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Research Article

Subspace-Based Localization and Inverse Scattering of Multiply Scattering Point Targets

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The nonlinear inverse scattering problem of estimating the locations and scattering strengths or reflectivities of a number of small, point-like inhomogeneities (targets) to a known background medium from single-snapshot active wave sensor array data is investigated in connection with time-reversal multiple signal classification and an alternative signal subspace method which is based on search in high-dimensional parameter space and which is found to outperform the time-reversal approach in number of localizable targets and in estimation variance. A noniterative formula for the calculation of the target reflectivities is derived which completes the solution of the nonlinear inverse scattering problem for the general case when there is significant multiple scattering between the targets. The paper includes computer simulations illustrating the theory and methods discussed in the paper.

1. INTRODUCTION

This research is concerned with signal subspace frameworks for inverse scattering with active wave sensor arrays of small, point-like inhomogeneities or perturbations to a background medium whose constitutive properties relevant to the particular remote sensing modality (e.g., (electromagnetic) permittivity, permeability, conductivity, (acoustic) sound speed, diffusion coefficient in (e.g., optic) radiative transfer-based sensing, etc.) are known. The problem under consideration comprises both localization of the inhomogeneities (targets) as well as determination of the perturbation strengths or target reflectivities from single-snapshot entries of a noisy scattering or multistatic response (MSR) matrix gathered by a generally noncoincident array of N_t point transmitters and N_r point receivers. Relevant applications are radar imaging [1, 2], subsurface sensing of buried targets [3, 4], nondestructive material testing [5, 6], microwave breast imaging [7, 8], and other biomedical applications [9].

We emphasize the particular scalar Helmholtz operator context, but the general developments apply in forms which differ only on the specifics of the Green function and the scattering potential operator [10, Chapter 9] to a variety of partial differential equations governing the source-field systems of interest. This includes the diffusion equation [11, Chapter 9] which is relevant to certain random media and has been used in time-reversal studies [12].

Thus we formulate in space-frequency $(\mathbf{r},\omega)$ domain signal subspace approaches for inverse scattering in the framework of the inhomogeneous Helmholtz equation

$$\left(\nabla^2 + k^2(\mathbf{r},\omega)\right)\psi(\mathbf{r},\omega) = \rho(\mathbf{r},\omega), \tag{1}$$

where ψ is the scalar field produced by a scalar source ρ, ∇^2 is the Laplacian operator, and $k^2(\mathbf{r},\omega) = k_0^2(\mathbf{r},\omega) - V(\mathbf{r},\omega)$ where $k_0(\mathbf{r},\omega)$ is the known wavenumber of the background medium (without the targets) and $V(\mathbf{r},\omega)$ is the sought-after scattering potential which for a scatterer composed of a collection of M point targets is of the form

$$V(\mathbf{r}) = \sum_{m=1}^{M} \tau_m \delta(\mathbf{r} - \mathbf{X}_m), \tag{2}$$

where here and henceforth the frequency variable ω is suppressed with the understanding that the results hold for a given frequency, and where $\delta(\cdot)$ is Dirac's delta function, $\mathbf{X}_m$, $m = 1, 2, \ldots, M$, are the unknown target positions, and $\tau_m \in \mathbf{C}$, $m = 1, 2, \ldots, M$, are the unknown, generally complex-valued reflectivities of the targets.

The present work expands the program of a previous contribution [13] coauthored by the present authors in which a two-step approach to this inverse scattering problem was proposed consisting of first estimating the target positions via time-reversal multiple signal classification (MUSIC) [14–18] (see also [19] for the conventional statistical MUSIC) and later solving for the target reflectivities by means of an iterative algorithm involving the Foldy-Lax multiple scattering model [20, 21]. Four aspects not treated in [13] are addressed in the present paper where (1) a more general formulation is established that is applicable to noncoincident arrays, (2) an alternative high-dimensional signal subspace approach is derived which corresponds for weakly scattering targets under spatially white Gaussian noise relevant to a Rayleigh fading environment to maximum likelihood (ML) estimation (see [22] and Section 4 of this paper) and which is shown to enable the localization of more targets (hence, the imaging of more object features) than the time-reversal approach, (3) solution uniqueness and performance questions are elucidated, including reference calculations of the Fisher information matrix (and of the companion Cramér-Rao bound (CRB)) relevant to the estimation of target positions and reflectivities which are valid under general multiple scattering, and (4) a noniterative formula for the determination of the target reflectivities is derived which holds even for the nonlinear, multiple scattering case. Early accounts of the high-dimensional signal subspace method can be found in conference proceedings authored by the present authors [23–26] and in a recent paper [22] which presents an equivalent approach, the present treatment differing from these contributions in that, in addition to the other aspects enumerated above, (1) it addresses the question of number of localizable targets, demonstrating how the high-dimensional signal subspace method can significantly enhance the number of localizable targets, particularly if the targets are weakly interacting, and (2) it comparatively studies under both multiple scattering and nonmultiple scattering conditions the performance of the method relative to time-reversal MUSIC and the pertinent CRB.

Unlike most previous work in this area, this paper addresses estimation not only of target locations but also of target reflectivities. That the singular values of the linear mapping K contain information about the reflectivities is obvious and has been the subject of well-known investigations [16, 27, 28]. Extraction of useful general features (not of the actual scattering potential) is addressed in [28]. In contrast, we are interested in this paper in extraction of the actual reflectivities. For Born-approximable targets this problem can be solved trivially once the target positions have been found [13, 18]. On the contrary, for the general multiple scattering regime the associated inversion is less straightforward due to the resulting nonlinearity of the reflectivities-to-MSR matrix mapping which traditionally would be handled via nonlinear optimization. Despite this nonlinearity, the latter problem is solved in this paper analytically, noniteratively (unlike in [13] which adopts the more conventional numerical iterations route). Here it is worthwhile pointing out that the noniterative solution of nonlinear inverse problems is a topic of much importance [29] which remains open in inverse scattering of general scatterers if one seeks to reconstruct both target support and constitutive properties or scattering potential. Our result in this direction provides a novel framework for the noniterative treatment of this problem which despite being emphasized here for the canonical case of point targets can also be applied to certain large scatterers whose response can be modeled using a computational grid.

The paper is organized as follows. Section 2 provides the forward scattering results upon which the remainder of the paper is built. In Section 3 a general form of time-reversal MUSIC for multiply scattering targets is developed that holds for noncoincident arrays. Section 4 is devoted to the high-dimensional signal subspace method. The Born approximation and exact multiple scattering cases are discussed separately. The noniterative analytical algorithm that solves for the target reflectivities after the target positions have been estimated is established in Section 5. The methods derived in the paper are illustrated numerically in Section 6. Conclusions are given in Section 7. Fundamental questions of linear independence of the Green function vectors for a given array upon which the signal subspace methods of the paper rely, which previous to this paper had been discussed in detail only in [14], are revisited with a reinterpretation in Appendices A–B. Appendix C presents the Fisher information matrix/CRB calculations relevant to the estimation of target positions and reflectivities under general multiple scattering conditions.

2. FORWARD SCATTERING FORMULATION

We consider a remote sensing system formed by a transmit array (the source ρ in (1)) having N_t point transmitters at the space points $\mathbf{R}_t(j)$, $j = 1, 2, \ldots, N_t$ and a receive array formed by N_r point receivers located at positions $\mathbf{R}_r(l)$, $l = 1, 2, \ldots, N_r$, which interrogates in a known background medium characterized by wavenumber $k_0(\mathbf{r})$ an unknown scattering object characterized by the scattering potential in (2). The respective MSR matrix $K : \mathcal{X} \equiv \mathbf{C}^{N_t} \rightarrow \mathcal{Y} \equiv \mathbf{C}^{N_r}$ governing the linear mapping from the transmit array excitation signal $\in \mathcal{X}$ to the scattered (total minus incident) field measured at the receive array $\in \mathcal{Y}$ is given by [10, Chapter 9]

$$K = \sum_{m=1}^{M} \tau_m g_{0,r}(\mathbf{X}_m) g_t^T(\mathbf{X}_m) = \mathcal{G}_{0,r} O \mathcal{G}_t^T \tag{3}$$

or, equivalently, by

$$K = \sum_{m=1}^{M} \tau_m g_r(\mathbf{X}_m) g_{0,t}^T(\mathbf{X}_m) = \mathcal{G}_r O \mathcal{G}_{0,t}^T, \tag{4}$$

where T denotes the transpose and where we have introduced the $N_t \times 1$ "transmit background Green function vector"

$$g_{0,t}(\mathbf{X}) = \Big[G_0(\mathbf{X}, \mathbf{R}_t(1)) \quad G_0(\mathbf{X}, \mathbf{R}_t(2)) \quad \cdots \quad G_0(\mathbf{X}, \mathbf{R}_t(N_t)) \Big]^T, \tag{5}$$

where $G_0(\mathbf{r},\mathbf{r}')$ is the (background) Green function of the Helmholtz operator $(\nabla^2 + k_0^2(\mathbf{r}))$ subject to the boundary conditions relevant to the problem at hand, say Sommerfeld's radiation condition, along with its receive counterpart, the $N_r \times 1$ "receive background Green function vector,"

$$g_{0,r}(\mathbf{X}) = \Big[G_0(\mathbf{R}_r(1),\mathbf{X}) \ G_0(\mathbf{R}_r(2),\mathbf{X}) \ \cdots \ G_0(\mathbf{R}_r(N_r),\mathbf{X})\Big]^T, \tag{6}$$

and where we have also introduced the $N_t \times 1$ "transmit total Green function vector,"

$$g_t(\mathbf{X}) = \Big[G(\mathbf{X},\mathbf{R}_t(1)) \ \ G(\mathbf{X},\mathbf{R}_t(2)) \ \ \cdots \ \ G(\mathbf{X},\mathbf{R}_t(N_t))\Big]^T, \tag{7}$$

where $G(\mathbf{r},\mathbf{r}')$ is the (total) Green function of the operator $(\nabla^2 + k^2(\mathbf{r}))$, along with the $N_r \times 1$ "receive total Green function vector,"

$$g_r(\mathbf{X}) = \Big[G(\mathbf{R}_r(1),\mathbf{X}) \ G(\mathbf{R}_r(2),\mathbf{X}) \ \cdots \ G(\mathbf{R}_r(N_r),\mathbf{X})\Big]^T. \tag{8}$$

