Hindawi Publishing Corporation
EURASIP Journal on Advances in Signal Processing
Volume 2007, Article ID 85606, 9 pages
doi:10.1155/2007/85606

Research Article

Regularizing Inverse Preconditioners for Symmetric Band Toeplitz Matrices

P. Favati,[1] G. Lotti,[2] and O. Menchi[3]

[1] *Istituto di Informatica e Telematica (IIT), CNR, Via G. Moruzzi 1, 56124 Pisa, Italy*
[2] *Dipartimento di Matematica, Università di Parma, Parco Area delle Scienze 53/A, 43100 Parma, Italy*
[3] *Dipartimento di Informatica, Università di Pisa, Largo Pontecorvo 3, 56127 Pisa, Italy*

Received 22 September 2006; Revised 31 January 2007; Accepted 16 March 2007

Recommended by Paul Van Dooren

Image restoration is a widely studied discrete ill-posed problem. Among the many regularization methods used for treating the problem, iterative methods have been shown to be effective. In this paper, we consider the case of a blurring function defined by space invariant and band-limited PSF, modeled by a linear system that has a band block Toeplitz structure with band Toeplitz blocks. In order to reduce the number of iterations required to obtain acceptable reconstructions, in [1] an inverse Toeplitz preconditioner for problems with a Toeplitz structure was proposed. The cost per iteration is of $O(n^2 \log n)$ operations, where n^2 is the pixel number of the 2D image. In this paper, we propose inverse preconditioners with a band Toeplitz structure, which lower the cost to $O(n^2)$ and in experiments showed the same speed of convergence and reconstruction efficiency as the inverse Toeplitz preconditioner.

1. INTRODUCTION

Many image restoration problems can be modeled by the linear system

$$A\mathbf{x} = \mathbf{b} - \mathbf{w}, \tag{1}$$

where $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{w}$ represent the original image, the observed image, and the noise, respectively. Matrix A is defined by the so-called *point spread function* (PSF), which describes how the image is blurred out. If the PSF is space invariant with respect to translation, that is, a single pixel is blurred independently of its location, and is bandlimited, that is, it has a local action, matrix A turns out to have a band block Toeplitz structure with band Toeplitz blocks (hereafter band BTTB structure).

Since A is generally ill-conditioned, the exact solution of the system

$$A\mathbf{y} = \mathbf{b} \tag{2}$$

may differ considerably from $\mathbf{x}$ even if $\mathbf{w}$ is small, and a *regularized* solution of (1) is sought. A widely used regularization technique [2–4] suggests solving (2) by employing the conjugate gradient (CG) method when A is positive definite or some of its generalizations for the nonpositive definite case. In fact, CG is a semiconvergent method: at first the iteration reconstructs the low frequency components of the original signal, then subsequently, the iteration also starts to recover increasing frequency components, corresponding to the noise. Thus the iteration must be stopped when the noise components start to interfere. A general purpose preconditioner, which reduces the condition number by clustering all the eigenvalues of the preconditioned matrix around 1, is not satisfactory in the present case. If it were applied, the signal subspace, generated by the eigenvectors corresponding to the largest eigenvalues, and the noise subspace, generated by the eigenvectors corresponding to the lowest eigenvalues, would be mixed up and the effect of the noise would appear before the image is fully reconstructed. In the present context, a good preconditioner should reduce the number of iterations required to reconstruct the information from the signal subspace, that is, it should only cluster the largest eigenvalues around 1, and leave the others out of the cluster.

This requires knowledge (or at least an estimate) of a parameter $\tau > 0$, called the *regularization parameter*, such that the eigenvalues of the matrix A which have a modulus greater than τ correspond to the signal subspace. Techniques which allow for an estimate of τ are described in the literature (see, e.g., [5]).

With a matrix A having a BTTB structure, the product $A\mathbf{z}$ (required in the application of CG) can be computed by means of the fast Fourier transform in $O(n^2 \log n)$ operations, where n^2 is the number of rows and columns of A. Then the construction of the preconditioner and its use should have costs not exceeding $O(n^2 \log n)$ operations. The preconditioners based on circulant matrices (see the extensive bibliography in [6]) satisfy this cost requirement, improve the convergence speed, and can be easily adapted to cope with the noise. The cost of the circulant preconditioners cannot be lowered when A has a band structure too, as in the present case. Band Toeplitz preconditioners, which have a cost per iteration of the same order as the cost of computing $A\mathbf{z}$ (i.e., $O(n^2)$), without any regularizing property, have been proposed in [7–9].

Band Toeplitz preconditioners with a regularizing property and with a cost per iteration $O(n^2)$ have been proposed in [10]. The reduction in cost was achieved by performing approximate spectral factorizations of a trigonometric bivariate polynomial which, through a fit technique, regularizes the symbol function associated with A. In this way, the preconditioner is expressed as the product of two band triangular factors.

Another strategy with the cost $O(n^2 \log n)$ consists in the use of an inverse Toeplitz preconditioner (see [11] for the general purpose preconditioner and [1] for the regularizing preconditioner).

In this paper, we consider some inverse preconditioners which have a band BTTB structure. We compare them with the inverse Toeplitz preconditioner of [1] and show that the reduction in cost per iteration to $O(n^2)$ operations does not imply a substantial decrease in the speed of convergence or in the reconstruction efficiency. The structure of matrix A is defined in detail in Section 2; three different banded preconditioners are described in Section 3, together with the inverse Toeplitz preconditioner. Then the banded preconditioners are tested and compared with the Inverse Toeplitz and the results are shown in Section 4.

