EURASIP Journal on Advances
in Signal Processing

RESEARCH **Open Access**

Nonuniform sampling and reconstruction of Diracs signal associated with linear canonical transform and its application

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Abstract

Sampling and reconstruction play a critical role in signal processing. The non-ideal sampling conditions motivate the development of the sampling theory. In this paper, associated with multiple non-ideal conditions, we discuss the nonuniform sampling and reconstruction of nonbandlimited signal in the linear canonical transform (LCT) domain with finite samples. The Diracs signal is nonbandlimited in the LCT domain but has the finite rate of innovation property. The sampling of the Diracs signal in the LCT domain is analyzed firstly. Secondly, the reconstruction of the signal with finite nonuniform samples is discussed, including two cases where the nonuniform sampling instants are known or unknown. Finally, the numerical experiment verifies the effect of the reconstruction algorithm, and the potential applications and generalized analysis indicate the value of the research.

Keywords: Linear canonical transform, Signal reconstruction, Nonbandlimited signal, Finite rate of innovation, Denoising

1 Introduction

The sampling and reconstruction are a crucial research interests in signal processing fields. Shannon sampling theorem indicates that a bandlimited signal in frequency or Fourier transform (FT) domain can be perfectly reconstructed by the uniform samples with a higher sampling rate than the Nyquist rate [1]. In the theorem, some restricted conditions for perfect reconstruction are required, such as the bandlimited signal in frequency domain, uniform sampling and infinite samples. In order to suitable for more complex situations, it need to relax the restricted conditions, which motives the developing of the sampling and reconstruction theory.

The linear canonical transform (LCT) is a generalized form of the FT and it includes three additional free parameters. For some nonbandlimited signals, they may be bandlimited in a specific LCT domain. Therefore, the sampling theorem associated with the LCT based on the Shannon sampling theorem was proposed and aroused attention [2, 3]. The sampling theorem associated with LCT has many research findings, including uniformly or nonuniformly sampling, finite samples, multichannel sampling and so on

[4–13]. Sharma and Sharma [14] discussed the signal reconstruction using undersampled signals taken in multiple LCT domains. Zhang et al. [15] studied the multichannel sampling expansions of LCT with explicit system functions and finite samples. Annaby et al. [16] incorporated two different regularization kernels in the sampling reconstruction of bandlimited signals to remarkably enhance the convergence rate in the LCT domain. Multichannel consistent sampling and reconstruction in the LCT domain were analyzed in reference [17]. Nonuniform sampling theorems of random signals in LCT domain were also discussed [18, 19]. However, these sampling theorems are all suitable for the bandlimited signals in the LCT domain. There are a few studies about the sampling and reconstruction of nonbandlimited signals in the LCT domain.

For the sampling and reconstruction of nonbandlimited signal, Liu et al. [20] introduced a sampling formula by constructing a class of function spaces, but there were no normative rules to determine the parameters. Shi et al. [21, 22] proposed the sampling theorem of the LCT without band-limiting constrain in the function spaces, which were derived based on Riesz bases. Bhandari and Marziliano [23] discussed the sampling and reconstruction of sparse signals in fractional Fourier transform (FRFT) sense based on the finite rate of innovation (FRI) theory. The sampling and reconstruction based on FRI theory is actually using the finite degree of freedom to represent the signal [24]. Xin et al. [25] discussed a novel Sub-Nyquist FRI sampling and reconstruction method in the LCT domain, but for the uniform sampling. Sub-Nyquist sampling system for pulse streams based on non-ideal filters was also analyzed in reference [26]. In [27], sampling theorems for certain types of nonbandlimited quaternionic signals were proposed, where the signals were reconstructed from its samples as well as the samples of its generalized Hilbert transforms associated with quaternion FT and LCT. These researches were presented from different sampling conditions and reconstruction methods, while we want to combine the nonbandlimted condition with nonuniform sampling and finite samples to develop the reconstruction theory.

In this paper, we will discuss the sampling and reconstruction of a nonbandlimited signal in the LCT domain using nonuniform samples and FRI theory. The FRI theory provides theoretical support for the reconstruction from finite samples. In the following analysis, the periodic Diracs signal will be as the research object and the potential application will be presented. Our research will have huge development potential. On the one hand, the analytical method is based on the LCT, it can be generalized to other time-frequency analysis transform tools and analyze the reconstruction in other transform domains. On the other hand, the analysis procedure can be naturally and conveniently generalized to more complex signals, such as splines and piecewise polynomial signals. A signal $s(t)$ is a periodic nonuniform splines of degree R with K knots in one period if and only if its $(R+1)$th derivative is a periodic stream of K weighted Diracs signal. Similarly, a signal $s(t)$ is a periodic piecewise polynomial with K pieces each of maximum degree R if and only if its $(R+1)$ derivative is a stream of differentiated Diracs signal [24]. That is to say, many complex signals can be derived into the Diracs signal. Therefore, the research of this paper has the potential to be generalized to more complex situations.

This paper is organized as follows. In Sect. 2, the related knowledge of the LCT and the FRI theory are introduced firstly. Nonuniform sampling and reconstruction analysis of Diracs signal with FRI in the LCT domain are discussed in detail in Sect. 3. Section 4 presents the numerical experiments and the potential applications. The conclusion follows in Sect. 5.