In (3)-(4) the $M \times M$ diagonal matrix

$$O = \operatorname{diag}(\tau_1, \tau_2, \ldots, \tau_M) \tag{9}$$

and the $N_t \times M$ matrices $\mathcal{G}_{0,t}$ and $\mathcal{G}_t$ are formed by aligning the $N_t \times 1$ vectors $g_{0,t}$ and g_t corresponding to the target positions, respectively, while the $N_r \times M$ matrices $\mathcal{G}_{0,r}$ and $\mathcal{G}_r$ are formed by aligning the respective $N_r \times 1$ vectors $g_{0,r}$ and g_r, thus, for example,

$$\mathcal{G}_{0,t} = \Big[g_{0,t}(\mathbf{X}_1) \ \ g_{0,t}(\mathbf{X}_2) \ \ \cdots \ \ g_{0,t}(\mathbf{X}_M)\Big]. \tag{10}$$

The Born approximation corresponds to using $G \approx G_0$ in the preceding analysis, which yields

$$K \approx \sum_{m=1}^{M} \tau_m g_{0,r}(\mathbf{X}_m) g_{0,t}^T(\mathbf{X}_m) = \mathcal{G}_{0,r} O \mathcal{G}_{0,t}^T. \tag{11}$$

To incorporate multiple scattering we adopt the framework of the Foldy-Lax model within which incident and total (incident plus scattered) fields at the target positions $\mathbf{X}_m$, $\psi_i(\mathbf{X}_m)$, and $\psi_T(\mathbf{X}_m)$, respectively, are related by [20, 21] (see also [11, pages 246–248])

$$\psi_T(\mathbf{X}_m) = \psi_i(\mathbf{X}_m) + \sum_{m' \neq m} \tau_{m'} G_0(\mathbf{X}_m, \mathbf{X}_{m'}) \psi_T(\mathbf{X}_{m'}) \tag{12}$$

so that assuming that the $M \times M$ matrix H defined by

$$H_{m,m'} = \delta_{m,m'} - \tau_{m'} G_0(\mathbf{X}_m, \mathbf{X}_{m'})(1 - \delta_{m,m'}) \tag{13}$$

(where $\delta_{\cdot,\cdot}$ is the Kronecker delta function) is nonsingular (which holds under mild conditions [30, page 201]) then

$$\Psi_T = H^{-1}\Psi_i, \tag{14}$$

where $\Psi_i = [\psi_i(\mathbf{X}_1) \ \ \psi_i(\mathbf{X}_2) \ \ \cdots \ \ \psi_i(\mathbf{X}_M)]^T$ with a similar expression for Ψ_T using ψ_T in place of ψ_i. It follows from (5)–(8), (12) with the substitutions $\psi_i \rightarrow G_0(\mathbf{X}_m, \mathbf{R}_t(j))$ (for transmit) or $G_0(\mathbf{R}_r(l), \mathbf{X}_m)$ (for receive) and $\psi_T \rightarrow G(\mathbf{X}_m, \mathbf{R}_t(j))$ or $G(\mathbf{R}_r(l), \mathbf{X}_m)$ that the total and background Green function vectors are related by

$$g_t(\mathbf{X}_m) = g_{0,t}(\mathbf{X}_m) + \sum_{m' \neq m} \tau_{m'} G_0(\mathbf{X}_m, \mathbf{X}_{m'}) g_t(\mathbf{X}_{m'}), \tag{15}$$

$$g_r(\mathbf{X}_m) = g_{0,r}(\mathbf{X}_m) + \sum_{m' \neq m} \tau_{m'} g_r(\mathbf{X}_{m'}) G_0(\mathbf{X}_{m'}, \mathbf{X}_m) \tag{16}$$

or, equivalently, from (13)-(14)

$$\begin{aligned} \mathcal{G}_t^T &= H^{-1}\mathcal{G}_{0,t}^T, \\ \mathcal{G}_r^T &= H^{-1}\mathcal{G}_{0,r}^T. \end{aligned} \tag{17}$$

Using (3)-(4), (17) the MSR matrix K can be expressed as (see also [31])

$$K = \mathcal{G}_{0,r} O H^{-1} \mathcal{G}_{0,t}^T = \sum_{m=1}^{M} \sum_{m'=1}^{M} A_{m,m'} g_{0,r}(\mathbf{X}_m) g_{0,t}^T(\mathbf{X}_{m'}), \tag{18}$$

where the generalized multiple scattering amplitudes $A_{m,m'} = \tau_m H^{-1}_{m,m'}$. Alternatively, applying successive substitutions in (15)-(16) to express the total Green function vectors as a series of the background Green function vectors and substituting this result into (3)-(4) one arrives at the formally convenient result

$$A_{m,m'} = \tau_m \delta_{m,m'} + \tau_m \tau_{m'} G(\mathbf{X}_m, \mathbf{X}_{m'}). \tag{19}$$

Given our assumption that the matrix H in (13) is invertible, it follows from these results, Sylvester's theorem (see [30, page 64], [32, page 126]) and the rank preservation theorem (see [30, page 61]), the latter being a corollary of Sylvester's theorem (see [32, page 128]), that the rank r_K of K obeys $r_K \leq \min(d_t, d_r)$ where the number of dimensions

$$\begin{aligned} d_t &\equiv \operatorname{rank}(\mathcal{G}_{0,t}) \\ &= \dim\{\operatorname{span}[g_{0,t}(\mathbf{X}_m),\ m = 1, 2, \ldots, M] \subseteq \mathcal{X}\} \\ &= \operatorname{rank}(\mathcal{G}_t) \end{aligned} \tag{20}$$

while

$$d_r \equiv \operatorname{rank}(\mathcal{G}_{0,r}) = \operatorname{rank}(\mathcal{G}_r). \tag{21}$$

It follows from results derived in [14] and discussed further in Appendix A of the present paper that with the exception of pathological rare cases (which we ignore next)

$$d_r = \min(M, N_r) \tag{22}$$

and, similarly,

$$d_t = \min(M, N_t) \tag{23}$$

so that the rank

$$r_K \leq \min(M, N_t, N_r). \tag{24}$$

Furthermore, under the same assumptions, if $M \leq \min(N_t, N_r)$ then actually

$$r_K = M \tag{25}$$

(this following from the same theorem in [30, page 64], and the associated lack of null space for $\mathcal{G}_{0,r}$).

3. TIME-REVERSAL MUSIC CONSIDERING MULTIPLE SCATTERING

Conventional signal subspace methods such as MUSIC rely on several signal realizations and, as such, cannot be applied to the present single-snapshot problem. Approaches for processing of single-snapshot passive array signals can be found in [33–35] all of which create pseudo-autocorrelation matrices via an approach analogous to that of spatial smoothing for decorrelation of coherent signals in MUSIC [36, 37]. Robustness aspects of single-snapshot direction finding have also been investigated [38]. But besides being applicable only to the passive case, the above methods hold only for the far field and for special (e.g., uniformly spaced, linear) array configurations, and require long data vectors for effectiveness. Extension of these methods to active array data follows readily using the concept of the coarray [39–41], but applicability remains limited to the far field and special configurations.

A method that has received much attention recently and that can be implemented with single-snapshot active array data of near or far field targets is the time-reversal MUSIC method [13–18, 42, 43], which blends ideas of standard MUSIC with the decomposition of the time-reversal operator technique [5, 9, 21, 42, 44, 45], also known simply as "time-reversal." The method is presented next, detailing necessary and sufficient conditions for applicability, for the general case of multiply scattering targets and noncoincident arrays, which generalizes more restricted results derived before in [13, 17, 18].

In particular, let σ_p^2 and v_p, where $p = 1, 2, \ldots, N_t$, respectively, represent the eigenvalues and eigenvectors of the transmit-mode time-reversal operator $K^\dagger K : \mathcal{X} \equiv \mathbf{C}^{N_t} \to \mathcal{X}$, where $\dagger$ denotes the adjoint. Also, let σ_p^2 and u_p, where $p = 1, 2, \ldots, N_r$, respectively, represent the eigenvalues and eigenvectors of the receive-mode time-reversal operator $KK^\dagger : \mathcal{Y} \equiv \mathbf{C}^{N_r} \to \mathcal{Y}$. Then $\mathbf{X}$ is a pole of the receive-mode time-reversal MUSIC pseudospectrum

$$P_r(\mathbf{X}) = \left[\sum_{\sigma_p = 0} \left| u_p^\dagger g_{0,r}(\mathbf{X}) \right|^2 \right]^{-1} \tag{26}$$

if and only if $\mathbf{X}$ coincides with any of the target positions $\mathbf{X}_m$, $m = 1, 2, \ldots, M$ if and only if $M < N_r$ and $M \leq N_t$.