2. PRELIMINARIES

We assume here that the original image has size $n\times n$, hence $\mathbf{x}$, $\mathbf{b}$, and $\mathbf{w}$ are n^2 vectors and A is an $n^2\times n^2$ matrix. Let the PSF describing the blurring be space invariant and bandlimited. The PSF can thus be represented by a *mask* of finite size $M = (m_{k,j})$, $-\mu \le k, j \le \mu$, with $\mu < n$. Matrix A has a band BTTB structure with bandwidth μ of the form

$$A = \begin{bmatrix} A_0 & A_1 & \dots & A_{n-1} \\ A_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_1 \\ A_{-n+1} & \dots & A_{-1} & A_0 \end{bmatrix}, \qquad A_k = O \text{ for } |k| > \mu, \tag{3}$$

where

$$A_k = \begin{bmatrix} a_{k,0} & a_{k,1} & \dots & a_{k,n-1} \\ a_{k,-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{k,1} \\ a_{k,-n+1} & \dots & a_{k,-1} & a_{k,0} \end{bmatrix}, \qquad a_{k,j} = \begin{cases} m_{k,j} & \text{for } |k|, |j| \le \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We assume that A is symmetric, that is, $m_{k,j} = m_{-k,-j}$ for $k, j = -\mu, \dots, \mu$. In addition, we assume that M is nonnegative and normalized, that is, $M \ge O$ and $\sum_{k,j} m_{k,j} = 1$.

We look for a preconditioner P, to be applied as follows:

$$PA\mathbf{y} = P\mathbf{b}. \tag{5}$$

Hence P is an inverse preconditioner, like the one introduced in [1].

If A is positive definite, system (5) is solved by CG. Otherwise, we assume that its eigenvalues verify $\lambda \ge -\tau$; in this case system (5) is solved by MR-II [2, 12] (we have chosen MR-II instead of CGNR because in our numerical experience CGNR appears to be slower even if skillfully preconditioned). Both CG and MR-II methods require one matrix-vector product per iteration. For BTTB matrices, the product can be computed by an ad hoc procedure relying on FFT, with cost $O(n^2 \log n)$. However, in our case, where a band is present, the direct computation, performed in $O(\mu^2 n^2)$ operations with μ constant, may be advantageous.

Even with a nonpositive definite A, the preconditioner P should be chosen positive definite and P^{-1} should approximate A in a regularizing way.

The *symbol* function of A is

$$f(\theta, \eta) = \sum_{k,j=-\mu}^{\mu} m_{k,j} e^{\mathbf{i}(k\theta + j\eta)}, \tag{6}$$

where $\mathbf{i}$ is the complex unit, such that $\mathbf{i}^2 = -1$. Since A is symmetric, f is a real function in the Wiener class. The classical Grenander and Szegő theorem [13, page 64] on the spectrum of symmetric Toeplitz matrices, extended to the 2D case in [14, Theorm 6.4.1], states that for any bounded function F uniformly continuous on $\mathbf{R}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n^2} F(\lambda_i(A)) = \frac{1}{4\pi^2} \iint_0^{2\pi} F(f(\theta, \eta))\, d\theta\, d\eta, \tag{7}$$

where $\lambda_i(A)$ are the eigenvalues of A. Moreover, if $f_{\min}$ and $f_{\max}$ are the minimum and maximum values of f, respectively, (in our case $f_{\max} = 1$) with $f_{\min} < f_{\max}$, then for any n,

$$f_{\min} < \lambda_i(A) < f_{\max} \quad \text{for } i = 1, \dots, n^2. \tag{8}$$

In particular, if f is *positive*, then $f_{\min} > 0$ and A is positive definite.

In order to construct a good preconditioner for matrix A, an approximate knowledge of the eigenvalues of A should be available. Given an integer $\mathcal{N}$, let

$$\mathcal{S}_{\mathcal{N}} = \left\{\theta_r = \frac{2r\pi}{\mathcal{N}},\ r = 0,\dots,\mathcal{N}-1\right\} \tag{9}$$

be a set of nodes. From the previous theorem, if $\mathcal{N}$ is large, the set of $\mathcal{N}^2$ values $f(\theta_r,\eta_s)$, with $(\theta_r,\eta_s) \in \mathcal{S}_{\mathcal{N}}^2$, can be assumed to be an acceptable approximation of the spectrum of the eigenvalues of A.

In reality, for $(\theta_r,\eta_s) \in \mathcal{S}_{\mathcal{N}}^2$, the values

$$\begin{aligned} f(\theta_r,\eta_s) &= \sum_{k,j=-\mu}^{\mu} m_{k,j} e^{\mathbf{i}(k\theta_r + j\eta_s)} \\ &= \sum_{k,j=-\mu}^{\mu} m_{k,j}\omega_{\mathcal{N}}^{kr+js}, \quad \omega_{\mathcal{N}} = e^{\mathbf{i}2\pi/\mathcal{N}}, \end{aligned} \tag{10}$$

are the eigenvalues of a 2D circulant matrix whose first row embeds the elements of the mask M which have been suitably rotated. Hence they can be computed using a two-dimensional fast Fourier transform ($\text{FFT}_{2\text{d}}$) of order $\mathcal{N}$. In fact, consider the $\mathcal{N}\times\mathcal{N}$ matrix R whose entries are

$$r_{k,j} = \begin{cases} m_{k,j} & \text{if } 0 \le k, j \le \mu, \\ m_{k,j-\mathcal{N}} & \text{if } 0 \le k \le \mu,\ \mathcal{N}-\mu \le j \le \mathcal{N}-1, \\ m_{k-\mathcal{N},j} & \text{if } \mathcal{N}-\mu \le k \le \mathcal{N}-1,\ 0 \le j \le \mu, \\ m_{k-\mathcal{N},j-\mathcal{N}} & \text{if } \mathcal{N}-\mu \le k, j \le \mathcal{N}-1, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Matrix $S = \mathcal{N}\cdot \text{FFT}_{2\text{d}}(R)$ contains the values $f(\theta_r,\eta_s)$ for $r,s = 0,\dots,\mathcal{N}-1$. The cost of this computation is $O(\mathcal{N}^2\log\mathcal{N})$. The computation of $f(\theta_r,\eta_s)$ for $r,s = 0,\dots,\mathcal{N}-1$, made by directly applying (10), has a cost $O(\mu^2\mathcal{N}^2)$, where μ does not depend on $\mathcal{N}$.