2 Preliminaries

2.1 The linear canonical transform

The LCT of a continuous signal $f(t)$ with real parameter $\mathbf{M} = (a, b; c, d)$ is defined as [2]

$$L_{\mathbf{M}}[f(t)](u) = F_{\mathbf{M}}(u) = \begin{cases} B_{\mathbf{M}} \int_{-\infty}^{\infty} f(t) K_{\mathbf{M}}(u,t) \mathrm{dt}, & b \neq 0 \\ \sqrt{d} e^{j(cd/2)u^2} f(\mathrm{du}), & b = 0 \end{cases} \tag{1}$$

where $K_{\mathbf{M}}(u,t) = e^{j\frac{1}{2}(\frac{a}{b}t^2 - \frac{2}{b}tu + \frac{d}{b}u^2)}$ is the kernel of the transform, $B_{\mathbf{M}} = \sqrt{1/(j2\pi b)}$, and $\det(\mathbf{M}) = ad - bc = 1$. Two successive LCTs with parameter matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ is another LCT with the matrix $\mathbf{M}_3 = \mathbf{M}_2\mathbf{M}_1$. Thus, the inverse transform of the LCT is given by the LCT with parameter $\mathbf{M}^{-1} = (d, -b; -c, a)$.

As the LCT of a signal is just a chirp multiplication when $b = 0$, which is of no particular interest, we do not discuss this case in our work. Without loss of generality, we assume $b \neq 0$ in this paper. From the definition of the LCT, it is obvious that the LCT with special parameters could reduce to the transforms what is familiar, such as the FT when $\mathbf{M} = (0, 1; -1, 0)$, the FRFT when $\mathbf{M} = (\cos\alpha, \sin\alpha; -\sin\alpha, \cos\alpha)$, the Fresnel transform when $\mathbf{M} = (1, b; 0, 1)$ and so on.

When the convolution operator associated with the LCT is defined as

$$f(t) *_{\mathbf{M}} g(t) = \sqrt{\frac{1}{j2\pi b}} e^{-j\frac{a}{2b}t^2} \left[\left(f(t) e^{j\frac{a}{2b}t^2} \right) * \left(g(t) e^{j\frac{a}{2b}t^2} \right) \right], \tag{2}$$

we can obtain the following convolution theorem

$$L_{\mathbf{M}}[f(t) *_{\mathbf{M}} g(t)] = e^{-j\frac{d}{2b}u^2} F_{\mathbf{M}}(u) G_{\mathbf{M}}(u), \tag{3}$$

where the symbol $*$ is the classical convolution operator, $*_{\mathbf{M}}$ denotes the linear canonical convolution operator, $F_{\mathbf{M}}(u)$ and $G_{\mathbf{M}}(u)$ are the LCTs of $f(t)$ and $g(t)$, respectively.

For a finite signal $f(t)$ in $(-\tau, \tau)$, its linear canonical transform series (LCTS) expansion with parameter $\mathbf{M} = (a, b; c, d)$ can be derived as [2]

$$f(t) = \sum_{n=-\infty}^{+\infty} \hat{f}_{\mathbf{M}}[n] \sqrt{\frac{j}{\tau}} e^{-j\frac{d}{2b}(n\frac{2\pi b}{\tau})^2 + j\frac{t}{b}(n\frac{2\pi b}{\tau}) - j\frac{a}{2b}t^2}, \tag{4}$$

where the LCTS expansion coefficients $\hat{f}_{\mathbf{M}}[n]$ is defined as

$$\hat{f}_{\mathbf{M}}[n] = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f(t) \sqrt{\frac{-j}{\tau}} e^{j\frac{d}{2b}(n\frac{2\pi b}{\tau})^2 - j\frac{t}{b}(n\frac{2\pi b}{\tau}) + j\frac{a}{2b}t^2} \mathrm{dt}, \tag{5}$$

where $t \in [-\tau/2, \tau/2]$ and τ is the length of the signal.

A continuous signal $f(t)$ is said to be bandlimited with bandwidth $\Omega_{\mathbf{M}}$ in the LCT domain with parameter matrix $\mathbf{M} = (a, b; c, d)$ when its transform spectrum in the LCT domain satisfies

$$F_{\mathbf{M}}(u) = 0, \text{ for } |u| > \Omega_{\mathbf{M}} \tag{6}$$

where $\Omega_{\mathbf{M}}$ is called the bandwidth of the signal *f*(*t*) in the LCT domain. For a continuous signal *f*(*t*) which is bandlimited $\Omega_{\mathbf{M}}$ in the LCT domain, it can be reconstructed from its uniform samples by the following uniform sampling theorem

$$f(t) = \exp\left(-j\frac{a}{2b}t^2\right)\sum_{n=-\infty}^{+\infty} f(nT)\exp\left[j\frac{a}{2b}(nT)^2\right]\mathrm{sinc}\left[\frac{\Omega_{\mathbf{M}}(t-nT)}{b}\right], \tag{7}$$

where T is the sampling interval and it satisfies the uniform sampling condition. The Nyquist rate associated with LCT is $\Omega_{\mathbf{M}}/(\pi b)$. More details about the properties of the LCT can be found in reference [2].

2.2 Signals with finite rate of innovation

The notion of the FRI sampling was first introduced by Vetterli et al. [24]. The parametric signal which has a finite number of degrees of freedom per unit time can be stated that it has the FRI property. Any real bandlimited signal can be seen as having $1/T$ degrees of freedom per unit of time, which is the number of samples per unit of time that can specify it. Thus, the number of degrees of freedom per unit of time is called the rate of innovation of a signal. For a signal with the FRI property, it can be interpreted as that the information of the signal can be presented by finite samples, which provides theoretical support for the reconstruction from finite samples.

The studies based on the rate of innovation theory have been researched a lot, including the sampling and reconstruction, its applications in signal denoising, ultrasound imaging and so on [28–30]. In this paper, we are interested in signals that have the FRI property, either on intervals or on average.

3 Nonuniform sampling and reconstruction of Diracs signal with FRI in the LCT domain

3.1 Nonuniform sampling of Diracs signal associated with the LCT

Most of the sampling theorems associated with the LCT is suitable to the signals which have a compact support in the LCT domain. In this paper, we focus on the nonbandlimited signal in the LCT domain and its reconstruction from finite nonuniform samples.