The proof of this result is as follows. The range of K is $S_r \equiv \{u_p, \sigma_p > 0\} \subseteq \text{span}[g_{0,r}(\mathbf{X}_m),\ m = 1, 2, \ldots, M] \subseteq \mathcal{Y}$ (refer to (3) and (18)) and, according to (24), has dimensionality $r_K \leq \min(M, N_t, N_r)$. Clearly $\mathbf{X}$ is a pole of this pseudospectrum if $\mathbf{X}$ coincides with any of the target positions if and only if $S_r = \text{span}[g_{0,r}(\mathbf{X}_m),\ m = 1, 2, \ldots, M] \subset \mathcal{Y}$ (strict subset), since if and only if this holds, any such Green function vector $g_{0,r}(\mathbf{X})$ is orthogonal to the nontrivial orthogonal complement $\{u_p, \sigma_p = 0\}$ of the range S_r of K in $\mathcal{Y}$. According to our discussion in (25) the conditions $M < N_r$ and $M \leq N_t$ are sufficient for this to hold. Also, necessarily for this to hold $r_K = \min(M, N_r) \leq \min(M, N_r, N_t)$ (which borrows from the discussion linked to (22)–(24)) and $r_K = \min(M, N_r) < N_r$ so that $M < N_r$ and $M \leq N_t$. This establishes the "if" part of the result. Now, one naturally wonders whether it is possible for a blind spot $\mathbf{X} \neq \mathbf{X}_m$, where $\mathbf{X}_m$ denotes any of the target positions, to exist such that $\text{span}[g_{0,r}(\mathbf{X}_m),\ m = 1, 2, \ldots, M,\ g_{0,r}(\mathbf{X}),\ \mathbf{X} \neq \mathbf{X}_m] = \text{span}[g_{0,r}(\mathbf{X}_m),\ m = 1, 2, \ldots, M]$, which would yield fictitious poles in the pseudospectrum in (26), making the inversion nonunique. This would hold if and only if there existed a configuration of $M + 1$ targets, the target positions $\mathbf{X}_m$ and the fictitious target position $\mathbf{X} \neq \mathbf{X}_m$ included, such that they were linearly dependent. Under the conditions required by the theorem, in particular, $M < N_r$, this can happen only for the unlikely configurations discussed in [14] and in Appendix A of this paper so that apart from such rare scenarios which we are ignoring in this paper this does not hold which completes the "only if" part of the result.

The corresponding transmit-mode version of the method is that point $\mathbf{X}$ is a pole of the transmit-mode time-reversal MUSIC pseudospectrum

$$P_t(\mathbf{X}) = \left[\sum_{\sigma_p = 0} \left| v_p^\dagger g_{0,t}^*(\mathbf{X}) \right|^2 \right]^{-1}, \tag{27}$$

(where $*$ denotes complex conjugation) if and only if $\mathbf{X}$ corresponds to one of the target locations $\mathbf{X}_m$, $m = 1, 2, \ldots, M$ if and only if $M < N_t$ and $M \leq N_r$. The above requirements tell us that *at least one* of the approaches (receive or transmit) will work if and only if $M \leq \min(N_t, N_r)$ and $M < \max(N_t, N_r)$. Furthermore, for the special case when $M < N_r$ and $M < N_t$, one can use the generalized time-reversal MUSIC pseudospectrum

$$P_{r,t}(\mathbf{X}) = \left[\sum_{\sigma_p = 0} \left| u_p^\dagger g_{0,r}(\mathbf{X}) \right|^2 + \left| v_p^\dagger g_{0,t}^*(\mathbf{X}) \right|^2 \right]^{-1} \tag{28}$$

which theoretically peaks at the correct target locations. Importantly, in the time-reversal MUSIC pseudospectra (26)–(28) only the background Green function vectors enter into play despite the generality of the development which considers multiple scattering.

4. HIGH-DIMENSIONAL SIGNAL SUBSPACE METHOD

The time-reversal MUSIC method discussed in the previous section is applicable as long as $M \leq \min(N_t, N_r)$ and

$M < \max(N_t, N_r)$. It is shown in this section that if one implements a different approach based on MUSIC-like steering not of a single target (as in (26)–(28)) but of all the M targets simultaneously (a multidimensional search) then it is actually possible to locate up to $N_r N_t - n(n-1)/2 - 1$ targets, where n is the number of coincident elements, as long as the targets are approximately describable by the Born approximation. For additive spatially white Gaussian noise, which in the form used next which is dictated by (34), (42) accounts for Rayleigh fading associated to a background environment with many small scatterers (see [46, pages 767–768], and [47]), this method corresponds to the ML estimator for the target locations [22]. The counterpart of the method for multiply scattering targets is also developed, and it is found that under these more general conditions the number of localizable targets becomes $\lceil (N_r N_t)^{1/2} \rceil - 1$ (where $\lceil x \rceil$ denotes the smallest integer $\geq x$) which is greater than or equal to the number corresponding to the time-reversal approach.

4.1. Born approximation case

Under the Born approximation the relevant MSR matrix K is given by (11) which we rewrite as

$$\bar{K} = \Pi(Q)\tau, \tag{29}$$

where the bar symbol over a matrix denotes the vectorized or stacked form of that matrix, the $N_r N_t \times M$ matrix $\Pi(Q) = [\bar{\Pi}_1 \;\; \bar{\Pi}_2 \;\; \cdots \;\; \bar{\Pi}_M]$ where

$$Q \equiv [\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M] \tag{30}$$

and the $N_r N_t \times 1$ vector

$$\bar{\Pi}_m = \operatorname{vec}\left[g_{0,r}(\mathbf{X}_m) g_{0,t}^T(\mathbf{X}_m) \right], \tag{31}$$

(where $\operatorname{vec}(\cdot)$ denotes vectorized or stacked form of a matrix) and where we have introduced the $M \times 1$ scattering amplitude vector $\tau = [\tau_1, \tau_2, \ldots, \tau_M]^T \in \mathcal{T} \equiv \mathbf{C}^M$.

The signal vector $\bar{K}$ belongs to the space $T \equiv \mathbf{C}^{N_r N_t}$. It follows from Appendix B which incorporates reciprocity considerations that as long as $M < N_r N_t - n(n-1)/2$ where n is the number of coincident transmitting and receiving elements, and with the exception of very specialized and unlikely target configurations which we ignore next, the propagators $\bar{\Pi}_m$ for different target positions are linearly independent and $\operatorname{rank}(\Pi(Q)) = M$. Thus assuming this condition next we introduce the M-dimensional signal subspace $S = \operatorname{span}(\bar{\Pi}_1, \bar{\Pi}_2, \ldots, \bar{\Pi}_M)$ of T spanned by the set of M propagators, and W, the orthogonal complement of S in T. Furthermore, if one hypothesizes a set of possible target locations (a steering vector), say, $Q' \equiv [\mathbf{X}'_1, \mathbf{X}'_2, \ldots, \mathbf{X}'_M]$, where $\mathbf{X}'_1$ is a hypothesized location for target 1, and so on, then one can compute the hypothesized propagators $\bar{\Pi}'_1, \bar{\Pi}'_2, \ldots, \bar{\Pi}'_M$ associated with Q' and hence the corresponding signal subspace $S' = \operatorname{span}(\bar{\Pi}'_1, \bar{\Pi}'_2, \ldots, \bar{\Pi}'_M)$. Next one can find, for example, by Gram-Schmidt orthonormalization, or by obtaining the eigenvectors of the matrix

$$\Pi(Q')\Pi^\dagger(Q') = \sum_{m=1}^{M} \bar{\Pi}'_m \bar{\Pi}'^\dagger_m \tag{32}$$

having zero singular value, a set of orthonormal basis vectors $c_1^{(Q')}, c_2^{(Q')}, \ldots, c_{N_r N_t - M}^{(Q')}$ spanning the orthogonal complement W' of S' in T. In the absence of noise, the projections $(c_i^{(Q')})^\dagger \bar{K} = 0$ for all $i = 1, 2, \ldots, N_r N_t - M$ if and only if the steering vector Q' coincides with the actual target locations, that is, $Q' = Q$ (or any permutation of the same positions therein). Then in the absence of noise the poles in the pseudospectrum

$$P(Q') = \frac{1}{\sum_{i=1}^{N_r N_t - M} \left| \left(c_i^{(Q')} \right)^\dagger \bar{K} \right|^2} \tag{33}$$

yield the M target locations exactly. We wish to mention that the necessary condition $M < N_r N_t - n(n-1)/2$ upon which this result is based is also sufficient, as can be established following a discussion analogous to that implemented in Section 3 for time-reversal MUSIC, while borrowing from Appendices A–B, but we will not dwell on this here. Instead, let us add noise w to the signal model (29) so that

$$\hat{\bar{K}} = \Pi(Q)\tau + w. \tag{34}$$

In the presence of noise, we substitute $\bar{K} \to \hat{\bar{K}}$ in (33), and the estimated Q' that maximizes $P(Q')$ is that value which also maximizes the signal-to-noise ratio (SNR) of the principal component of $\hat{\bar{K}}$ as given by the projection of $\hat{\bar{K}}$ onto the associated signal subspace S'. In the absence of noise this method works perfectly but in the presence of noise the resolution diminishes as the pole of $P(Q')$ is smoothed out.