3. REGULARIZING INVERSE PRECONDITIONERS

Let $\tau > 0$ be the regularization parameter (chosen in such a way that $\lambda_i(A) \ge -\tau$ for $i = 1,\dots,n^2$). Define

$$\begin{aligned} \Gamma_\tau &= \{(\theta,\eta) \in [0,2\pi]^2 : f(\theta,\eta) \ge \tau\}, \\ f_\tau(\theta,\eta) &= \begin{cases} f(\theta,\eta) & \text{for } (\theta,\eta) \in \Gamma_\tau, \\ \tau & \text{otherwise.} \end{cases} \end{aligned} \tag{12}$$

Function $f_\tau(\theta,\eta)$ is continuous and strictly positive on $[0,2\pi]^2$. We can then define the functions

$$\begin{aligned} g_\tau(\theta,\eta) &= \frac{1}{f_\tau(\theta,\eta)}, \\ h_\tau(\theta,\eta) &= g_\tau(\theta,\eta)f(\theta,\eta). \end{aligned} \tag{13}$$

Function $h_\tau(\theta,\eta)$ assumes value 1 on Γ_τ and values $f(\theta,\eta)/\tau < 1$ elsewhere.

Let

$$c_{k,j} = \frac{1}{4\pi^2}\iint_0^{2\pi} g_\tau(\theta,\eta)e^{-\mathbf{i}(k\theta+j\eta)}d\theta\, d\eta \tag{14}$$

be the (k,j)th Fourier coefficient of $g_\tau(\theta,\eta)$ and let

$$\sum_{k,j=-\infty}^{\infty} c_{k,j}e^{\mathbf{i}(k\theta+j\eta)} \tag{15}$$

be the trigonometric expansion of $g_\tau(\theta,\eta)$. Since $g_\tau(\theta,\eta)$ is a continuous periodic function on $[0,2\pi]^2$ and has a bounded generalized derivative, $g_\tau(\theta,\eta)$ is equal to its trigonometric expansion, which is uniformly convergent.

Let G_τ and H_τ be the $n^2\times n^2$ BTTB matrices whose symbols are $g_\tau(\theta,\eta)$ and $h_\tau(\theta,\eta)$, respectively. Since A is symmetric, G_τ is symmetric as well, that is, $c_{k,j} = c_{-k,-j}$. In accordance with Grenander and Szegő theorem, for $n\to\infty$, matrix H_τ has a cluster of eigenvalues around 1 corresponding to the eigenvalues of A greater or equal to τ. The other eigenvalues are generally not clustered and have a modulus lower than 1. By direct computation, it is easy to verify that matrix $G_\tau A - H_\tau$ has rank $\rho = 4\mu(n-\mu)$. Then for $n\to\infty$ also matrix $G_\tau A$ has a cluster around 1. No more than 2ρ eigenvalues of $G_\tau A$ leave the cluster of H_τ and in particular no more than ρ become greater than $\max h_\tau = 1$ (see [15, Theorem 10.3.1 and Corollary 10.3.2]). Many similar results can be found in the literature on preconditioners for Toeplitz systems (see, e.g., [1, 5, 6, 11, 16, 17]).

It follows that for a sufficiently large n, matrix G_τ would be a good regularizing inverse preconditioner. In general, the trigonometric expansion of $g_\tau(\theta,\eta)$ is not finite and G_τ does not have a band structure. On the contrary, the preconditioners we are interested in should have a band BTTB structure, which would lead to a cost per iteration $O(n^2)$.

3.1. Least-squares approximation

In this subsection, we examine different banded approximations of G_τ which can be obtained through a fit procedure. Similar procedures have been followed in [10, 16] for the construction of banded direct preconditioners.

The choice of the bandwidth of the preconditioner should take into consideration the rate of decay of $c_{k,j}$ for growing indices k and j: the faster the decay, the smaller the bandwidth. Since function f is bandlimited with bandwidth μ, it is reasonable to expect that a bandwidth close to μ can be chosen. We look for a preconditioner with the same bandwidth μ as the given matrix A. This choice is also influenced by computational considerations and its suitability is supported by the numerical experimentation of Section 4. In any case, what follows would hold for any choice of constant value of the bandwidth.

Let $\mathcal{P}_\mu$ be the set of bivariate trigonometric polynomials of the form

$$p(\theta,\eta) = \sum_{k,j=-\mu}^{\mu} d_{k,j}e^{\mathbf{i}(k\theta+j\eta)}, \tag{16}$$

such that $p(\theta,\eta) > 0$ for any (θ,η). We consider the problem

$$\min_{p\in\mathcal{P}_\mu} ||w(\theta,\eta)(p(\theta,\eta) - g_\tau(\theta,\eta))||, \quad (17)$$

where $w(\theta,\eta) > 0$ is a weight function (we choose the Euclidean norm).

Various choices of the weight $w(\theta,\eta)$ can be considered.

(1) If $w(\theta,\eta) \equiv 1$, the absolute error is minimized, that is, problem (17) becomes

$$\min_{p\in\mathcal{P}_\mu} ||p(\theta,\eta) - g_\tau(\theta,\eta)||. \quad (18)$$

In this way, all the values of $g_\tau(\theta,\eta)$ are given the same importance when the fit is computed.

