low" angle quantization uncertainty $\delta \theta_{\mathrm{ADC}, n, m, \text { high }}$ and "high" angle quantization uncertainty $\delta \theta_{\mathrm{ADC}, n, m, \text { low }}$ for an $N$-bit ADC.

### 3.3 Angle quantization error

We introduce the $\operatorname{ADC}$ quantization sampling error $\delta \theta_{s, n, m}$ in an $N$-bit ADC as the difference between the actual angle of the sampled signal $\theta_{\mathrm{ADC}, n, m}$ and the angle $\theta_{s, n, m}$ reported by the ADC for this sampled signal, such that

$$
\begin{equation*}
\delta \theta_{s, n, m}=\theta_{s, n, m}-\theta_{\mathrm{ADC}, n, m} . \tag{24}
\end{equation*}
$$

Figure 8c shows the angle quantization error $\delta \theta_{s, n, m}$ in an $N$-bit ADC.
The sequence of arc segments between the pairs of points intersecting the maximum signal follow from Tables 3 and 4 for the $N$-bit ADC in order of increasing angle as
$\widehat{\theta_{k, k+1} \theta_{k+1, k+2}}$, where $\theta_{k=0,1}=0$.

## 4 Application to $\mathbf{N}=\mathbf{1}, \mathbf{2}, 3$, and 10-bit ADCs

Figures 9 and 10 show the angle quantization error and output for a (a) 1-bit, (b) 2-bit, and (c) 3-bit ADC, respectively, where the amplitude of the input complex signal takes on the maximum amplitude. For the 1 -bit ADC , there is one bit that intersects the complex signal, namely $p=1$. For the 2 -bit ADC , there are three bits (in the first quadrant) that intersect the complex signal taking on the maximum signal amplitude; these bits are $p=\{1,2,3\}$.
For the first quadrant of the 3-bit ADC, seven bits of the ADC intersect the signal with maximum radius and seven corresponding arc segments with lengths given by the sequence in order of increasing angle as

$$
\widehat{\theta_{k, k+1} \theta_{k+1, k+2}}=\left\{\widehat{\theta_{0,1} \theta_{1,2}}, \widehat{\theta_{1,2} \theta_{2,3}}, \widehat{\theta_{2,3} \theta_{3,4}}, \widehat{\theta_{3,4} \theta_{4,5}}, \widehat{\theta_{4,5} \theta_{5,6}}, \widehat{\theta_{5,6} \theta_{6,7}}, \widehat{\theta_{6,7} \theta_{7,8}}\right\} .
$$

For the 3-bit ADC, the actual values of the angular quantization ranges for each $p^{\text {th }}$ bit (assuming a complex signal with maximum signal amplitude) can be written as

$$
\begin{aligned}
& \left\{0 \arctan \frac{1}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-1^{2}}}\right. \\
& \arctan \frac{\widehat{1}}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-1^{2}}} \arctan \frac{2}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-2^{2}}} \\
& \arctan \frac{2}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-2^{2}}} \arctan \frac{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-3^{2}}}{3} \\
& \arctan \frac{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-3^{2}}}{3} \arctan \frac{3}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-3^{2}}} \\
& \arctan \frac{3}{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-3^{2}}} \arctan \frac{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-2^{2}}}{2} \\
& \arctan \frac{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-2^{2}}}{2} \arctan \frac{\sqrt{\left(\frac{2^{3}}{2}\right)^{2}-1^{2}}}{1}
\end{aligned}, .
$$

Table 2 shows the values of the quantization angles $\theta_{\mathrm{ADC}, n, m}$ and quantization signals $s_{\mathrm{ADC}, n, m}$ of the 3 -bit ADC . The values of the remaining quantization angles for the other three quadrants can be obtained using symmetry).


Fig. 9 Angle quantization error of $\mathbf{a}$ 1-bit, $\mathbf{b} 2$-bit, and $\mathbf{c} 3$-bit ADCs, where the amplitude of the input complex signal takes on the maximum amplitude


Fig. 10 Digital output of a 1-bit, b 2-bit, and c 3-bit ADCs, where the amplitude of the input complex signal takes on the maximum amplitude