Next we show that for the particular case when the noise term w in (34) corresponds to zero mean, spatially white Gaussian noise of variance σ^2, the method described in connection with (33) coincides with the ML estimator for the target positions derived recently in [22]. Our starting point is (34). One readily finds along the usual lines [48] that the ML estimates of the generalized target coordinate Q and of the scattering amplitude vector τ are obtained via the minimization

$$\min_{Q,\tau} \| \hat{\bar{K}} - \Pi(Q)\tau \|, \tag{35}$$

where $\| \cdot \|$ denotes the Euclidean norm. As is well known [48, 49], for any given Q, the scattering amplitude vector $\hat{\tau}_Q$ that minimizes the norm in (35) is of the form

$$\hat{\tau}_Q = \Pi^+(Q)\hat{\bar{K}}, \tag{36}$$

where $\Pi^+(Q) = [\Pi^\dagger(Q)\Pi(Q)]^{-1}\Pi^\dagger(Q)$ is the pseudoinverse of $\Pi(Q)$. By substituting this result into (35) while recalling that [49]

$$\Pi(Q)\hat{\tau}_Q = P_R \hat{\bar{K}}, \tag{37}$$

where P_R is an operator that projects the data space T onto the range of $\Pi(Q)$, which is given in terms of the singular system (λ_p, V_p, U_p) of $\Pi(Q)$ (where $\lambda_p, V_p \in \mathcal{T}$ and $U_p \in T$ are the corresponding singular values, object singular vectors and data singular vectors, resp.) by

$$P_R = \sum_{\lambda_p > 0} U_p(Q) U_p^\dagger(Q), \tag{38}$$

one finds that the ML estimate $\hat{Q}$ of the target locations is given by (see also [22, equation (33)])

$$\hat{Q} = \arg \min_{Q} ||(I - P_R)\hat{\bar{K}}|| = \arg \min_{Q} ||P_{NR}\hat{\bar{K}}||, \quad (39)$$

where I is the $N_rN_t \times N_rN_t$ identity matrix and P_{NR} is the projection operator onto the orthogonal complement of the range of $\Pi(Q)$, which is also the null space of the operator $\Pi(Q)\Pi^{\dagger}(Q)$ [49] and is defined by

$$P_{NR} = \sum_{\lambda_p=0} U_p(Q)U_p^{\dagger}(Q). \quad (40)$$

Then from (39)-(40) the ML estimator

$$\hat{Q} = \arg\min_{Q} \sum_{\lambda_p=0} | U_p^{\dagger}(Q)\hat{\bar{K}} |^2, \quad (41)$$

which is exactly the estimation method described in (33) with the substitution $\bar{K} \rightarrow \hat{\bar{K}}$ since the singular vectors $U_p, \lambda_p = 0$ of $\Pi(Q)$ span the same subspace as the vectors c_i^Q, $i = 1, 2, \ldots, N_rN_t - M$ in (33), which completes the proof. Finally, the ML estimate of the target scattering amplitude vector τ is given by substituting $Q \rightarrow \hat{Q}$ in (36) (see also [22, equation (34)]). We consider next the full multiple scattering case.

4.2. *Multiple scattering case*

The starting point of the multiple scattering generalization is (18) and its companion equation (19) which we rewrite as

$$\hat{\bar{K}} = \Gamma(Q)A(Q, \tau) + w, \quad (42)$$

where $\Gamma(Q)$ is the $N_rN_t \times M^2$ matrix whose columns are the $N_rN_t \times 1$ vectors obtained by stacking of the matrices

$$\Gamma_{m,m'} = g_{0,r}(\mathbf{X}_m)g_{0,t}^T(\mathbf{X}_{m'}), \quad m, m' = 1, 2, \ldots, M, \quad (43)$$

and where $A(Q, \tau)$ is the $M^2 \times 1$ vector having entries $A_{m,m'}$, $m = 1, 2, \ldots, M$, $m' = 1, 2, \ldots, M$.

By comparing expressions (11), (34) and (18)-(19), (42) corresponding to the Born approximated and non-Born-approximated, multiple scattering cases, respectively, we realize that unlike in the Born-approximated case which involves a sum of only M propagators Π_m, the corresponding expression (42) in the multiple scattering case comprises a total of M^2 propagators $\Gamma_{m,m'}$. From this and the results in Appendix B, it follows that the applicability condition in the multiple scattering case becomes $M^2 < N_rN_t$, in particular, the high-dimensional method of this section functions for the localization of up to $\lceil (N_tN_r)^{1/2} \rceil - 1$ targets. Under this condition the obvious generalization of the method of the preceding subsections, with $A(Q, \tau)$ taking the role of τ and $\Gamma(Q)$ taking the role of $\Pi(Q)$, becomes the estimation of the generalized target coordinate Q via the maximizing of the pseudospectrum

$$P(Q) = \frac{1}{\sum_{\lambda_p=0} | \mathcal{U}_p^{\dagger}\hat{\bar{K}} |^2}, \quad (44)$$

where $\mathcal{U}_p, \lambda_p = 0$ are the zero singular value eigenvectors of the matrix $\Gamma(Q)\Gamma^{\dagger}(Q)$. Furthermore, this is also the ML estimate from the point of view of estimation of the parameters Q and $A(Q, \tau)$ (where the model dependence of A on Q and τ is ignored), in which case the associated ML estimate $\hat{A}(Q, \tau)$ of the generalized scattering amplitude vector $A(Q, \tau)$ is obtained by substituting the value of Q found in (44) into

$$\hat{A}(Q, \tau) = \Gamma^{+}(Q)\hat{\bar{K}}. \quad (45)$$

This approach has been derived independently and in a different form in [22, pages 236-237].

5. NONITERATIVE SCATTERING AMPLITUDE INVERSION

The problem of estimating the reflectivities τ_m, $m = 1, 2, \ldots, M$ after the target locations $\mathbf{X}_m$, $m = 1, 2, \ldots, M$, have been found consists of the inversion of the nonlinear mapping of the reflectivities τ_m to the MSR matrix K as specified, for example, in (18) and (19). This nonlinear inversion has been tackled in a recent paper [13] by means of an iterative algorithm. Given the nonlinear nature of the problem, it is not obvious that under certain conditions it might be actually possible to carry out the inversion via an explicit formula, that is, a noniterative algorithm, in place of iterative approaches. It is shown next that despite the nonlinearity of the associated forward mapping, such a reconstruction formula does exist if $M \leq \min(N_t, N_r)$ and can be implemented rather trivially once the target positions have been estimated via time-reversal MUSIC or other approaches.

If $M \leq \min(N_t, N_r)$, the background Green function vectors form a linearly independent set so that

$$\sum_{m=1}^{M} \alpha_m g_{0,r}(\mathbf{X}_m) = 0 \quad \text{iff } \alpha_m = 0, \ m = 1, 2, \ldots, M \quad (46)$$

with a similar condition holding for the respective transmit vectors. To arrive at the desired formula, let us consider an "active target isolation," consisting of generating the unique vector $K^{+}g_{0,r}(\mathbf{X}_1)$, where $K^{+} : \mathcal{Y} \rightarrow \mathcal{X}$ denotes the pseudoinverse (defining the normal solution of minimum L^2 norm [49]) of the linear mapping $K : \mathcal{X} \rightarrow \mathcal{Y}$, which when used as excitation at the transmit array yields an output at the receive array equal to the background Green function vector $g_{0,r}(\mathbf{X}_1)$ corresponding to target 1. Thus the entire received signal associated to this vector arises from target 1 only, a desirable property, as we will see, in isolating the effect of that target alone. It now follows from expression (3) for the MSR matrix K that

$$\sum_{m=1}^{M} \tau_m g_{0,r}(\mathbf{X}_m)g_t^T(\mathbf{X}_m)K^{+}g_{0,r}(\mathbf{X}_1) = g_{0,r}(\mathbf{X}_1) \quad (47)$$

which in view of (46) further translates into

$$\begin{aligned} \tau_1 g_t^T(\mathbf{X}_1)K^{+}g_{0,r}(\mathbf{X}_1) &= 1, \\ g_t^T(\mathbf{X}_m)K^{+}g_{0,r}(\mathbf{X}_1) &= 0, \quad m = 2, 3, \ldots, M. \end{aligned} \quad (48)$$

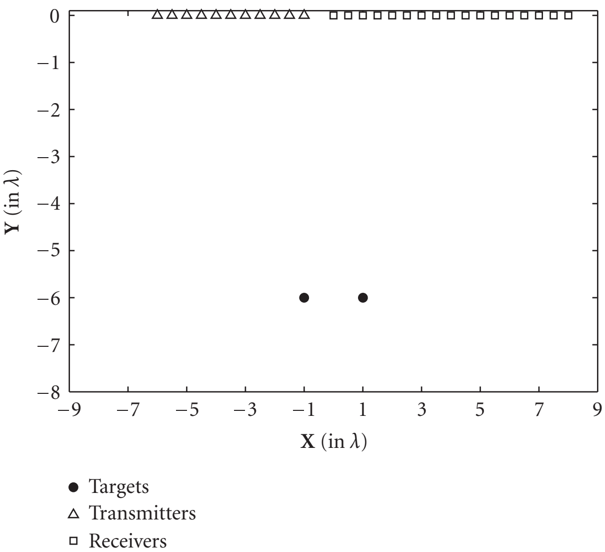

Figure 1: Geometry for the simulation in 2D space of subspace-based localization and inverse scattering of two targets.

By applying the Foldy-Lax model expression (16) to the total Green function vector $g_t(\mathbf{X}_1)$, and substituting the obtained result in the top equation in (48) while recalling the constraint imposed by the bottom equation in (48), one arrives at the more convenient statement

$$\tau_1 g_{0,t}^T(\mathbf{X}_1) K^+ g_{0,r}(\mathbf{X}_1) = 1 \tag{49}$$

involving only the known MSR matrix K and the known background Green function vectors $g_{0,r}(\mathbf{X}_1)$ and $g_{0,t}(\mathbf{X}_1)$ evaluated at the known (e.g., via time-reversal MUSIC or the high-dimensional signal subspace method) target position $\mathbf{X}_1$, from which the unknown coefficient τ_1 can be readily (and uniquely) computed. Equation (49) is the sought-after reconstruction formula. By using the formula (49) for all targets ($m = 1, 2, \ldots, M$) one thereby completes the desired inversion which under no noise is guaranteed to yield no error. The analysis above has been validated with several numerical examples during the course of this investigation, some of which are given in the following section.