(2) We can get a better result if we put more emphasis on the greatest values of $f_\tau(\theta,\eta)$. In fact, the largest eigenvalues of A are transformed into eigenvalues of the preconditioned matrix which are clustered around 1, while the smallest eigenvalues of A are transformed into eigenvalues lower than 1, which can lie anywhere, provided they are outside the cluster. This result can be obtained by putting $w(\theta,\eta) = f_\tau(\theta,\eta)$. In this way, the relative error is minimized, that is, problem (17) becomes

$$\min_{p\in\mathcal{P}_\mu} \left\|\frac{p(\theta,\eta) - g_\tau(\theta,\eta)}{g_\tau(\theta,\eta)}\right\| = \min_{p\in\mathcal{P}_\mu} ||p(\theta,\eta) f_\tau(\theta,\eta) - 1||. \quad (19)$$

(3) Since $\tau \le f_\tau(\theta,\eta) \le 1$ for any (θ,η), the largest values of $f_\tau(\theta,\eta)$ are even more weighted by choosing a function similar to the Chebyshev weight of the form

$$w(\theta,\eta) = \left(1 - \varphi f_\tau^2(\theta,\eta)\right)^{-1/2} \quad (20)$$

for a constant φ slightly lower than 1 (in our experiments we took $\varphi = 0.99$).

The solution of problem (17) can be approximated by a constrained discrete least-squares procedure on the $\mathcal{N}^2$ nodes $(\theta_r,\eta_s) \in \mathcal{S}^2_{\mathcal{N}}$, with $\mathcal{N} > 2\mu + 1$ and independent from n. Let $\hat{p}(\theta,\eta)$ be the polynomial thus computed. The preconditioner we look for is generated by $\hat{p}(\theta,\eta)$ and, according to [18], we call it an *optimal* preconditioner when it is obtained by solving problem (18) and a *superoptimal* preconditioner when it is obtained by solving problem (19). We call the third one a *Chebyshev* preconditioner.

Let P be the $n^2\times n^2$ BTTB matrix generated by the symbol $\hat{p}(\theta,\eta)$. The cluster around 1 of the preconditioned matrix is modified when G_τ is replaced by P. Let

$$\nu = \max_{(\theta,\eta)\in\Gamma_\tau} \left|\hat{p}(\theta,\eta) - g_\tau(\theta,\eta)\right|. \quad (21)$$

Thus

$$\left|\hat{p}(\theta,\eta) f(\theta,\eta) - h_\tau(\theta,\eta)\right| < \nu \quad \text{for any } (\theta,\eta) \in \Gamma_\tau. \quad (22)$$

Hence matrix K_τ whose symbol function is $\hat{p}(\theta,\eta)f(\theta,\eta)$ has a cluster of eigenvalues around 1 (corresponding to the eigenvalues of A greater or equal to τ) of size ν and the matrix $PA - K_\tau$ has rank ρ. As before, we can conclude that at most 2ρ eigenvalues leave the cluster of K_τ.

3.2. Unconstrained approximation

First, we examine the approximation one would obtain if the constraint $p(\theta,\eta) > 0$ were not imposed. The coefficients $\hat{d}_{k,j}$ of $\hat{p}(\theta,\eta)$ satisfy the $(2\mu+1)^2 \times (2\mu+1)^2$ linear system

$$\begin{aligned}\sum_{k,j=-\mu}^{\mu} d_{k,j} \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 e^{\mathbf{i}((k+k')\theta_r + (j+j')\eta_s)} \\ = \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 g_{r,s} e^{\mathbf{i}(k'\theta_r + j'\eta_s)} \quad \text{for } k', j' = -\mu,\dots,\mu,\end{aligned} \quad (23)$$

where $w_{r,s} = w(\theta_r,\eta_s)$ and $g_{r,s} = g_\tau(\theta_r,\eta_s)$. When the nodes are chosen in $\mathcal{S}^2_{\mathcal{N}}$, system (23) becomes

$$\begin{aligned}\sum_{k,j=-\mu}^{\mu} d_{k,j} \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 \omega_{\mathcal{N}}^{r(k+k')+s(j+j')} \\ = \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 g_{r,s} \omega_{\mathcal{N}}^{rk'+sj'} \quad \text{for } k', j' = -\mu,\dots,\mu.\end{aligned} \quad (24)$$

The elements of the coefficient matrix of the system only depend on the sums $k + k'$ and $j + j'$ of the indices. Hence this matrix is a block Hankel matrix and the system can be solved by special fast techniques [19]. The computation of the required entries, once the values $f_{r,s}$ have been computed, has a cost $O(\mu^2\mathcal{N}^2)$ if the sums are directly computed and a cost $O(\mathcal{N}^2 \log \mathcal{N})$ if the computation is made through the Fourier transforms.

When the weight $w(\theta,\eta) \equiv 1$ is chosen, we have

$$\hat{d}_{k,j} = \frac{1}{\mathcal{N}^2} \sum_{r,s=0}^{\mathcal{N}-1} g_{r,s} \omega_{\mathcal{N}}^{-(rk+sj)} \quad \text{for } k, j = -\mu,\dots,\mu. \quad (25)$$

The following theorem connects the polynomial $\hat{p}(\theta,\eta)$ with the coefficients $\hat{d}_{k,j}$ given in (25) to a finite approximation of the trigonometric polynomial (15).