Table 2 ADC angle quantization $\theta_{A D C, n, m}$ and $A D C$ signal quantization $s_{A D C, n, m}$ (first quadrant)

|  |  |  |  | $\left(x_{n}=n+\frac{1}{2}, y_{m}=m+\frac{1}{2}\right)$ | $s_{\text {ADC, } n, m}=\sqrt{\left(x_{n}\right)^{2}+\left(y_{m}\right)^{2}}$ | $\theta_{\mathrm{ADC}, n, m}=\arctan \left(\frac{y_{m}}{x_{n}}\right)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N=1$ | 1-bit ADC | $n$ | $m$ | $\left(x_{n}, y_{m}\right)$ | $s_{\text {ADC, }, \text {, m }}$ | $\theta_{\text {ADC, }, \text {, } m}$ |
|  | $(0,0)$ | 0 | 0 | \{0.5, 0.5\} | $S_{A D C, 1,1}=\sqrt{(0.5)^{2}+(0.5)^{2}}$ | $\theta_{\text {ADC, }, 1,1} \simeq 0.2500 \pi$ |
| $N=2$ | 2-bit ADC | $n$ | m | $\left(x_{n}, y_{m}\right)$ | $S_{\text {ADC, }, \text {, m }}$ | $\theta_{\text {ADC, }, \text {, m }}$ |
|  | $(1,0)$ | 1 | 0 | $\{1.5,0.5\}$ | $S_{A D C, 1,0}=\sqrt{(1.5)^{2}+(0.5)^{2}}$ | $\theta_{\text {ADC, }, 1,0} \simeq 0.1024 \pi$ |
|  | $(1,1)$ | 1 | 1 | $\{1.5,1.5\}$ | $S_{A D C, 1,1}=\sqrt{(1.5)^{2}+(1.5)^{2}}$ | $\theta_{\text {ADC, }, 1,1} \simeq 0.2500 \pi$ |
| fv | $(0,1)$ | 0 | 1 | $\{0.5,1.5\}$ | $S_{A D C, 0,1}=\sqrt{(0.5)^{2}+(1.5)^{2}}$ | $\theta_{\text {ADC, }, 0,1} \simeq 0.3976 \pi$ |
| $N=3$ | 3-bit ADC | n | $m$ | $\left(x_{n}, y_{m}\right)$ | SADC, $n, m$ | $\theta_{\text {ADC, }, \text {, m }}$ |
|  | $(3,0)$ | 3 | 0 | $\{3.5,0.5\}$ | $S_{A D C, 3,0}=\sqrt{(3.5)^{2}+(0.5)^{2}}$ | $\theta_{\text {ADC, }, 3,0} \simeq 0.0455 \pi$ |
|  | $(3,1)$ | 3 | 1 | \{3.5, 1.5\} | $s_{\text {ADC, } 3,1}=\sqrt{(3.5)^{2}+(1.5)^{2}}$ | $\theta_{\text {ADC, }, 3,1} \simeq 0.1289 \pi$ |
|  | $(3,2)$ | 3 | 2 | $\{3.5,2.5\}$ | $S_{A D C, 3,2}=\sqrt{(3.5)^{2}+(2.5)^{2}}$ | $\theta_{\text {ADC,3,2 }} \simeq 0.1974 \pi$ |
|  | $(2,2)$ | 2 | 2 | $\{2.5,2.5\}$ | $S_{A D C, 2,2}=\sqrt{(2.5)^{2}+(2.5)^{2}}$ | $\theta_{A D C, 2,2} \simeq \frac{\pi}{4}$ |
|  | $(2,3)$ | 2 | 3 | $\{2.5,3.5\}$ | $S_{A D C, 2,3}=\sqrt{(2.5)^{2}+(3.5)^{2}}$ | $\theta_{\text {ADC }, 2,3} \simeq 0.3026 \pi$ |
|  | $(1,3)$ | 1 | 3 | \{1.5, 3.5\} | $s_{\text {ADC, } 1,3}=\sqrt{(1.5)^{2}+(3.5)^{2}}$ | $\theta_{\text {ADC, }, 1,3} \simeq 0.3711 \pi$ |
|  | $(0,3)$ | 0 | 3 | \{0.5, 3.5\} | $s_{\text {ADC, }, 0,3}=\sqrt{(0.5)^{2}+(3.5)^{2}}$ | $\theta_{\text {ADC }, 0,3} \simeq 0.4545 \pi$ |