6. COMPUTER SIMULATIONS

This section presents the results of computer simulations of the time-reversal MUSIC and high-dimensional signal subspace methods presented in Sections 3 and 4, respectively, as well as of the formula for the direct reconstruction of the target reflectivities presented in Section 5, which together solve the full inverse scattering problem for point targets. In Sections 6.1 and 6.2 we simulate interrogation of two targets ($M = 2$) in two-dimensional (2D) free space using a noncoincident array system formed by 11 half-wavelength-separated transmitters and 17 half-wavelength-separated receivers (refer to Figure 1 for the geometry of the experiments where triangles, squares, and circles indicate the transmitters, receivers, and targets, resp.). Unless otherwise stated, this system configuration is throughout maintained next. In all the simulations we take the wavelength $\lambda = 1$ so that $k_0 = 2\pi$. For the simulation experiments the scattering amplitude is set to $\tau_1 = 0.03$ and $\tau_2 = 0.04$ for the Born-approximated case and to $\tau_1 = 3$ and $\tau_2 = 4$ for the multiple scattering case. It is assumed that the measured MSR matrix K is contaminated by a single snapshot of additive white Gaussian noise whose variance σ^2 is related to the SNR by

$$\sigma^2 = \frac{\|K\|_F}{N_r N_t \,\mathrm{SNR}}, \tag{50}$$

where $\|K\|_F$ is the Frobenius norm of the 17×11 MSR matrix K.

The pertinent background Green function is

$$G_0(\mathbf{R}, \mathbf{R}') = H_0(k_0 |\mathbf{R} - \mathbf{R}'|), \tag{51}$$

where $H_0(\cdot)$ is the Hankel function of order zero pertinent to outgoing waves in the far zone.

In Sections 6.1 and 6.2 we pay equal attention to the two different regimes of weak scatterers where the Born approximation model is valid, and of strong scatterers interacting according to the Foldy-Lax model. To quantify the level of multiple scattering we consider the index

$$\eta = \frac{||K_b - K||_F}{\|K\|_F}, \tag{52}$$

where K_b is a reference MSR matrix applicable to the same configuration under the Born approximation. As an additional indicator, we also consider the $M \times M$ scattering potential matrix $\{A_{m,m'}\}$ (see (18)-(19)) which is essentially diagonal for Born-approximable targets. This matrix has off-diagonal couplings under more general multiple scattering. For example, for the multiple scattering case using $\tau_1 = 3$ and $\tau_2 = 4$ we have $\eta = 0.7891$ and

$$\{A_{m,m'}\} = \begin{bmatrix} 2.1794 - 1.3079i & 0.5326 - 2.2246i \\ 0.5326 - 2.2246i & 2.9059 - 1.7438i \end{bmatrix}, \tag{53}$$

while for the Born-approximated case $\eta = 0.0074$ and

$$\{A_{m,m'}\} = \begin{bmatrix} 0.0300 - 0.0000i & 0.0002 - 0.0002i \\ 0.0002 - 0.0002i & 0.0400 - 0.0000i \end{bmatrix}. \tag{54}$$

We also wish to mention that while for the time-reversal MUSIC simulations the number of targets may be estimated from the number of nonnegligible singular values of the MSR matrix K, for the high-dimensional signal subspace method this is not possible. On the other hand, preprocessing via information theoretic criteria (e.g., Akaike information [50], minimum description length [51]) can be used to estimate the number of sources (refer to [52–54] for other treatments of this "source enumeration problem"). For uniformity, it is assumed in the following that the number M of targets is known, with the understanding that in practice one may utilize complementary methods to first tackle the prior enumeration problem.

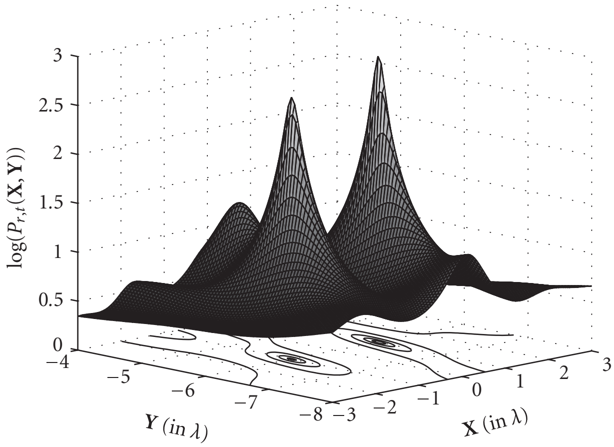

Figure 2: Generalized time-reversal MUSIC pseudospectrum for the multiple scattering case for SNR = 20 dB.

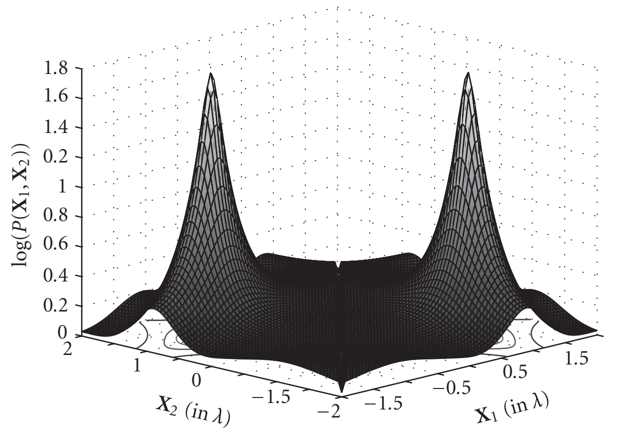

Figure 3: High-dimensional signal subspace method pseudospectrum for the multiple scattering case for SNR = 20 dB.

6.1. Pseudospectra

Figure 2 illustrates the pseudospectrum $P_{r,t}$ ($\mathbf{X} = (X, Y)$) as defined by (28) for the multiple scattering case under a 20 dB SNR. For this noise level the pseudospectrum peaks at the correct target positions; however, noticeable spurious peaks appear with a level comparable to those of the correct target positions. In fact, for higher levels of noise these spurious peaks dominate, seriously affecting the target position estimates. Figure 3 shows, for the correct value of the targets' Y coordinate, the corresponding high-dimensional signal subspace method pseudospectrum (44) for 20 dB SNR. An advantage of the high-dimensional signal subspace method established in Section 4 is in the number of localizable targets. For example, Figure 4 shows, for the Born approximated case, the pseudospectrum for the geometry shown in Figure 1 corresponding to MSR data gathered using only the 4 central array elements (two receivers and two transmitters). Under these conditions time-reversal MUSIC is limited to detecting one target only. On the contrary, the high-dimensional algorithm is capable of detecting two targets (theoretically, up to 3 targets) as illustrated in the figure.

6.2. Comparative study

In this subsection we compare (again, for the same system shown in Figure 1) the performance under different noise levels of the target location approaches associated to the pseudospectra derived in this paper in (26), (28), (33), and (44) as well as the CRB derived in Appendix C. To simplify the analysis and reduce the computational overhead, in the following the Y coordinate of the targets will be fixed to the correct value, assuming that this (range) parameter has been estimated a priori, and the respective search will be carried out only for the unknown X coordinate. For each of the methods that we are testing we calculate an estimate of the respective variance from 50 statistical replications of each noise level according to

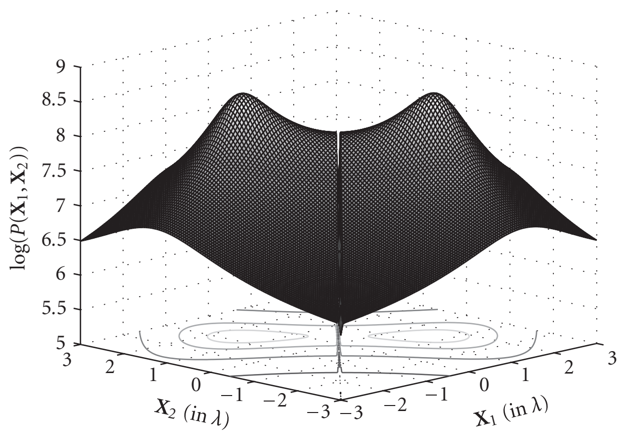

Figure 4: High-dimensional pseudospectrum for two antennas detecting two Born-approximated targets located at $\mathbf{X}_1 = (X_1 = -1,\ Y_1 = -6)$ and $\mathbf{X}_2 = (X_2 = 1,\ Y_2 = -6)$ for SNR = 50 dB.

$$\widehat{\sigma}^2 = E\Big[\big(X_m - \widehat{X}_m\big)^2\Big], \tag{55}$$

where $E[\cdot]$ denotes the expected value, X_m is the actual value of the mth target's X coordinate, and $\widehat{X}_m$ is the estimated value. In the following illustrations, the lowest value of SNR considered was chosen around the critical value for which the signal subspace methods appear to break down, for example, when the pseudospectrum no longer works due to spurious peaks.