Theorem 1. *The polynomial $\hat{p}(\theta,\eta)$, which approximates the minimum of $\|p(\theta,\eta) - g_\tau(\theta,\eta)\|$ among all the bivariate trigonometric polynomials of degree μ by discretizing on $\mathcal{N}^2$ nodes, coincides with the approximate truncated expansion of $g_\tau(\theta,\eta)$:*

$$\tilde{p}(\theta,\eta) = \sum_{k,j=-\mu}^{\mu} \tilde{c}_{k,j} e^{\mathbf{i}(k\theta+j\eta)}, \quad (26)$$

where the coefficients $\tilde{c}_{k,j}$ are computed by applying the rectangular rule to (14) on the set of nodes $(\theta_r,\eta_s) \in \mathcal{S}^2_{\mathcal{N}}$, that is,

$$\tilde{c}_{k,j} = \frac{1}{\mathcal{N}^2} \sum_{r,s=0}^{\mathcal{N}-1} g_\tau(\theta_r,\eta_s) e^{-\mathbf{i}(k\theta_r + j\eta_s)} \quad \text{for } k, j = -\mu,\dots,\mu. \quad (27)$$

Proof. Let $\mathcal{N} > 2\mu+1$ (we assume, without loss of generality, that $\mathcal{N}$ is even). According to [20, Section 9.2.2], the polynomial

$$q(\theta,\eta) = \sum_{k,j=-\mathcal{N}/2+1}^{\mathcal{N}/2} \tilde{c}_{k,j} e^{\mathbf{i}(k\theta+j\eta)}, \quad (28)$$

with the coefficients $\widetilde{c}_{k,j}$ given in (27) interpolates $g_\tau(\theta,\eta)$ on the $\mathcal{N}^2$ nodes $(\theta_r,\eta_s) \in \mathcal{S}^2_{\mathcal{N}}$, and the polynomial (26) with the coefficients given by (27) (i.e., the truncation at the μth term of (28)) coincides with the polynomial $\widehat{p}(\theta,\eta)$, which realizes the minimum of $\|p(\theta,\eta) - g_\tau(\theta,\eta)\|$ discretized on the same $\mathcal{N}^2$ nodes. □

The use of the rectangular rule is suggested in [11].

3.3. Enforcing the positivity

Even if all the values $g_{r,s}$ are positive, the polynomial obtained by solving system (24) is not guaranteed to satisfy the positivity constraint $p(\theta,\eta) > 0$. We could impose the Karush-Kuhn-Tucker conditions to problem (17) discretized on all the $\mathcal{N}^2$ nodes. Unfortunately, this approach, besides being computationally demanding, would not suffice, because of the oscillations characteristic of a trigonometric polynomial. On the other hand, the most dangerous oscillations are those occurring near the minimum point of function g_τ, that is, in the neighborhood of $(0,0)$. We expect this phenomenon to happen more frequently with the optimal preconditioner, since in the case of the superoptimal and Chebyshev preconditioners this problem is, to some extent, prevented by the presence of a heavy weight in the neighborhood of $(0,0)$. Other oscillations frequently occur near the points where the function f is cut by τ, but they do not appear to threaten the positivity of the fit, due to the large values of $1/\tau$ required in the applications.

These considerations suggest a heuristic approach privileging the positivity in $(0,0)$. Since the necessary condition $p(0,0) > 0$ is too weak, we replace it by the stronger condition $p(0,0) \geq p_{\min}$ for a suitable constant $p_{\min} > 0$ and neglect other positivity conditions. The new simpler problem is then solved by a constrained discrete least squares procedure. The coefficients $\widehat{d}_{k,j}$ and the Karush-Kuhn-Tucker parameter ψ satisfy

$$\begin{gathered}\sum_{k,j=-\mu}^{\mu} d_{k,j} \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 \omega_{\mathcal{N}}^{r(k+k')+s(j+j')} \\ = \sum_{r,s=0}^{\mathcal{N}-1} w_{r,s}^2 g_{r,s} \omega_{\mathcal{N}}^{rk'+sj'} + \psi \quad \text{for } k', j' = -\mu,\dots,\mu, \\ \psi\left(\sum_{k,j=-\mu}^{\mu} d_{k,j} - p_{\min}\right) = 0, \quad \psi \geq 0, \\ \sum_{k,j=-\mu}^{\mu} d_{k,j} - p_{\min} \geq 0.\end{gathered} \tag{29}$$

The coefficients $\widehat{d}_{k,j}$ found by solving (24) correspond to the null value of the parameter ψ and can be accepted if $\sum_{k,j=-\mu}^{\mu} \widehat{d}_{k,j} \geq p_{\min}$. Otherwise, the equation $\sum_{k,j=-\mu}^{\mu} d_{k,j} = p_{\min}$ is added to the first $2\mu + 1$ equations and the enlarged system is solved.

3.4. The inverse Toeplitz preconditioner

The approach followed in this paper is similar to the one proposed in [1], where the preconditioner does not have a band structure, since its bandwidth is set to n, and $\mathcal{N}$ is set to $2n$. In this case, the values $f(\theta_r,\eta_s)$ are the eigenvalues of the circulant matrix whose first row elements are the entries of R defined in (11). The values $g_\tau(\theta_r,\eta_s)$ are set equal to the inverse of the eigenvalues, modified for the regularization. Actually, in [1] when $f(\theta_r,\eta_s) < \tau$ these values are set to 1 instead of $1/\tau$, but we believe that a continuous function in (14) makes the approximation of the integral more effective (see also [21]). The preconditioner P, called *inverse Toeplitz* preconditioner, is then extracted from the circulant matrix with $g_\tau(\theta_r,\eta_s)$ as eigenvalues. The cost for both the construction of P and per iteration is $O(n^2 \log n)$.

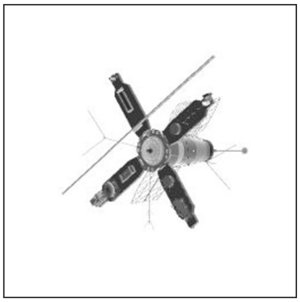
Within circulant preconditioners with regularizing properties, superoptimal preconditioners have been proposed in [22, 23]. They are independent of the regularization parameter τ and have a cost per iteration of $O(n^2 \log n)$.