For a Dirac impulse $\delta(t)$, its LCT with parameters $\mathbf{M} = (a, b; c, d)$ is

$$L_{\mathbf{M}}[\delta(t)](u) = \hat{\delta}_{\mathbf{M}}(u) = \sqrt{\frac{1}{j2\pi b}}e^{j\frac{d}{2b}u^2}, \tag{8}$$

which is a chirp function and nonbandlimited in any LCT domain. For simplicity, the periodic Diracs signal will be as the research object in the following analysis, as it can be naturally and conveniently generalized to more complex signals, such as splines and piecewise polynomial signals. Then, we discuss the sampling of a continuous time periodic stream of Diracs by an acquisition device which deploys a sinc-based low-pass filter. A stream of K Diracs periodized with period τ is presented as

$$x(t) = \sum_{n=-\infty}^{+\infty} \sum_{k=0}^{K-1} c_k \delta(t - t_k - n\tau), \tag{9}$$

where unknown weights $\{c_k\}$ and arbitrary shifts $\{t_k\}$ satisfy $t_{n+K} = t_n + \tau$ and $c_{n+K} = c_n$, $\forall n \in \mathbb{Z}$. Based on the theory of FRI, the above signal has $2K$ degrees of freedom per period and its rate of innovation is $\rho = 2K/\tau$.

Based on the theory of LCTS, we can represented the periodic Diracs signal. Let $\Psi(t) = \sum_{n\in\mathbb{Z}} \delta(t - n\tau)$ be the τ-period Diracs comb, then its shifted form can be expressed as $\Psi_k(t) = \sum_{n\in\mathbb{Z}} \delta(t - t_k - n\tau) = \Psi(t - t_k)$. Based on the definition of the LCTS, $\Psi_k(t)$ can be expressed as follows

$$\Psi_k(t) = \sum_{n=-\infty}^{+\infty} \hat{\Psi}_{k\mathbf{M}}[n] \sqrt{\frac{j}{\tau}} e^{-j\frac{d}{2b}(n\frac{2\pi b}{\tau})^2 + j\frac{t}{b}(n\frac{2\pi b}{\tau}) - j\frac{a}{2b}t^2}, \tag{10}$$

where the LCTS expansion coefficients $\hat{\Psi}_{k\mathbf{M}}[n]$ can be derived as

$$\begin{aligned} \hat{\Psi}_{k\mathbf{M}}[n] &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} \delta(t - t_k) \sqrt{\frac{-j}{\tau}} e^{j\frac{d}{2b}(n\frac{2\pi b}{\tau})^2 - j\frac{t}{b}(n\frac{2\pi b}{\tau}) + j\frac{a}{2b}t^2} \mathrm{dt} \\ &= \sqrt{\frac{-j}{\tau}} e^{j\frac{d}{2b}(n\frac{2\pi b}{\tau})^2 - j\frac{t_k}{b}(n\frac{2\pi b}{\tau}) + j\frac{a}{2b}{t_k}^2}. \end{aligned} \tag{11}$$

Thus, Eq. (10) can be simplified as

$$\Psi_k(t) = \frac{1}{\tau} \sum_{n\in\mathbb{Z}} e^{j\frac{a}{2b}({t_k}^2 - t^2)} e^{j\frac{2\pi}{\tau}n(t - t_k)}. \tag{12}$$

Concisely, the periodic stream of Diracs $x(t)$ can be written as

$$x(t) = \sum_{k=0}^{K-1} c_k \Psi_k(t) = \frac{1}{\tau} \sum_{k=0}^{K-1} c_k \sum_{n\in\mathbb{Z}} e^{j\frac{a}{2b}({t_k}^2 - t^2)} e^{j\frac{2\pi}{\tau}n(t - t_k)}. \tag{13}$$

In practice, the uniform sampling is usually not realized because of multiple interfering factors and only finite samples can be used. In the following, we will analyze the nonuniform sampling and reconstruction of the Diracs signal associated with the LCT and FRI theory.

Be similar to the Shannon framework, the signal is being observed through a low-pass filter $h_B(t) = e^{-j\frac{a}{2b}t^2} \mathrm{sinc}(\frac{B}{b}t)$ and the sampling kernel is set to be $h_B(t - T_n) = e^{-j\frac{a}{2b}t^2} \mathrm{sinc}(\frac{B}{b}(t - T_n))$, where T_n is the sampling instant for $n = 0, 1, 2, \ldots, N-1$. Besides, the LCT of $h_B(t)$ can be derived as

$$L_{\mathbf{M}}[h_B(t)] = \frac{1}{\sqrt{j2\pi b}} e^{j\frac{d}{2b}u^2} \mathrm{rect}\Big(\frac{u}{2\pi B}\Big). \tag{14}$$

Thus, we can deduce that $h_B(t)$ is compactly supported over $[-B\pi, B\pi]$. We sample $x(t)$ by prefiltering with anti-aliasing filter $h_B(-t)$ followed by nonuniform sampling, i.e., $y(t) = x(t) *_{\mathbf{M}} h_B(-t)$ is sampled nonuniformly at locations T_n, which is illustrated in Fig. 1.