Figure 5 shows the variances of the position estimates of target 2 versus SNR for the general multiple scattering case for which $\tau_1 = 3$ and $\tau_2 = 4$. The high-dimensional signal subspace method estimates are the best, relative to those of the other methods, but the time-reversal MUSIC approach also performs well despite the presence of significant multiple scattering, as predicted by the theory in Section 3. Similar results (not shown) were obtained for target 1. The

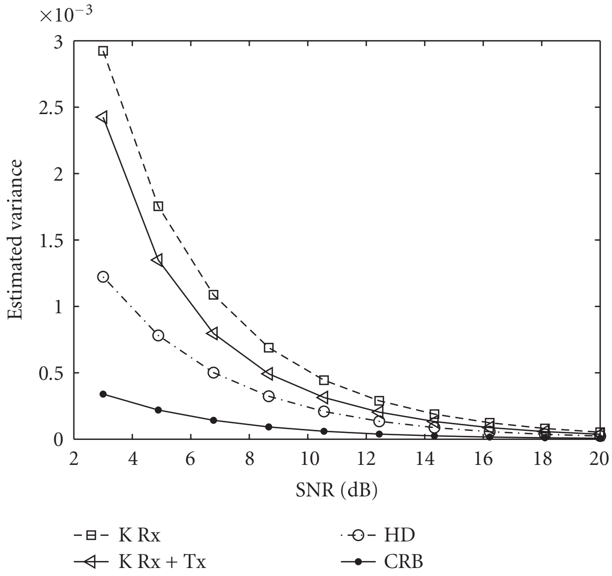

FIGURE 5: Estimated variance for the estimate of the position of target 2 versus SNR corresponding to the location under significant multiple scattering conditions of two targets located at $\mathbf{X}_1 = (X_1 = -1,\ Y_1 = -6)$ and $\mathbf{X}_2 = (X_2 = 1,\ Y_2 = -6)$. The different plots correspond to the receive-mode time-reversal MUSIC method (K Rx), the transmit-plus-receive time-reversal MUSIC method (K Rx+Tx), the high-dimensional method (HD), and the CRB. The search was done on the X coordinate only, the Y coordinate being fixed during the simulations at the correct value.

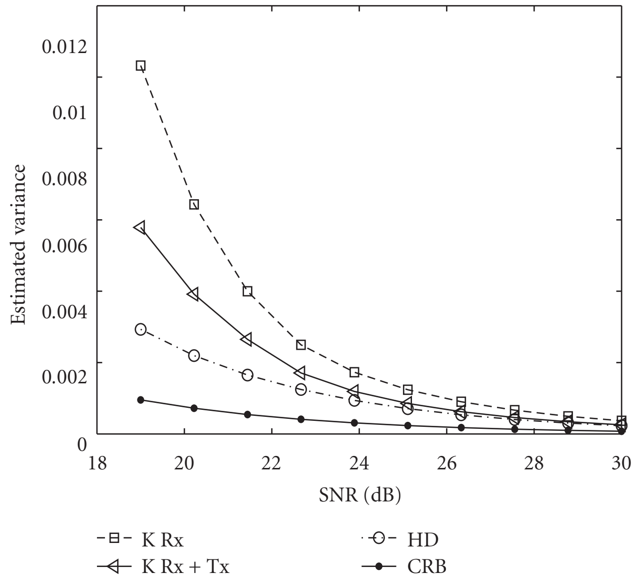

FIGURE 6: Estimated variance for the estimates of the position of target 2 versus SNR corresponding to the location under Born-approximable conditions of two targets located at $\mathbf{X}_1 = (X_1 = -1,\ Y_1 = -6)$ and $\mathbf{X}_2 = (X_2 = 1,\ Y_2 = -6)$ (refer to the caption of Figure 5 for details applicable also to this figure).

corresponding plot for the Born approximation case where $\tau_1 = 0.03$ and $\tau_2 = 0.04$ is shown in Figure 6. In this illustration, again the high-dimensional approach outperforms the other approaches.

6.3. Scattering amplitude reconstruction

In this subsection we compare the variances of the noniterative algorithm proposed in Section 5, the estimate obtained from ignoring the multiple scattering between the targets and using (36), and the CRB.

The variances are estimated according to the expression

$$E\Big[\big(\Re(\hat{\tau}_m - \tau_m)\big)^2\Big], \tag{56}$$

where $\Re[\cdot]$ denotes the real part, $\hat{\tau}_m$ is the estimate of the scattering potential of target m, and τ_m is the actual value.

The estimated variance for the estimation of the reflectivities and the X coordinate of the targets after 50 instances of noise is plotted in Figure 7. Although not shown in the plots, the performance of the noniterative method is comparable to that of the iterative algorithm proposed in [13] despite its requirement of a single calculation. The results suggest correlation between position and reflectivity estimation errors and also show that the estimation error when the Born approximation is wrongly assumed can be significant, even under no noise.

7. CONCLUDING REMARKS

This work investigated signal subspace methods for inverse scattering of point targets embedded in a known background medium from knowledge of the MSR matrix as measured by a general array of wave transmitters and receivers. The target location methods presented in the paper were comparatively characterized partly analytically and partly numerically. It was shown that the high-dimensional signal subspace method outperforms the time-reversal approach in number of localizable targets if the Born approximation holds. The version of the method for multiply scattering targets can also enhance in certain cases the number of localizable targets. The high-dimensional signal subspace approach was also found to outperform the time-reversal approach in localization error (variance). The problem of estimating the reflectivities was solved by means of a direct, noniterative formula which holds even if there is nonnegligible multiple scattering in which case the problem is nonlinear.

A drawback of the high-dimensional signal subspace method is its high computational intensity. Thus for M targets the high-dimensional signal subspace method requires a search in $2M$ ($3M$) dimensions in the 2D (3D) cases. On the other hand, this difficulty can be dealt with partly via prior approaches rendering initial values for the searches such as time-reversal MUSIC. The time-reversal MUSIC estimation for the 2D (3D) case requires an exhaustive search in 2D (3D) dimensions. For instance, as long as time-reversal MUSIC or another method is applicable, one can use its estimate as an initial value for, for example, a fixed point iteration routine (e.g., Newton, quasi-Newton), or other nonlinear

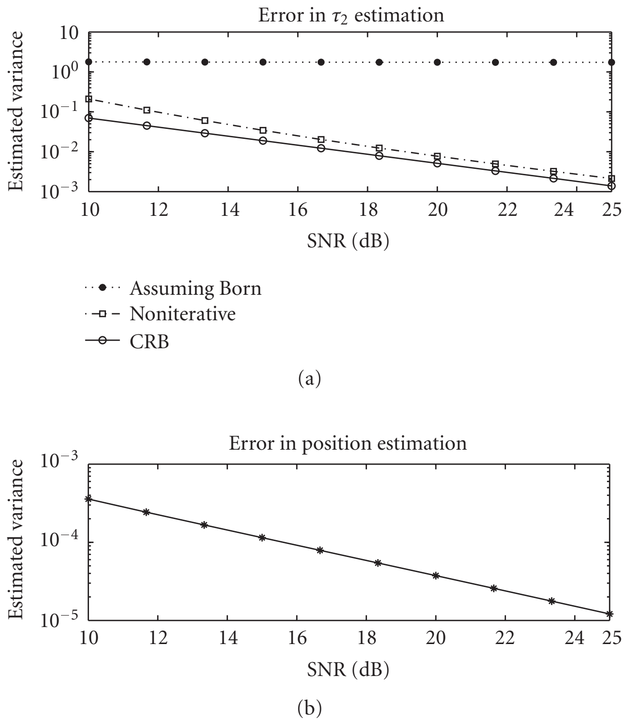

FIGURE 7: Variance (logarithmic scale) of the estimate of the scattering amplitude and the position of target 2 under significant multiple scattering for both the noniterative estimation algorithm and the Born-approximated estimate (36). The targets are located at the positions $\mathbf{X}_1 = (X_1 = -1,\ Y_1 = -6)$ and $\mathbf{X}_2 = (X_2 = 1,\ Y_2 = -6)$. Also shown is the CRB of the estimation of the real part of the scattering potential. The plot shows the average after 50 instances of random noise.

optimization approach yielding the estimate of the high-dimensional signal subspace method. If there are more targets than sensors, the MUSIC technique breaks down, while, for example, under the Born approximation the high-dimensional method remains useful as long as $M < N_t N_r - n(n-1)/2$ where n is the number of coincident elements across arrays. In this case genetic algorithms can be used which significantly reduce computational burden [26].

A limitation of the signal subspace methods discussed in this paper is their need for a priori knowledge of the background medium properties, in particular, the background Green function. Another limitation is the strong sensitivity to noise and other perturbations. Both of these limitations are expected to diminish by implementing time domain versions of the imaging methods considered in the present paper which are expected to be statistically more stable in moderately cluttered environments due to self-averaging over individual realizations of the medium as has been shown for time reversal in [31, 55] (also relevant is the experiment in [56]). Furthermore, for densely cluttered environments one is forced to abandon the coherent signal regime which was the focus of our presentation and formulate, for example, a radiative transfer-based type of imaging where the Green function of the present treatment must be substituted by the Green function relevant to the partial differential equations governing this type of imaging, for example, the diffusion equation pertinent to certain kinds of such cluttered media (see [12] for the key ideas). In this connection we also wish to point out that the vector electromagnetic version of the methods is conceptually similar to that given in this paper for scalar fields if one substitutes the scalar Green functions and scattering potential operator of this work by the respective electromagnetic dyadic Green functions and (generally dyadic) scattering potential operator (the key formal tools can be found in [57, 58], see also [20, pages 516–518], for the pertaining vector form of the Foldy-Lax model).

We are currently working on the generalization for certain classes of extended (nonpoint-like) targets, particularly piecewise constant scattering potentials, of many of the methods established in this work and plan to report the associated results in the future. Background for the treatment of the envisioned generalization is contained, for example, in the work of Tsihrintzis and Devaney [59] on ML localization of a known strongly scattering object, of Zhao [60] and Hou et al. [61] on time-reversal localization of an extended target, of Poon et al. [62] on electromagnetic information channels (where we will consider, e.g., multipole or other extended object modes instead of the singular (point-like) scatterer modes of the present work) and, more recently, of Pierri et al. [63] on shape reconstruction beyond the physical optics model which has, in fact, connection to some of our linear-to-nonlinear signal model extensions such as the time-reversal generalization to multiply scattering targets. Another natural and important line of continuation of the present effort is further performance analysis which will benefit from work on perturbation analysis of signal subspace methods [64, 65].