Table 3 ADC angle quantization range for complex signal with maximum signal amplitude in 1-bit and 2-bit ADCs

| $N \text {-bit }$ ADC | $a_{p, p+1}$ | $b_{p, p+1}$ | $c_{\text {max }}=s_{\text {max }}$ | Angle $\theta_{\text {p, }+1}$ | ADC angle quantization range in $p^{\text {th }}$ bit |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N=1$ | $\begin{aligned} & a_{p, p+1} \\ & a_{0,1}=0 \end{aligned}$ | $\begin{aligned} & b_{p, p+1} \\ & b_{0,1}=0 \end{aligned}$ | $\begin{aligned} & 2^{N-1} \\ & s_{\max }=1 \end{aligned}$ | $\begin{aligned} & \theta_{p, p+1} \\ & \tan \theta_{0,1}=0 \end{aligned}$ | $\begin{aligned} & \Delta \theta_{p} \\ & \Delta \theta_{p=1}=\widehat{\theta_{0,1} \theta_{1,2}}=\theta_{1,2}-\theta_{0,1} \end{aligned}$ |
| $N=2$ | $\begin{aligned} & a_{p, p+1} \\ & a_{1,2} \\ & a_{2,3}=1 \end{aligned}$ | $\begin{aligned} & b_{p, p+1} \\ & b_{1,2}=1 \\ & b_{2,3} \end{aligned}$ | $\begin{aligned} & 2^{N-1} \\ & s_{\text {max }}=2 \end{aligned}$ $s_{\max }=2$ | $\begin{aligned} & \theta_{p, p+1} \\ & \tan \theta_{1,2}=\frac{1}{\sqrt{\left(\frac{2^{2}}{2}\right)^{2}-1^{2}}} \\ & \tan \theta_{2,3}=\frac{\sqrt{\left(\frac{2^{2}}{2}\right)^{2}-1^{2}}}{1} \end{aligned}$ | $\Delta \theta_{p}$ $\Delta \theta_{p=1}=\widehat{\theta_{0,1} \theta_{1,2}}=\theta_{1,2}-0=\arctan \frac{1}{\sqrt{\left(\frac{2}{2}\right)^{2}-1^{2}}}$ $\Delta \theta_{p=2}=\widehat{\theta_{1,2} \theta_{2,3}}=\theta_{2,3}-\theta_{1,2}=\frac{\sqrt{\left(\frac{2^{2}}{2}\right)^{2}-1^{2}}}{1} .$ |

Tables 3 and 4 show the angle quantization range for each bit of the 1-bit, 2-bit, and 3-bit ADCs. The tables list the values of the angles at the limits of the angle quantization range for each bit (the limits correspond to the values of the angles of the triangles with sides $\left\{a_{p, p+1}, b_{p, p+1}\right\}$ intersecting the signal with maximum signal amplitude (first quadrant) in each ADC). Here, the notation $\{p, p+1\}$ denotes the two adjacent bits having sides that are intersected by the signal with maximum amplitude that has length $s_{\max }$ given by

$$
\begin{equation*}
s_{\max }=c_{\text {maximum signal }}, \tag{25}
\end{equation*}
$$

$$
\begin{align*}
& =\frac{2^{N}}{2}  \tag{26}\\
& =4 \text { bits. } \tag{27}
\end{align*}
$$

For the case in which the sampled angle is within any of the given arcs, the corresponding (quantized) angle reported by the ADC will be $\theta_{\mathrm{ADC}, n, m}=\left\{\theta_{\mathrm{ADC}, 3,0}, \theta_{\mathrm{ADC}, 3,1}, \theta_{\mathrm{ADC}, 3,2}\right.$, $\left.\theta_{\mathrm{ADC}, 2,2}, \theta_{\mathrm{ADC}, 2,3}, \theta_{\mathrm{ADC}, 1,3}, \theta_{\mathrm{ADC}, 0,3}\right\}$, respectively, which take on the corresponding values $\left\{0.0455 \pi, 0.1289 \pi, 0.1974 \pi, \frac{\pi}{4}, 0.3026 \pi, 0.3711 \pi, 0.4545 \pi\right\}$, as shown in Tables 3 and 4 .
Figure 11 shows the output of a 10 -bit ADC as a function of input angle for input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians. The dashed sloped line indicates the result for an ideal ADC in which there is no quantization error. The upper figure, Fig. 11a, shows that the angle reported by the ADC for the entire range of 0 to $\frac{\pi}{4}$ radians. In Fig. 11b, the staircase structure is evident, and the figure shows that the angle reported by the ADC takes on quantized values. The horizontal stairs shown in the figure have nonuniform length because the arclength of the sampled values through each bit of the ADC is not uniform since the ADC is modeled as a square array.