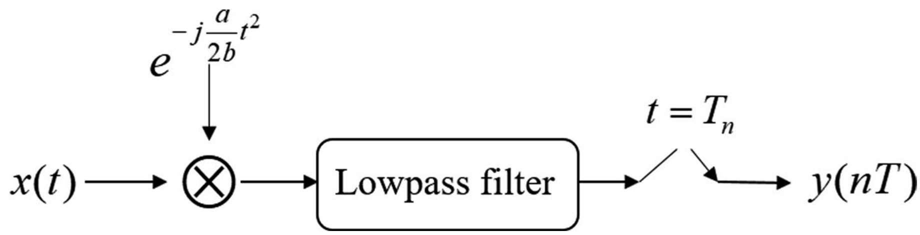

Fig. 1 The prefiltering and sampling of the signal

Furthermore, the signal $x(t)$ in Eq. (13) can be expressed as the following

$$x(t) = e^{-j\frac{a}{2b}t^2}\sum_{n\in\mathbb{Z}} p[n]e^{j\frac{2\pi}{\tau}nt}, \tag{15}$$

where $p[n] = \frac{1}{\tau}\sum_{k=0}^{K-1} a_k{\gamma_k}^n$, $a_k = c_k e^{j\frac{a}{2b}{t_k}^2}$, and $\gamma_k = e^{-j\frac{2\pi}{\tau}t_k}$. It indicates that, although $x(t)$ is nonbandlimited in the LCT domain, it can be completely described by the knowledge of $p[n]$ which is a linear combination of K complex exponentials. Therefore, the samples can be expressed as

$$\begin{aligned} y(T_n) &= x(t)*_{\mathbf{M}}h_B(-t)\big|_{t=T_n} \\ &= \sqrt{\frac{1}{j2\pi b}}e^{-j\frac{a}{2b}t^2}\left[\sum_{m\in\mathbb{Z}} p[m]e^{j\frac{2\pi}{\tau}mt} * \operatorname{sinc}\left(-\frac{B}{b}t\right)\right]\Big|_{t=T_n} \\ &= \sqrt{\frac{1}{j2\pi b}}e^{-j\frac{a}{2b}(T_n)^2}\sum_{m\in\mathbb{Z}} p[m]\int e^{j\frac{2\pi}{\tau}ms}\operatorname{sinc}\left[-\frac{B}{b}(T_n - s)\right]\mathrm{ds}. \end{aligned} \tag{16}$$

Using the following equivalent formula

$$\int e^{j\frac{2\pi}{\tau}ms}\cdot\operatorname{sinc}\left[-\frac{B}{b}(T_n - s)\right]\mathrm{ds} = \frac{\mathrm{b}}{\mathrm{B}}\mathrm{rect}\left(\frac{\mathrm{bm}}{\mathrm{B}\tau}\right)\mathrm{e}^{\mathrm{j}\frac{2\pi}{\tau}\mathrm{m}(\mathrm{T_n})}, \tag{17}$$

we can deduce that

$$\begin{aligned} y(T_n) &= \frac{\sqrt{-j2\pi b}}{2\pi B}e^{-j\frac{a}{2b}(T_n)^2}\sum_{m\in\mathbb{Z}} p[m]\mathrm{rect}\left(\frac{bm}{B\tau}\right)e^{j\frac{2\pi}{\tau}m(T_n)} \\ &= \frac{\sqrt{-j2\pi b}}{2\pi B}e^{-j\frac{a}{2b}(T_n)^2}\sum_{m=-A_{\mathbf{M}}}^{A_{\mathbf{M}}} p[m]e^{j\frac{2\pi}{\tau}m(T_n)}, \end{aligned} \tag{18}$$

where $A_{\mathbf{M}} = \left\lfloor \frac{B\tau}{2b} \right\rfloor$ represents the largest integer smaller than or equal to $\frac{B\tau}{2b}$. Then, the following nonuniform sampling theorem can be summarized as:

Theorem. Let $x(t)$ be a τ-periodic stream of Diracs weighted by coefficients $\{c_k\}$ and locations $\{t_k\}$ with FRI $\rho = 2K/\tau$. Let the sampling kernel $h_B(t)$ be an ideal low-pass filter which is bandlimited to $[-B\pi, B\pi]$ in the LCT domain, where B is chosen such that $B \geq \rho$. If the filtered version of $x(t)$, i.e., $y(t) = x(t)*_{\mathbf{M}}h_B(-t)$ is sampled nonuniformly at locations T_n, the samples

$$y(T_n) = x(t) *_{\mathbf{M}} h_B(-t)|_{t=T_n}, \text{for } n = 0, 1, \ldots N-1, \tag{19}$$

are a sufficient characterization of $x(t)$, provided that $N \geq 2A_{\mathbf{M}} + 1$ and $A_{\mathbf{M}} = \lfloor B\tau/(2b) \rfloor$.

Remark: In the above analysis, how to choose the parameters of the LCT is not given. Actually, for the stream of Diracs signal, the parameters are no restricted. In any LCT domain, the Diracs signal is nonbandlimited. The parameters are only related to the ideal low-pass filter. In the following potential application, we will also point out that the choice of the parameters changes with the practical applications.

3.2 Reconstruction analysis of Diracs signal with finite nonuniform samples

Next we consider the issue of reconstruction of $x(t)$ from the nonuniform samples $y(T_n)$ in the LCT domain. We discuss the reconstruction from two cases: the nonuniform sampling instants are known or unknown.

Case 1 When the nonuniform sampling instants are known.

Having mentioned in the above subsection that the signal $x(t)$ can be completely described by $p[n]$ which is the linear combination of K complex exponentials in Eq. (15). Therefore, as the sampling instants T_n are known, the sequence $p[m]$ can be obtained immediately by utilizing the enough signal samples $y(T_n)$ in Eq. (18). Based on the theory that the signal $p[n] = \frac{1}{\tau}\sum_{k=0}^{K-1} \alpha_k {\gamma_k}^n$ can be annihilated by the filter $G(z) = \prod_{k=0}^{K-1}(1-\gamma_k z^{-1}) = \sum_{l=0}^{K} g[l]z^{-l}$ [24], the problem of calculating $\{\alpha_k\}_{k=0}^{K-1}$ and $\{\gamma_k\}_{k=0}^{K-1}$ are equivalent to find a suitable polynomial $G(z) = \prod_{k=0}^{K-1}(1-\gamma_k z^{-1}) = \sum_{l=0}^{K} g[l]z^{-l}$ whose coefficients $g[l]$ annihilate $p[n]$, i.e.