3.5. Analysis of the cost per iteration

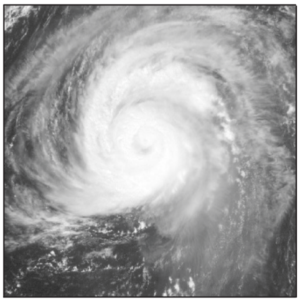
The cost we analyze here takes into account the complexity of one iteration of the preconditioned methods, neglecting the cost for the construction of the preconditioner, which is made only once. Each iteration requires two matrix-vector products, one by the coefficient matrix and one by the preconditioner. The product by a banded preconditioner, with bandwidth μ, has a cost upper bounded by $c_b = (2\mu+1)^2 n^2$. The product by the inverse Toeplitz preconditioner requires two applications of the discrete Fourier transform (one direct and one inverse) to a vector of size $(2n)^2$, representing the first column of a block circulant matrix of double dimension, and one componentwise multiplication of vectors of size $(2n)^2$ (see [12] for details). By using the standard complexity bound of $5N \log_2 N$ operations for the radix-2 FFT algorithm applied to a vector of size N, and by dropping the lower order terms, we see that the cost of the product for the Inverse Toeplitz preconditioner amounts to $c_T = (2 \times 5 \log_2 (2n)^2 + 1)(2n)^2$. It follows that $c_b < c_T$ if $\mu < \sqrt{10 \log_2 (2n)^2 + 1} - 1/2$. For example, in the case $n = 1024$, $c_b < c_T$ for $\mu \leq 14$.

4. NUMERICAL EXPERIMENTS

The aim of the experiments was to test the effectiveness of the banded preconditioners. In other words, we wanted to check whether the preconditioned method can obtain reconstructions comparable with those of the unpreconditioned method at a lower computational cost. In order to be able to compare the results objectively (i.e., numerically), we worked in a simulated context where an exact solution was assumed to be available and the error of the reconstructions could be computed at any iteration. We also wanted to compare the performance of the banded preconditioners with that of the inverse Toeplitz preconditioner.



(a)



(b)

Figure 1: Original images.

The experiments performed with positive definite matrices showed that the number of iterations required by an unpreconditioned CG to obtain acceptable reconstructions is very small, especially for higher noise levels. Hence, in the positive definite case the use of a preconditioner does not provide much of a margin for improvement. For this reason, below we only show the results obtained by applying the preconditioned MR-II to the symmetric indefinite problems, where more iterations are generally required.

4.1. The test problems

Two images were used for the experiments. The first was the 128×128 image shown in Figure 1(a). This data, widely used in the literature for testing image restoration algorithms, can be found in the package *RestoreTools* [24]. The second was the 1024×1024 meteorological image shown in Figure 1(b), which can be found in the Monterey Naval Research Laboratory site [25].

We considered one mask obtained by measurements and three analytically defined masks. The first one, Mask 1, was the mask used in [24], truncated at bandwidth $\mu = 8$. The three others were of the form

$$m_{i,j} = \gamma \exp\left(-\alpha(i+j)^2 - \beta(i-j)^2\right), \quad i,j = -\mu,\ldots,\mu, \tag{30}$$

where α, β, γ are positive parameters. The entries of M were scaled by the constant γ in such a way that $\sum_{i,j} m_{i,j} = 1$. Once again the bandwidth was set to $\mu = 8$. The masks have different properties, according to the choice of parameters α and β. The following choices were considered: Mask 2 for $\alpha = 0.04$ and $\beta = 0.02$, Mask 3 for $\alpha = 0.01$ and $\beta = 0.4$, Mask 4 for $\alpha = 0.019$ and $\beta = 0.017$. Mask 4 is a smooth approximation of Mask 1.

The noisy image $\mathbf{b}$ was obtained by computing $A\mathbf{x} + \mathbf{w}$, where $\mathbf{w}$ is a vector of randomly generated entries, with normal distribution and mean 0, scaled in such a way that the noise level $\ell = \|\mathbf{w}\|_2 / \|A\mathbf{x}\|_2$ was equal to an assigned quantity $\ell = 10^{-t}$, with $t \in [2, 4]$.

In general, for a given noise level, smoother masks, such as the exponential ones, required less iterations to achieve an acceptable reconstruction than nonsmooth ones, like Mask 1.

4.2. Selection of parameters

The banded preconditioners depend on three parameters: the regularization parameter τ, the number $\mathcal{N}^2$ of nodes for the fit, and the constant $p_{\min}$ used to enforce the positivity of the fit.

As is well known, a suitable value of the parameter τ is fundamental for the efficiency of any regularizing preconditioner. To find such a value, two different lines could be followed: (a) in a simulated context one can find the best value of τ, that is, that particular value for which the preconditioner computes an acceptable solution in the minimum number of iterations, and (b) even in a simulated context one can use a practical approach, employing one of the procedures described in the literature, such as a method based on the L-curve [1] or the more general method based on the FFT of the right-hand side noisy vector [5]. For a given problem, line (a) may lead to different values of τ according to the particular preconditioner used, and this would prevent an objective comparison, which would be useful for solving problems arising in nonsimulated contexts.

We preferred a practical technique and used the one described in Section 5 of [5]. It allowed us to estimate the dimension of the noise and signal subspaces by only exploiting the information derived from the observed image and matrix A, independently of the preconditioner. This technique generally leads to reasonable values for the regularization parameter τ. The values of τ found in this way are aimed at only clusterizing the eigenvalues that correspond to the signal subspace, leaving the eigenvalues of the transient and noise subspaces outside. In reality, the presence of the outliers alters the situation somewhat. For the test problems taken into consideration, we verified that for the computed values of τ, the condition $-\tau \le f_{\min}$ holds, where $f_{\min}$ is the minimum value of the symbol function f.

Regarding parameter $\mathcal{N}$, we note that great accuracy in the approximation of the coefficients $\hat{d}_{k,j}$ of $\hat{p}(\theta,\eta)$ is not required, due to the fact that this polynomial is in any case an approximation of $g_\tau(\theta,\eta)$. Thus the choice of a suitable value of $\mathcal{N}$ is not so critical, as the ad hoc experiment in the next subsection shows. As a matter of fact, it appears that the speed of convergence of the preconditioned method does not vary much when $\mathcal{N}$ is increased, suggesting that a choice of $\mathcal{N}$ not much greater than the bound $2\mu + 2$ is adequate.