Figure 12 shows the angle quantization range (a), angle quantization uncertainty (low) (b), and angle quantization uncertainty (high) (c) of a 10-bit ADC as a function of input angle between 0 and $\frac{\pi}{4}$ radians. The results are obtained by sampling the input signal at 10,000 data points. The results in (a) show that the angle quantization range increases from approximately $6 \frac{\mathrm{rad}}{\pi}$ at small angles and increase to approximately $8 \frac{\mathrm{rad}}{\pi}$ as the angle approaches $\frac{\pi}{4}$ in the first half quadrant. At the same time, the results also show that the angle quantization range can take on much smaller values, close to $0 \frac{\mathrm{rad}}{\pi}$, when the sampled signal is close to one of the corners of a bit in the ADC.

The results in Fig. 12 show that the angle quantization ranges "high" and "low" take on values close to $\approx \pm 3 \frac{\mathrm{rad}}{\pi}$ for small angles and that the range of the angle quantization range tends to increase as the angle increases. The figures also show the underlying structure in the "high" and "low" angle quantization ranges. This structure reflects the underlying square structure in the model of the ADC that is capturing the sampled signal.

Figure 13a and b show histograms of the angle quantization uncertainty (Fig. 12) to the lower edge of the bit containing the maximum signal and to the upper edge of the bit, respectively, containing the maximum signal of a 10 -bit ADC as a function of input angle for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians with 300 bins. The red lines show fits with the extreme value distribution with mean 0.000336 and standard deviation $6.4 \times 10^{-5}$ in Fig. 13a and mean 0.000334 and standard deviation $6.3 \times 10^{-5}$ in Fig. 13b.

As the number of bits in the ADC increases, the size of each bit decreases, and the nonuniformity of where the circle representing the signal is intersecting each bit has a greater variation, and the value reported by the ADC becomes more sensitive to the actual sample taken of the signal.
Figure 14 shows the angle quantization error of a 10 -bit ADC as a function of input angle for input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians. The results in this figure show that the angle quantization error is centered at zero $\frac{\mathrm{rad}}{\pi}$
Table 4 ADC angle quantization range for complex signal with maximum signal amplitude in a 3-bit ADC



Fig. 11 Output (solid dots) of a 10-bit ADC as a function of input angle for an input signal sampled at 10,000 data points for angles between $\mathbf{a} 0$ and $\frac{\pi}{4}$ radians and $\mathbf{b} 0.1$ to 0.104 radians showing the quantization by the ADC. The dashed line indicates the result for an ideal ADC (no quantization error). The horizontal dotted blue lines indicate the quantization levels of the ADC; these levels are not visible in (a)
and takes on values in the range $\approx \pm 3 \frac{\mathrm{rad}}{\pi}$. The results in the figure also show that the range of values increase as the angle increases to $\frac{\pi}{2}$ to $\approx \pm 3 \frac{\mathrm{rad}}{\pi}$. The figures also show the underlying structure in the angle quantization error that reflects the underlying square structure in the model of the ADC.
Figure 15 shows histograms of angle quantization error of a 10 -bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for (a) 100 bins, (b) 200 bins, and (c) 300 bins. The results in these figures show that the histograms are centered on an angle quantization error equal to zero and range as large as $\approx \pm 4 \frac{\mathrm{rad}}{\pi}$. The red curves in the histograms show a fit to the Normal distribution (Gaussian) with mean $-3.2 \times 10^{-7}$ and standard deviation 0.00018145 .
Figure 16 shows the (a) signal quantization for a 10 -bit ADC with 10,000 samples as a function of angle from 0 to $\frac{\pi}{4}$ radians and (b) difference in signal quantization, where the difference is obtained by subtracting the signal (radius) to the lower left corner of the


Fig. 12 Angle quantization range (a), angle quantization uncertainty (to lower edge of bit containing the maximum signal) (b), and angle quantization uncertainty (c) of a 10-bit ADC as a function of input angle for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians


Fig. 13 Histograms of the angle quantization uncertainty shown in Fig. $12 \mathbf{a}$ to lower bit edge containing the maximum signal and $\mathbf{b}$ to upper bit edge containing the maximum signal of a 10 -bit ADC as a function of input angle for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians with 300 bins. The red lines show fits with the extreme value distribution with mean 0.000336 and standard deviation $6.4 \times 10^{-5}$ in (a) and mean 0.000334 and standard deviation $6.3 \times 10^{-5}$ in (b)

ADC square (containing the signal) from the signal (radius) to the upper right corner of the same square. The results show that the value of the difference ranges from 1 to $\frac{\sqrt{2}}{2}$.
Figure 17 shows histograms of signal quantization error of a 10-bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for (a) 100 bins, (b) 200 bins, and (c) 300 bins. The results in these figures show that the histograms are