$$\sum_{l=0}^{K} g[l]p[n-l] = 0, \ \forall n \in \mathbb{Z}. \tag{20}$$

In matrix form, the system in Eq. (20) is equivalent to

$$\begin{pmatrix} \vdots & \vdots & \cdots & \vdots \\ p[0] & p[-1] & \cdots & p[-K] \\ p[1] & p[0] & \cdots & p[-K+1] \\ \vdots & \vdots & \ddots & \vdots \\ p[K] & p[K-1] & \cdots & p[0] \\ \vdots & \vdots & \cdots & \vdots \end{pmatrix} \times \begin{pmatrix} g[0] \\ g[1] \\ \vdots \\ g[K] \end{pmatrix} = 0. \tag{21}$$

We can note that finding $g[l]$ is equivalent to find a corresponding vector $\mathbf{g} = (g[0], g[1], \ldots, g[K])^{\mathrm{T}}$ that forms a null space of a suitable submatrix of $p[n]$, i.e., $\mathbf{P}^{(2A_{\mathbf{M}}+1-K)\times(K+1)}$, which is essentially the set $\mathbf{Null}(\mathbf{P}) = \{\mathbf{g} \in \mathbb{R}^{K+1} : \mathbf{P}\cdot\mathbf{g} = 0\}$. Suppose N values $p[n]$ are available. Since there are $K+1$ unknown filter coefficients, we need at least $K+1$ equations and therefore, N must be greater or equal to $2K+1$.

Having computed the weights $g[l]$, the values γ_k are obtained by finding roots of the polynomial $G(z)$ which in turn give one set of innovative parameters, i.e., t_k. To determine the weights α_k, it suffices to take K equations in $p[n] = \frac{1}{\tau}\sum_{k=0}^{K-1}\alpha_k {\gamma_k}^n$ and solves the system for α_k. Next, we can compute c_k. Once all of the t_k are obtained, we have $c_k = \alpha_k e^{-j\frac{a}{2b}t_k^2}$. To find the α_k on the other hand, we need to solve the Vandermonde system of equations $\mathbf{Va} = \mathbf{P}$, where $\mathbf{V}$ is the Vandermonde matrix with elements $V_{i,j} = \gamma_j^{i-1}$, for $i = 1, \ldots, K$ and $j = 0, \ldots, K-1$, vectors $\mathbf{a} = (\alpha_0, \alpha_1, \ldots, \alpha_{K-1})^{\mathrm{T}}$ and $\mathbf{p} = (p[0], p[1], \ldots, p[K-1])^{\mathrm{T}}$. This solution is unique since $\gamma_k \neq \gamma_l, \forall k \neq l$. Therefore, all of the unknown parameters of the Diracs signal in Eq. (9) are obtained, i.e., the original signal is reconstructed.

The sampling and reconstruction process of Diracs signal associated with the LCT is summarized in the following Algorithm 1.

Algorithm 1 Sampling and Reconstruction

Input:

The original signal and its period, $x(t)$ and τ;
The parameters of LCT, $\mathbf{M} = (a, b; c, d)$;
The bandwidth of the lowpass filter, B;
Sampling interval, T;

Output:

Obtaining the samples of the signal $y(T_n)$ by Eq.(16);
Determine the value of $A_{\mathbf{M}}$ by $A_{\mathbf{M}} = \left\lfloor \frac{B\tau}{2b} \right\rfloor$;
Calculating $p[m]$ by Eq.(18) for $m = -A_{\mathbf{M}}, -A_{\mathbf{M}}+1, \cdots, A_{\mathbf{M}}$;
Computing the coefficients $g[l]$ of annihilating $p[m]$ by Eq.(21);
Obtaining the values γ_k by finding the roots of the polynomial $G(z) = \prod_{k=0}^{K-1}(1-\gamma_k z^{-1}) = \sum\limits_{l=0}^{K} g[l]z^{-l}$;
Calculating the shifts t_k by $\gamma_k = e^{-j\frac{2\pi}{\tau}t_k}$;
Determine a_k in $p[n] = \frac{1}{\tau}\sum\limits_{k=0}^{K-1} a_k {\gamma_k}^n$;
Computing the weights c_k by the relationship $a_k = c_k e^{j\frac{a}{2b}{t_k}^2}$;
return The construction of the signal $x(t) = \sum\limits_{k=0}^{K-1} c_k \sum\limits_{n=-\infty}^{+\infty} \delta(t - t_k - n\tau)$.

Case 2 When the nonuniform sampling instants are unknown.

In most applications, the samples cannot be collected uniformly for many reasons, such as the sampling jitters and observation errors from the inherent sampling mechanism. These inevitable errors result in the actual sampling instants are unknown. The actual sampling instants can be expressed as $T_n = nT_0 + \varepsilon_n$, where $n = 0, 1, 2, \ldots, N-1$, T_0 denotes the uniform sampling interval and ε_n is the sampling error at this instant, i.e., the unknown nonuniform samples are similar to the uniform samples with a little jitters. Usually the sampling jitters are unmeasurable, therefore the sampling instants are unknown. Directly utilizing the reconstruction algorithm proposed in the above subsection is infeasible. The sequence $p[m]$ cannot be obtained immediately from the samples in Eq. (18), because there are many unknown variables in the equation. In order to resolve this question, we still use the algorithm 1 to reconstruct the original signal but based on the hypothesis that the sampling instants are uniform approximately. The relationship between the samples and $p[m]$ is modified as

$$y_n = \frac{\sqrt{-j2\pi b}}{2\pi B} e^{-j\frac{a}{2b}(nT_0)^2} \sum_{m=-A_{\mathbf{M}}}^{A_{\mathbf{M}}} p[m] e^{j\frac{2\pi}{\tau}m(nT_0)}, \tag{22}$$

which can also be represented as

$$y_n = \frac{\sqrt{-j2\pi b}}{2\pi B} e^{-j\frac{a}{2b}(nT_0)^2} \mathbf{H}_n \mathbf{P}, \tag{23}$$

where

$$\mathbf{H}_n = (e^{j\frac{2\pi}{\tau}(-A_{\mathbf{M}})nT_0}, e^{j\frac{2\pi}{\tau}(-A_{\mathbf{M}}+1)nT_0}, \ldots, e^{j\frac{2\pi}{\tau}(A_{\mathbf{M}})nT_0}),$$
$$\mathbf{P} = (p[-A_{\mathbf{M}}], p[-A_{\mathbf{M}}+1], \ldots, p[A_{\mathbf{M}}])^T.$$

Then, the reconstructed signal can be obtained based on the subsequent steps in algorithm 1.