APPENDICES

A. ELABORATION CONCERNING (22)–(24)

This appendix discusses for a given array of N_r receivers whose positions $\mathbf{R}_r(l)$, $l = 1, 2, \ldots, N_r$, are fixed, the question of the possible existence of certain configurations of target positions $\mathbf{X}_m$, $m = 1, 2, \ldots, M$, for which the resulting receive background Green function vectors $g_{0,r}(\mathbf{X}_m)$, $m = 1, 2, \ldots, M$, are linearly dependent. Of particular interest are the conditions under which $d_r \equiv \dim\{g_{0,r}(\mathbf{X}_1), g_{0,r}(\mathbf{X}_2), \ldots, g_{0,r}(\mathbf{X}_M)\} < \min(M, N_r)$. The analysis and results apply to the transmit background Green function vectors after obvious substitutions.

It is not hard to show that the necessary and sufficient condition for $d_r < M$ if $M \le N_r$ is

$$\det\left[h_C(\mathbf{X}_1)\ \ h_C(\mathbf{X}_2)\ \ \cdots\ \ h_C(\mathbf{X}_M)\right] = 0, \qquad (A.1)$$

where $C = (\alpha_1, \alpha_2, \ldots, \alpha_M)$ is any of the $N_r!/[(N_r - M)!M!]$ combinations of M elements of $\{\mathbf{R}_r(1), \mathbf{R}_r(2), \ldots, \mathbf{R}_r(N_r)\}$ and the associated $h_C(\mathbf{X}_m) = [g_{0,r}(\alpha_1, \mathbf{X}_m)\ g_{0,r}(\alpha_2, \mathbf{X}_m)\ \cdots\ g_{0,r}(\alpha_M, \mathbf{X}_m)]^T$. Also, the necessary and sufficient condition for $d_r = \mathrm{rank}(\mathcal{G}_{0,r}) = \mathrm{rank}(\mathcal{G}_{0,r}^T) < N_r$ if $M > N_r$ is

$$\det\left[g_{0,r}(\beta_1) \quad g_{0,r}(\beta_2) \quad \cdots \quad g_{0,r}(\beta_{N_r})\right] = 0, \tag{A.2}$$

where $(\beta_1, \beta_2, \ldots, \beta_{N_r})$ is any of the $M!/[(M-N_r)!N_r!]$ combinations of N_r elements of $\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M\}$. If and only if both conditions in (A.1)-(A.2) are disobeyed for the given array for any set of target positions $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_M)$, then (22) holds. The conditions in question hold only in pathological rare configurations [14] so that, as in other studies [13–15, 17, 18, 22, 31, 66], we assume in this paper that they do not hold so that (22) holds.

For example, for $M = 2 \leq N_r$ the condition in (A.1) becomes with reference to (6)

$$\frac{G_0(\mathbf{R}_r(i), \mathbf{X}_1)}{G_0(\mathbf{R}_r(l), \mathbf{X}_1)} = \frac{G_0(\mathbf{R}_r(i), \mathbf{X}_2)}{G_0(\mathbf{R}_r(l), \mathbf{X}_2)} \tag{A.3}$$

which must hold for any combination of bistatic subarrays labelled $(i, l) \in \{1, 2, \ldots, N_r\} \times \{1, 2, \ldots, N_r\}$, $\times$ denoting Cartesian product. In 3D free space, for which $G_0(\mathbf{R}_r(l), \mathbf{X}_m)$ is of the form $-e^{ik_0 d_{lm}}/4\pi d_{lm}$ where k_0 is the free space wavenumber and

$$d_{lm} \equiv \left|\mathbf{R}_r(l) - \mathbf{X}_m\right|, \tag{A.4}$$

this implies that for any pair (i, l) representing a particular bistatic subarray of the full array both targets must lie in a hyperbola of the form [14]

$$d_{i1} - d_{l1} = d_{i2} - d_{l2} = \text{const} \tag{A.5}$$

whose foci are the positions $\mathbf{R}_r(i)$ and $\mathbf{R}_r(l)$, and in the geometric place defined by

$$\log d_{i1} - \log d_{l1} = \log d_{i2} - \log d_{l2} = \text{const}, \tag{A.6}$$

which is formally similar to the so-called ovals of Cassini of bistatic radar [67, pages 70–72], but where in the latter addition takes the place of subtraction above so that the curves associated to (A.6) correspond to contours of constant SNR *difference*. Returning to the hyperbola in (A.5), we find it to have the form of the so-called isorange-*difference* contours of bistatic radar which are perpendicular, at intersection points, to the pertinent isorange contours which are ellipses [67, pages 60–62]. This means that target 1 must be simultaneously in the same isorange-difference and isoSNR-difference contours defined, for each bistatic subarray within the full array, by (A.5) and (A.6), respectively, as target 2, which is very unlikely. A complementary counterpart of this discussion is the manifestation of (A.2) for $N = 2 < M$, where one obtains in general

$$\frac{G_0(\mathbf{R}_r(1), \mathbf{X}_1)}{G_0(\mathbf{R}_r(2), \mathbf{X}_1)} = \frac{G_0(\mathbf{R}_r(1), \mathbf{X}_m)}{G_0(\mathbf{R}_r(2), \mathbf{X}_m)} = \text{const}, \tag{A.7}$$

where m represents any of the M targets. In free space this becomes

$$d_{11} - d_{21} = d_{1m} - d_{2m} = \text{const}, \tag{A.8}$$

$$\log d_{11} - \log d_{21} = \log d_{1m} - \log d_{2m} = \text{const}. \tag{A.9}$$

Thus given the two receivers, then all the M targets must lie in the intersection of the same isorange-difference and iso-SNR-difference curves defined for those two receivers by (A.8) and (A.9), respectively, which is a very specialized situation.

B. RANK OF Γ AND Π

The general situation is assumed wherein the transmit array has N_t elements, the receive array has N_r elements, and n of these elements are coincident. The linear independence of the columns of the $N_r N_t \times M^2$ matrix Γ (see (42) and (43)) for the case where $M \leq N_m = \min(N_r, N_t)$ can be shown by rewriting it in terms of Kronecker products:

$$\begin{aligned}\Gamma &= \mathcal{G}_{0,t} \otimes \mathcal{G}_{0,r} \\ &= \left[g_{0,t}(\mathbf{X}_1) \otimes g_{0,r}(\mathbf{X}_1) \;\; g_{0,t}(\mathbf{X}_1) \otimes g_{0,r}(\mathbf{X}_2) \;\cdots\; g_{0,t}(\mathbf{X}_M) \otimes g_{0,r}(\mathbf{X}_M)\right],\end{aligned} \tag{B.1}$$

where $\mathcal{G}_{0,t}$ and $\mathcal{G}_{0,r}$ are defined by our discussion in (10) and $\otimes$ indicates the Kronecker or direct product. In this case the rank of $\mathcal{G}_{0,t}$ and $\mathcal{G}_{0,r}$ is M as we explained in Section 2 and in Appendix A and, in particular, the different background Green function vectors are linearly independent. On the other hand, it is easy to show that the rank of the Kronecker product of two matrices is equal to the product of the rank of each matrix [68, page 246], so that the rank of Γ is M^2 and all columns are linearly independent. This also implies that for the Born approximated case (Section 4.1) the $N_r N_t \times M$ matrix Π has rank M since its columns are a subset of the set of columns of Γ. Note also that if $M^2 > N_r N_t$ the rank of Γ is $N_r N_t$ and the set of columns becomes linearly dependent.

A particularly useful result occurs for the Born approximation case when $N_m < M < N_r N_t - n(n-1)/2$. The $N_r N_t \times M$ propagator matrix Π is given by

$$\begin{aligned}\Pi &= \mathcal{G}_{0,t} \odot \mathcal{G}_{0,r} \\ &= \left[g_{0,t}(\mathbf{X}_1) \otimes g_{0,r}(\mathbf{X}_1) \;\; g_{0,t}(\mathbf{X}_2) \otimes g_{0,r}(\mathbf{X}_2) \;\cdots\; g_{0,t}(\mathbf{X}_M) \otimes g_{0,r}(\mathbf{X}_M)\right],\end{aligned} \tag{B.2}$$

where $\odot$ represents the Khatri-Rao matrix product [15].

In matrix $\mathcal{G}_{0,t}$ any set of N_t columns is linearly independent while in matrix $\mathcal{G}_{0,r}$ any set of N_r columns is linearly independent (see Appendix A). Each of these sets spans the column space of the corresponding matrices, that is,

$$\begin{aligned} g_{0,r}(\mathbf{X}_i) &\in \operatorname{span}\left[g_{0,r}(\mathbf{X}_1), \ldots, g_{0,r}(\mathbf{X}_{N_r})\right], \\ & \qquad i = N_r + 1, \ldots, M, \\ g_{0,t}(\mathbf{X}_i) &\in \operatorname{span}\left[g_{0,t}(\mathbf{X}_1), \ldots, g_{0,t}(\mathbf{X}_{N_t})\right], \\ & \qquad i = N_t + 1, \ldots, M. \end{aligned} \tag{B.3}$$

From (B.3) we have for $l = N_m + 1, \dots, M$

$$g_{0,t}(\mathbf{X}_l) \otimes g_{0,r}(\mathbf{X}_l) = \left(\sum_{j=1}^{N_t} a_{j,l} g_{0,t}(\mathbf{X}_j)\right) \otimes \left(\sum_{k=1}^{N_r} b_{k,l} g_{0,r}(\mathbf{X}_k)\right) = \sum_{j=1}^{N_t}\sum_{k=1}^{N_r} a_{j,l} b_{k,l} g_{0,t}(\mathbf{X}_j) \otimes g_{0,r}(\mathbf{X}_k), \tag{B.4}$$

where $\{a_{j,l}\}$ and $\{b_{k,l}\}$ are not-all-zero scalars.