Fig. 14 Angle quantization error of a 10-bit ADC as a function of input angle for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians
centered on an angle quantization error equal to 512 and range approximately from 511.3 to 512.7. The red curves in the histograms show a fit to the Normal distribution (Gaussian) with mean 511.98 and standard deviation 0.29.
Figure 18 shows histograms of the product of the signal quantization error and the angle quantization error of a 10 -bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for (a) 100 bins, (b) 200 bins, and (c) 300 bins. The red line in the figures shows fits using the kernel distribution with bandwidth $1.45 \times 10^{-5}$.

## 5 Conclusions

The main contribution of this paper is a method to determine estimates of the angle quantization error in $N$-bit ADCs with the aim toward obtaining a quantitative understanding of the inherent nature of the quantization error which has the potential to lead to additional capabilities in error vector magnitude (EVM) calculations and, ultimately, to correct for the error. In this way, this approach could have the desirable effect to improve the ADC performance closer to the quantum limit, which is desirable for RF signal estimation applications in congested environments. The signal-to-noise ratio can be obtained by dividing the signal (radius of the circle) by whichever noise characteristic is of interest (such as angle quantization error and signal quantization error).
Results obtained with the proposed method show that the magnitude of the angle quantization uncertainty and error take on values that are less than $\sim 5 \times 10^{-4} \pi$ radians. The results for the 10 -bit ADC show that the magnitude of the angle quantization uncertainty, which is the difference in angle (within the bit that captures the input signal) and the value of the angle reported by the ADC , is less than $\sim 5 \times 10^{-4} \pi$ radians.


Fig. 15 Histograms of angle quantization error of a 10 -bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for a 100 bins, b 200 bins, and c 300 bins. The red curves in the histograms show a fit to the Normal distribution (Gaussian) with mean $-3.2 \times 10^{-7}$ and standard deviation 0.00018145


Fig. 16 a Signal quantization for a 10-bit ADC with 10,000 samples as a function of angle from 0 to $\frac{\pi}{4}$ radians. b Difference in signal quantization, where the difference is obtained by subtracting the signal (radius) to the lower left corner of the ADC square (containing the signal) from the signal (radius) to the upper right corner of the same square

The results also show that the magnitude of the angle quantization error is less than $\sim 5 \times 10^{-4} \pi$ radians.

## 6 Future work

Quantization error is inherent in the signal conversion process, and there is a need to quantify the quantization error $[32,33]$ introduced by an ideal ADC in the conversion of complex signals. In addition, a quantitative understanding of the inherent nature of the quantization error has the potential to lead to methodologies to correct for the error and thereby could have the desirable effect to improve the ADC performance toward the quantum limit [14].

The method that we are using to estimate the weak signal parameters is for the situation in which there are $N=2$ simultaneous signals, and we are currently starting to estimate weak signal parameters for the situations in which $N=3$ and $N=4$ because


Fig. 17 Histograms of signal quantization error of a 10-bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for a 100 bins, b 200 bins, and $\mathbf{c} 300$ bins. The red curves in the histograms show a fit to the Normal distribution (Gaussian) with mean 511.98 and standard deviation 0.29
these cases are of interest in developing an electronic warfare receiver. The method is not yet able to respond in a completely blind scenario; however, the method to quantify error introduced by an ADC is general and can be applied for any signal since the signal amplitude will intersect at least one bit of the ADC. Recognizing this, future work will explore more common scenarios and additionally take into account the effect of the quantized sampled instantaneous frequency on the estimation of the weak signal parameters (See green box in Fig. 3). Future work will consider applications of the


Fig. 18 Histograms of the product of the signal quantization error and the angle quantization error of a 10-bit ADC for an input signal sampled at 10,000 data points for angles between 0 and $\frac{\pi}{4}$ radians, for a 100 bins, b 200 bins, and c 300 bins. The red line in the figures show fits using the kernel distribution with bandwidth $1.45 \times 10^{-5}$
zero-crossing phase calculation approaches to the situation in which the signal is timevarying, such as a multi-tier detection process [2] and to the situation in which the weak signal is linear FM pulsed waveform.

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## Author contributions

The authors equally contributed to the paper. M.L. wrote the drafts of the paper, code, and figures. A. Q. and C. C. read the paper and gave feedback.

## Availability of data and materials

Please contact author for data requests.

## Declarations

## Competing interests

The authors declare that they have no competing interests.
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