Obviously, the reconstruction result contains some errors inevitably. This is due to that the uniform samples are used in the reconstruction instead of the nonuniform samples in Eq. (22). In fact, it should be

$$\begin{aligned} y(T_n) &= \frac{\sqrt{-j2\pi b}}{2\pi B} e^{-j\frac{a}{2b}(nT_0+\varepsilon_n)^2} \sum_{m=-A_{\mathbf{M}}}^{A_{\mathbf{M}}} p[m] e^{j\frac{2\pi}{\tau}m(nT_0+\varepsilon_n)} \\ &= \frac{\sqrt{-j2\pi b}}{2\pi B} e^{-j\frac{a}{2b}(nT_0+\varepsilon_n)^2} \mathbf{H}_n \mathbf{E}_n \mathbf{P}, \end{aligned} \tag{24}$$

where $\mathbf{E}_n = \mathrm{diag}(\mathrm{e}^{\mathrm{j}\frac{2\pi}{\tau}(-\mathrm{A}_{\mathbf{M}})\varepsilon_\mathrm{n}}, \mathrm{e}^{\mathrm{j}\frac{2\pi}{\tau}(-\mathrm{A}_{\mathbf{M}}+1)\varepsilon_\mathrm{n}}, \ldots, \mathrm{e}^{\mathrm{j}\frac{2\pi}{\tau}(\mathrm{A}_{\mathbf{M}})\varepsilon_\mathrm{n}})$ is named as error matrix. But it is irresolvable because of the unmeasurable jitters. Compared Eqs. (23) and (24), it can be seen that calculating $p[m]$ from Eq. (23) immediately, all of the sampling errors will be added in the calculation result $p[m]$. The errors will be propagated in the following calculations. So in order to improve the effect of the reconstruction, we should first reduce or compensate the error of the calculated $p[m]$. One of the solutions is estimating or simulating the sampling jitters ε_n and substituting into Eq. (24) to revise the value of $p[m]$. The jitters error analysis and compensation methods can refer to the previous related research in [31, 32].

4 Numerical experiments and applications

4.1 Reconstruction of Diracs signal

The reconstruction of Diracs signal based on the nonuniform sampling and reconstruction algorithm proposed above is presented in this section. The Diracs signal in the experiment includes three impulse signals. The period of the signal is set to be 1 s. In the time interval [0,1], the detailed expression of the simulated Diracs signal is $x(t) = 5\delta(t-0.2) + \delta(t-0.4) + 3\delta(t-0.6)$. It is clear that the number of the parameters is 6, i.e. $c_0 = 5$, $c_1 = 1$, $c_2 = 3$, $t_0 = 0.2$, $t_1 = 0.4$, $t_2 = 0.6$. Therefore, the rate of innovation of the signal is $\rho = 2K/\tau = 6$.

Applying the proposed LCT-based reconstruction algorithm (where the parameters of the LCT is simply chosen as (2,0.5;0,0.5)), the reconstruction result compared with the original signal is illustrated in the following Fig. 2, where the reconstructed sampling instants are 0.2, 0.4 and 0.6. The reconstructed weights are 4.9713, 0.9943 and 2.9828. From the comparison, we can see that the reconstruction performance is very well, which proves the availability of the proposed algorithm.