Finally, we might think that tuning a good value for $p_{\min}$ is difficult, because the polynomial $p(\theta,\eta)$ obtained from small values of $p_{\min}$ may be nonpositive, and polynomials corresponding to large values of $p_{\min}$ may be unsuitable for our preconditioning purposes, even if they are positive. But the experiment showed that it is not so difficult. In fact, in the case of the superoptimal and Chebyshev preconditioners we obtained satisfactory results without having to apply the heuristic approach proposed in Section 3.3. Moreover, in the case of the optimal preconditioner, even the small translation caused by setting $p_{\min} = 1$ was sufficient to get a positive polynomial $p(\theta,\eta)$.

Table 1: Number of iterations varying $\mathcal{N}$ for Mask 1, with $\tau = 0.07$ for $\ell = 10^{-2}$, $\tau = 0.05$ for $\ell = 10^{-2.5}$, and $\tau = 0.03$ for $\ell = 10^{-3}$.

Noise level	10^{-2}			$10^{-2.5}$			10^{-3}		
$\mathcal{N}$	18	24	30	18	24	30	18	24	30
Optimal	8	10	12	22	26	29	58	73	65
Superopt.	7	7	7	17	21	19	42	52	48
Chebyshev	8	10	12	22	26	27	57	69	64

Table 2: Number of iterations varying $\mathcal{N}$ for Mask 2, with $\tau = 0.1$ for $\ell = 10^{-3}$, $\tau = 0.09$ for $\ell = 10^{-3.5}$, and $\tau = 0.08$ for $\ell = 10^{-4}$.

Noise level	10^{-3}			$10^{-3.5}$			10^{-3}		
$\mathcal{N}$	18	24	30	18	24	30	18	24	30
Optimal	10	11	10	25	26	24	68	72	68
Superopt.	10	10	10	25	24	24	68	67	67
Chebyshev	10	10	10	25	25	25	68	69	68

4.3. Performance measures

Each problem was first solved without preconditioning in order to determine the reconstruction efficiency limit. By denoting with $\mathbf{x}^{(i)}$ the vector obtained at the ith iteration starting with $\mathbf{x}^{(0)} = \mathbf{0}$ and with $e^{(i)} = \|\mathbf{x}^{(i)} - \mathbf{x}\|_2/\|\mathbf{x}\|_2$ the relative error, we considered the minimum error $e_m = \min_i e^{(i)}$. The quantity $E = 1.05e_m$ is taken as the *reference* value, in the sense that any approximated image with an error lower than E is considered as an *acceptable* reconstruction. The index I of the first acceptable iteration is the *reference* index. The value I appears to be very close to the number of iterations that can be made before the noise starts to contaminate the reconstructed image. Since the cost per iteration of a banded preconditioned method is twice the cost of the unpreconditioned one, preconditioners computing acceptable reconstructions with a number of iterations lower than $I/2$ are considered effective.

The results obtained in three different sets of experiments are summarized in the tables, where the minimum iteration numbers κ such that $e^{(\kappa)} \leq E$ are shown. The caption of each table lists, for each noise level, the corresponding τ. The heuristic described in Section 3.3 was required only for the optimal preconditioner and it was applied with $p_{\min} = 1$.

A first set of experiments was carried out on the first image in order to analyze the effects of the choice of $\mathcal{N}$ on the performance of the banded preconditioners. The masks used here were Mask 1 for noise levels 10^{-2}, $10^{-2.5}$ and 10^{-3}, and Mask 2 for noise levels 10^{-3}, $10^{-3.5}$, and 10^{-4}. The three values $2\mu+2$, $2\mu+8$, and $2\mu+14$ were chosen for $\mathcal{N}$. The results are shown in Tables 1 and 2. It appears that the different values of $\mathcal{N}$ do not affect the results much, hence a value not much greater than $2\mu+2$ is suggested for $\mathcal{N}$.

The second set of experiments too was carried out on the first image. All the masks and the banded preconditioners were considered, together with the inverse Toeplitz preconditioner. The value $\mathcal{N} = 24$ was chosen. The results are shown in Tables 3 and 4. We observe that the overall behavior of the banded preconditioners does not differ much from that of the inverse Toeplitz preconditioner and shows comparable reconstruction efficiency and speed of convergence. In particular, we note that the margin for improvement increases when the noise level decreases, as shown in Table 4, and that in general the superoptimal preconditioner can be advised.

Table 3: Number of iterations for all the methods. Mask 1, with $\tau = 0.07$ for $\ell = 10^{-2}$, $\tau = 0.05$ for $\ell = 10^{-2.5}$, and $\tau = 0.03$ for $\ell = 10^{-3}$. Mask 2, with $\tau = 0.18$ for $\ell = 10^{-2}$, $\tau = 0.14$ for $\ell = 10^{-2.5}$, and $\tau = 0.1$ for $\ell = 10^{-3}$.