In order for the set of M columns to be dependent we must be able to find a set of scalars $\{A_i\}$ not all zeros such that

$$\sum_{i=1}^{M} A_i g_{0,t}(\mathbf{X}_i) \otimes g_{0,r}(\mathbf{X}_i) = 0, \tag{B.5}$$

where here 0 denotes the $N_t N_r \times 1$ zero vector.

Using (B.4),

$$\sum_{l=1}^{N_m}\left[A_l + \sum_{i=N_m+1}^{M} A_i a_{l,i} b_{l,i}\right] g_{0,t}(\mathbf{X}_l) \otimes g_{0,r}(\mathbf{X}_l) + \sum_{j=1}^{N_t}\sum_{k=1}^{N_r} g_{0,t}(\mathbf{X}_j) \otimes g_{0,r}(\mathbf{X}_k)\left(1-\delta_{j,k}\right)\sum_{i=N_m+1}^{M} A_i a_{j,i} b_{k,i} = 0. \tag{B.6}$$

Since the two terms are independent, one arrives at the conditions

$$A_l + \sum_{i=N_m+1}^{M} A_i a_{l,i} b_{l,i} = 0, \qquad \sum_{i=N_m+1}^{M} A_i a_{j,i} b_{k,i} = 0 \tag{B.7}$$

for all $l = 1, 2, \dots, N_m$, $j \neq k$, $j = 1, 2, \dots, N_t$, and $k = 1, 2, \dots, N_r$.

Note that if reciprocity holds, this meaning that the order of the arguments of the background Green function is inconsequential, as is the case in the present scalar Helmholtz operator-based formulation, and we have n coincident elements, then $n(n-1)/2$ rows of the vectors in (B.5) are repeated. This causes the system of equations represented by conditions (B.7) to correspond to $N_r N_t - n(n-1)/2$ nonredundant linear equations with M unknowns A_i. As long as $M < N_r N_t - n(n-1)/2$ this is an overdetermined system of equations which will be obeyed only for very specialized situations. In our computer simulation geometries this particular situation never occurred. Finally, note that if $M > N_r N_t - n(n-1)/2$, we have more unknowns than equations and there exists a nontrivial solution to the system.

C. FISHER INFORMATION MATRIX/CRB

Consider a general array of N_r receiver and N_t transmitter elements interrogating in 2D space M targets at positions $\mathbf{X}_m = (X_m, Y_m)$, $m = 1, 2, \dots, M$ having complex-valued reflectivities $\tau_m = \tau_m^{(r)} + \imath\tau_m^{(i)}$, where here $\imath \equiv \sqrt{-1}$ while i denotes the imaginary part. The noisy observations are modeled as

$$\hat{\bar{K}} = \bar{K}(\theta) + w, \tag{C.1}$$

where

$$\theta = \left[X_1, \dots, X_M, Y_1, \dots, Y_M, \tau_1^{(r)}, \dots, \tau_M^{(r)}, \tau_1^{(i)}, \dots, \tau_M^{(i)}, \sigma^2\right]^T \tag{C.2}$$

is the vector of parameters to be estimated, w is complex white Gaussian noise with variance σ^2, and $\bar{K}(\theta)$ is the vectorized version of the matrix K (see (3)) with components

$$\bar{K}_i(\theta) = \sum_{m=1}^{M} \tau_m G_0\left(\mathbf{R}_r(u), \mathbf{X}_m\right) G\left(\mathbf{X}_m, \mathbf{R}_t(v)\right), \quad i = 1, \dots, N_r N_t, \tag{C.3}$$

where u and v are indices that depend on i (as detailed in (3), (6)-(7)).

The Fisher information matrix relevant to this problem is given by (see [69, page 525])

$$\left[I(\theta)\right]_{ij} = \frac{N_r N_t}{\sigma^4}\delta_{i,j}\delta_{i,4M+1} + \frac{2}{\sigma^2}\Re\left[\sum_{n=1}^{N_r N_t} \frac{\partial \bar{K}_n^*(\theta)}{\partial\theta_i}\frac{\partial \bar{K}_n(\theta)}{\partial\theta_j}\right]. \tag{C.4}$$

From (C.3) we find that

$$\begin{aligned}
\frac{\partial \bar{K}_i(\theta)}{\partial\sigma^2} &= 0, \\
\frac{\partial \bar{K}_i(\theta)}{\partial X_j} &= \tau_j \frac{\partial G_0\left(\mathbf{R}_r(u), \mathbf{X}_m\right)}{\partial X_j} G\left(\mathbf{X}_j, \mathbf{R}_t(v)\right) + \sum_{m=1}^{M} \tau_m G_0\left(\mathbf{R}_r(u), \mathbf{X}_m\right)\frac{\partial G\left(\mathbf{X}_m, \mathbf{R}_t(v)\right)}{\partial X_j}, \\
\frac{\partial \bar{K}_i(\theta)}{\partial\tau_j^{(t)}} &= \xi(t) G_0\left(\mathbf{R}_r(u), \mathbf{X}_m\right) G\left(\mathbf{X}_j, \mathbf{R}_t(v)\right) + \sum_{m=1}^{M} \tau_m G_0\left(\mathbf{R}_r(u), \mathbf{X}_m\right)\frac{\partial G\left(\mathbf{X}_m, \mathbf{R}_t(v)\right)}{\partial\tau_j^{(t)}},
\end{aligned} \tag{C.5}$$

where t denotes r (standing for real) or i (standing for imaginary) and $\xi(r) = 1$ and $\xi(i) = \imath$.

The total Green function is defined by (17) and the derivatives of this Green function are determined by differentiating the Foldy-Lax equation

$$G\left(\mathbf{X}_m, \mathbf{R}_t(v)\right) = G_0\left(\mathbf{X}_m, \mathbf{R}_t(v)\right) + \sum_{m' \neq m} \tau_{m'} G_0\left(\mathbf{X}_m, \mathbf{X}_{m'}\right) G\left(\mathbf{X}_{m'}, \mathbf{R}_t(v)\right). \tag{C.6}$$

This leads to

$$\begin{bmatrix} \dfrac{\partial G(\mathbf{X}_1,\mathbf{R}_t(\nu))}{\partial X_j} \\ \dfrac{\partial G(\mathbf{X}_2,\mathbf{R}_t(\nu))}{\partial X_j} \\ \vdots \\ \dfrac{\partial G(\mathbf{X}_{j-1},\mathbf{R}_t(\nu))}{\partial X_j} \\ \dfrac{\partial G(\mathbf{X}_j,\mathbf{R}_t(\nu))}{\partial X_j} \\ \dfrac{\partial G(\mathbf{X}_{j+1},\mathbf{R}_t(\nu))}{\partial X_j} \\ \vdots \\ \dfrac{\partial G(\mathbf{X}_M,\mathbf{R}_t(\nu))}{\partial X_j} \end{bmatrix} = H^{-1} \begin{bmatrix} \tau_j \dfrac{\partial G_0(\mathbf{X}_1,\mathbf{X}_j)}{\partial X_j} G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \tau_j \dfrac{\partial G_0(\mathbf{X}_2,\mathbf{X}_j)}{\partial X_j} G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \vdots \\ \tau_j \dfrac{\partial G_0(\mathbf{X}_{j-1},\mathbf{X}_j)}{\partial X_j} G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \dfrac{\partial G_0(\mathbf{X}_j,\mathbf{R}_t(\nu))}{\partial X_j} + \sum_{m'\neq j} \tau_{m'} \dfrac{\partial G_0(\mathbf{X}_j,\mathbf{X}_{m'})}{\partial X_j} G(\mathbf{X}_{m'},\mathbf{R}_t(\nu)) \\ \tau_j \dfrac{\partial G_0(\mathbf{X}_{j+1},\mathbf{X}_j)}{\partial X_j} G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \vdots \\ \tau_j \dfrac{\partial G_0(\mathbf{X}_M,\mathbf{X}_j)}{\partial X_j} G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \end{bmatrix}, \tag{C.7}$$

$$\begin{bmatrix} \dfrac{\partial G(\mathbf{X}_1,\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \\ \dfrac{\partial G(\mathbf{X}_2,\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \\ \vdots \\ \dfrac{\partial G(\mathbf{X}_{j-1},\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \\ \dfrac{\partial G(\mathbf{X}_j,\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \\ \dfrac{\partial G(\mathbf{X}_{j+1},\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \\ \vdots \\ \dfrac{\partial G(\mathbf{X}_M,\mathbf{R}_t(\nu))}{\partial \tau_j^{(t)}} \end{bmatrix} = \xi(t) H^{-1} \begin{bmatrix} G_0(\mathbf{X}_1,\mathbf{X}_j) G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ G_0(\mathbf{X}_2,\mathbf{X}_j) G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \vdots \\ G_0(\mathbf{X}_{j-1},\mathbf{X}_j) G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ 0 \\ G_0(\mathbf{X}_{j+1},\mathbf{X}_j) G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \\ \vdots \\ G_0(\mathbf{X}_M,\mathbf{X}_j) G(\mathbf{X}_j,\mathbf{R}_t(\nu)) \end{bmatrix}.$$

Finally, the CRB is obtained by replacing the expressions above in (C.4), inverting the resulting Fisher matrix, and taking the diagonal elements [69, page 40].

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