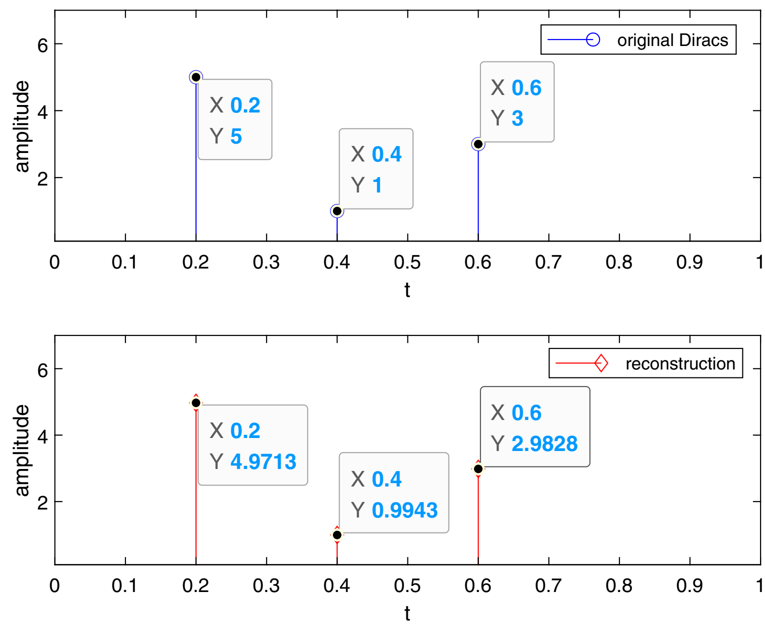

Fig. 2 Comparison of the original and reconstructed Diracs signal

4.2 Potential application of the proposed reconstruction algorithm in the LCT domain

In the above subsection, we give a simple example to verify the performance of the proposed algorithm. We can consider the following problem: give a τ-periodic chirp signal corrupted by additive impulsive noise, $f(t) = s(t) + x(t)$, where $s(t)$ is bandlimited to $[-L_{\mathbf{M}}, L_{\mathbf{M}}]$ in the LCT domain, and $x(t)$ is the impulsive noise which is nonbandlimited in the LCT domain. Thus the rate of innovation of $f(t)$ is $\rho_f = \frac{2(L_{\mathbf{M}}+K)+1}{\tau}$. By using the proposed algorithm, $x(t)$ can be reconstructed from $\hat{f}_{\mathbf{M}}[n] = \hat{x}_{\mathbf{M}}[n], n \in [L_{\mathbf{M}} + 1, L_{\mathbf{M}} + 2K]$, which is outside the bandwidth of the signal $s(t)$. And then the LCTS coefficients of $s(t)$ can be obtained, i.e. $s_{\mathbf{M}}[n] = \hat{f}_{\mathbf{M}}[n] - \hat{x}_{\mathbf{M}}[n], n \in [-L_{\mathbf{M}}, L_{\mathbf{M}}]$. Thus the signal $s(t)$ can be recovered and we achieve the goal of denoising $f(t)$.

In the following experiment, the denoising of a chirp signal with periodic Diracs noise is simulated. The original and noisy signals are illustrated in Fig. 3, where the noisy signal is a chirp signal $s(t) = real(2 * exp(j * 1.2 * t + j * 0.6 * t^2))$ mixed with a periodic Diracs signal $x(t) = 2\delta_1 - 6\delta_2 + 5\delta_3$. In the LCT domain with $\mathbf{M} = (1.2, 1; 0.2, 0.2)$, the energy of the original signal is mostly concentrated in a finite range while the noise is distributed in the whole range as shown in Fig. 4. It is clearly that the LCT parameters is chosen based on the original clean signal, i.e., the original clean signal is bandlimited in the LCT domain with parameter $\mathbf{M}$. Outside the bandwidth of the original signal, the spectrum of the noisy signal is exact or approximate to the spectrum of the noise, such as near the two boundaries in Fig. 4. Based on the information outside the bandwidth of the original signal, the Diracs noise can be reconstructed utilizing the proposed reconstruction algorithm in the LCT domain. Then the denoised signal can be obtained.

Figure 5 shows the denoising effects with proper and improper parameters compared with the clean original signal. It seems that when the parameters of the LCT are proper, i.e., $\mathbf{M} = (1.2, 1; 0.2, 0.2)$, the denoising result is satisfactory except near the edge of the signal. The reasons why the reconstruction near the edge is imperfect can

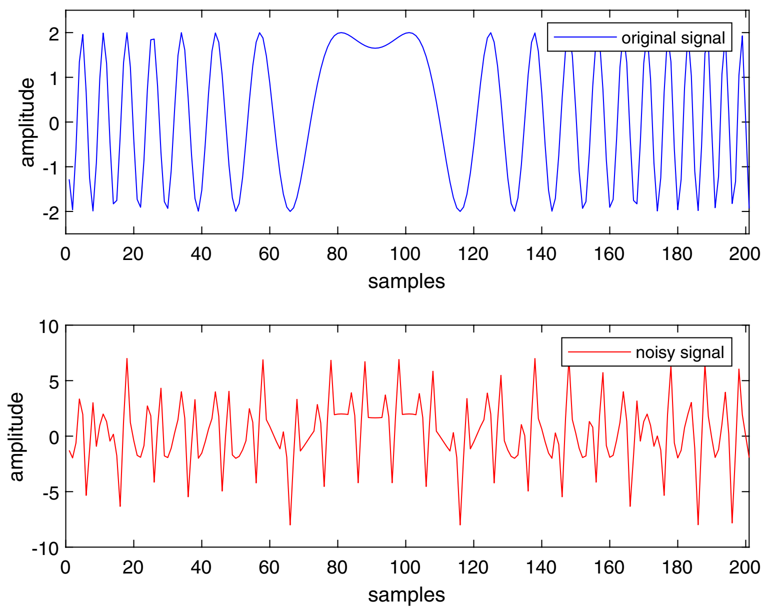

Fig. 3 The original signal and noisy signal

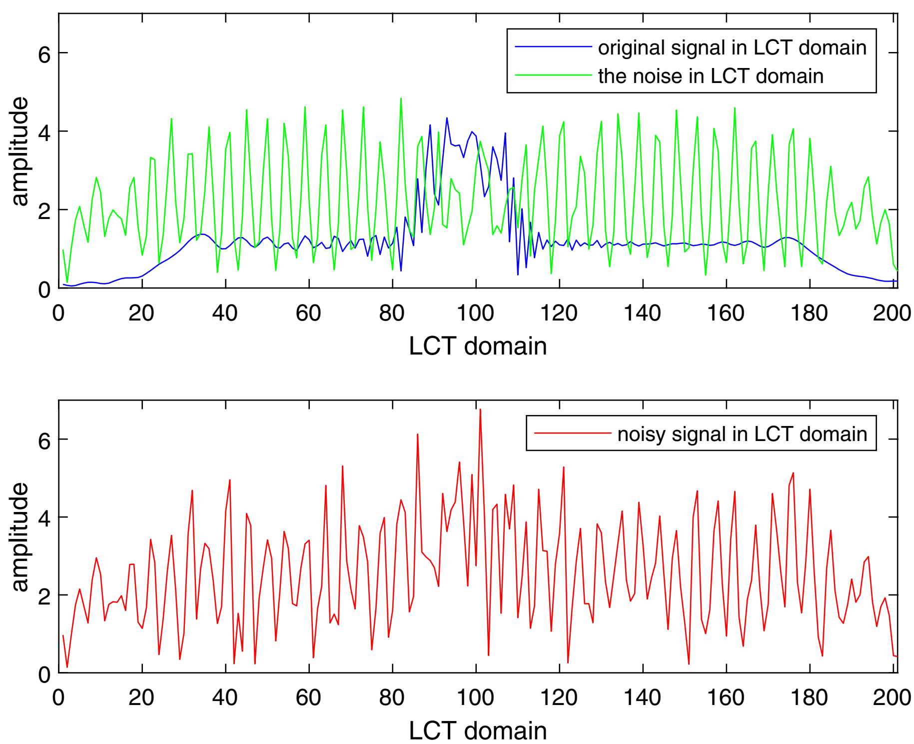

Fig. 4 The original signal, noise and noisy signal in the LCT domain

be summarized as the follows. First, a time-limited signal with a stream of Diracs are simulated in the experiment. The spectrum of the clean signal in the LCT domain is extended in the whole domain and cannot be only limited in a short range. Therefore, the spectrum of the Diracs in the LCT domain includes some errors, i.e., mixed some

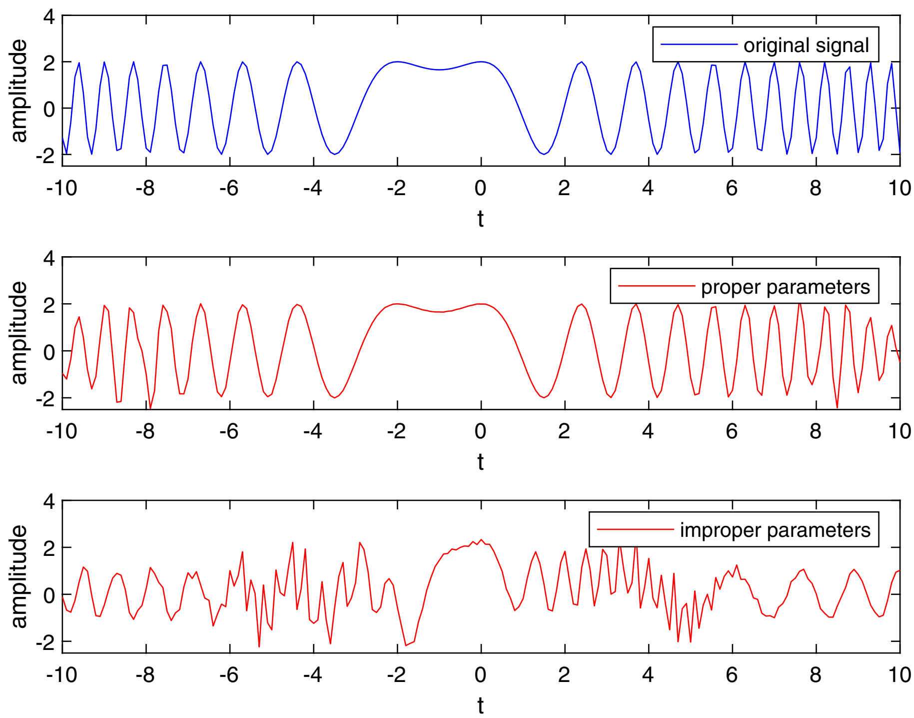

Fig. 5 Comparison of the original and reconstructed signals with proper and improper parameters

that of the signal. Second, based on the reconstruction algorithm of the Diracs signal, the numerical calculation errors are existed and may be propagated in the whole calculating process. At last, the previous errors will affect the subsequent reconstruction of the clean signal. The estimation of the signal in the LCT domain exist errors and the result of the invert LCT will not reach the satisfactory reconstruction effect. In the future work, we will improve the reconstruction algorithm and the denoising effect. The numerical calculation method can be improved to decrease the propagation of the errors or compensate the estimated errors. Accordingly, the result of the reconstruction and the denoising will be more effective. Then, it will be much validity and comparability when compared with other relative denoising methods. Moreover, when the parameters of the LCT are chosen as $\mathbf{M} = (1.5, 0.5; 1, 1)$, which is quite different from the proper parameters, the reconstruction result is very different from the original signal in Fig. 5. It also indicates the importance of the parameters and they must match with the original signal.

We also notice that the above denoising application is suitable for the bandlimited signal in LCT domain with the additive impulsive noise. The type of the noise seems to be restricted. However, some complicated noise may could be modeled as a period of nonuniform splines or piecewise polynomials. A signal $s(t)$ is a periodic nonuniform splines of degree R with K knots in one period if and only if its $(R+1)$th derivative is a periodic stream of K weighted Diracs signal. Similarly, a signal $s(t)$ is a periodic piecewise polynomial with K pieces each of maximum degree R if and only if its $(R+1)$ derivative is a stream of differentiated Diracs signal [24]. Therefore, we can see that some transformation of the noise is quite related to the Diracs signal. The reconstruction of the nonuniform splines or piecewise polynomials with FRI in the LCT domain can be researched based on the analysis in Sect. 3. Then it will expand the scope of denoising application. The above potential theory and application will be researched in our further work.

5 Conclusion

This paper proposed a novel nonuniform sampling and reconstruction of the Diracs signal in the LCT domain using the FRI theory. A signal with the FRI property means that finite samples of the signal could include all effective information of the original signal. Therefore, the reconstruction result can be achieved from finite samples without considering the truncation error. The reconstruction from the finite nonuniform samples with two cases were discussed, including the sampling instants were known or unknown. The final experiment presented the effect of the reconstruction algorithm. The potential applications of the study were analyzed. There are still more error analysis and generalized research need to be study in detail and this will be our future research.

Abbreviations

FT	Fourier transform
FRFT	Fractional Fourier transform
LCT	Linear canonical transform
FRI	Finite rate of innovation

Acknowledgements
The authors thank the editor and anonymous reviewers for their helpful comments and valuable suggestions

Author contributions
LX proposed the algorithm and was a major contributor in writing the manuscript. All authors read and approved the final manuscript.

Funding
This work was supported by the National Natural Science Foundation of China (No. 61901248) and Natural Science Foundation of Shanxi Province.

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Received: 18 April 2023 Accepted: 16 August 2023
Published online: 29 August 2023

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