Noise level	Mask 1			Mask 2		
	10^{-2}	$10^{-2.5}$	10^{-3}	10^{-2}	$10^{-2.5}$	10^{-3}
Ref. index I	24	63	169	12	20	29
Optimal	10	26	73	6	8	11
Superopt.	7	21	52	5	7	10
Chebyshev	10	26	69	5	8	10
Inv. Toep.	6	19	49	4	7	9

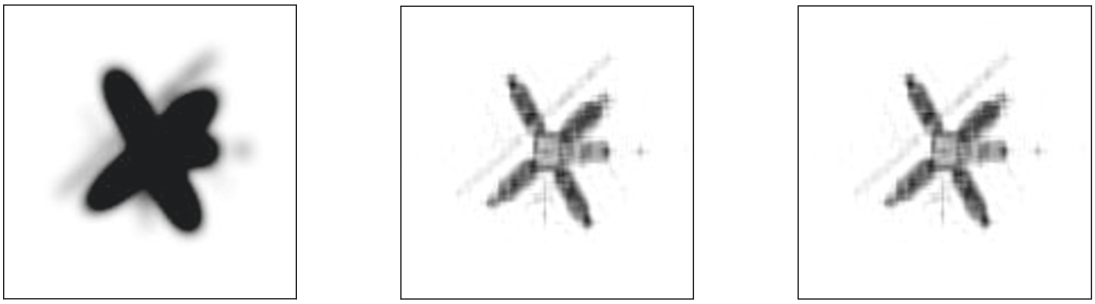
Table 4: Number of iterations for all the methods. Mask 3, with $\tau = 0.12$ for $\ell = 10^{-3}$, $\tau = 0.1$ for $\ell = 10^{-3.5}$, and $\tau = 0.08$ for $\ell = 10^{-4}$. Mask 4, with $\tau = 0.08$ for $\ell = 10^{-3}$, and $\tau = 0.06$ for $\ell = 10^{-3.5}$, $\tau = 0.04$ for $\ell = 10^{-4}$.

Noise level	Mask 3			Mask 4		
	10^{-3}	$10^{-3.5}$	10^{-4}	10^{-3}	$10^{-3.5}$	10^{-4}
Ref. index I	53	155	485	44	146	655
Optimal	24	62	180	15	49	222
Superopt.	21	58	175	13	47	206
Chebyshev	23	61	180	15	49	207
Inv. Toep.	21	59	183	12	47	207

Figure 2(a) shows the noisy image, obtained by blurring the original image of Figure 1(a) with Mask 4 and noise level $10^{-3.5}$, together with the images reconstructed with the inverse Toeplitz preconditioner (Figure 2(b)) and with the superoptimal preconditioner (Figure 2(c)). They are both applied with the value τ and the number of iterations indicated in Table 4. The two reconstructions appear to be very similar.

The third set of experiments was aimed at showing that the equivalence (in terms of the numbers of iterations required to get the same acceptable reconstruction) of the banded preconditioners and the inverse Toeplitz preconditioner, verified for the size $n = 128$, also holds for larger dimensions, which are of interest in the applications. For this purpose, the second image with size $n = 1024$ was chosen. Mask 3 and the three noise levels $\ell = 10^{-3}$, $\ell = 10^{-3.5}$, and $\ell = 10^{-4}$ were considered. The value $\mathcal{N} = 20$ was chosen. In Table 5, the results of the comparison between the superoptimal preconditioner and the inverse Toeplitz preconditioner are shown.

The numbers of iterations required by the two preconditioners are comparable. The cost of the matrix-vector product is $c_b = 289\ 2^{20}$ for the superoptimal and $c_T = 884\ 2^{20}$ for inverse Toeplitz, hence $c_T \sim 3c_b$.



(a) (b) (c)

Figure 2: (a) Image blurred with Mask 4 and noise level $10^{-3.5}$, (b) reconstructed images with inverse Toeplitz preconditioner and (c) with superoptimal preconditioner.

Table 5: Number of iterations required for a large image. Mask 3, with $\tau = 0.1$ for $\ell = 10^{-3}$, $\tau = 0.08$ for $\ell = 10^{-3.5}$, and $\tau = 0.06$ for $\ell = 10^{-4}$.

Noise level	10^{-3}	$10^{-3.5}$	10^{-4}
Ref. index I	14	31	66
Superopt.	5	12	26
Inv. Toep.	6	12	25

5. CONCLUSIONS

The proposed banded preconditioners appear to be effective compared to the unpreconditioned method. They show the same performances as the inverse Toeplitz preconditioner, but the cost per iteration of a banded preconditioner is $O(n^2)$ operations, while the cost per iteration of the inverse Toeplitz preconditioner is $O(n^2 \log n)$. The constants hidden in the O notation are such that the banded preconditioners result competitive with the inverse Toeplitz preconditioner already for sizes of practical interest.

REFERENCES

[1] M. Hanke and J. Nagy, "Inverse Toeplitz preconditioners for ill-posed problems," *Linear Algebra and Its Applications*, vol. 284, no. 1–3, pp. 137–156, 1998.

[2] M. Hanke, *Conjugate Gradient Type Methods for Ill-Posed Problems*, Pitman Research Notes in Mathematics, Longman, Harlow, UK, 1995.

[3] M. Hanke, "Iterative regularization techniques in image restoration," in *Mathematical Methods in Inverse Problems for Partial Differential Equations*, Springer, New York, NY, USA, 1998.

[4] P. E. Hansen, *Rank-Deficient and Discrete Ill-Posed Problems*, SIAM Monographs on Mathematical Modeling and Computation, SIAM, Philadelphia, Pa, USA, 1998.

[5] M. Hanke, J. Nagy, and R. Plemmons, "Preconditioned iterative regularization for ill-posed problems," in *Numerical Linear Algebra and Scientific Computing*, L. Reichel, A. Ruttan, and R. S. Varga, Eds., pp. 141–163, de Gruyter, Berlin, Germany, 1993.

[6] X.-Q. Jin, *Developments and Applications of Block Toeplitz Iterative Solvers*, Kluwer Academic Publishers, Dordrecht, The Netherlands; Science Press, Beijing, China, 2002.

[7] R. H. Chan and P. Tang, "Fast band-Toeplitz preconditioner for Hermitian Toeplitz systems," *SIAM Journal on Scientific Computing*, vol. 15, no. 1, pp. 164–171, 1994.

[8] X.-Q. Jin, "Band Toeplitz preconditioners for block Toeplitz systems," *Journal of Computational and Applied Mathematics*, vol. 70, no. 2, pp. 225–230, 1996.

[9] S. Serra Capizzano, "Optimal, quasi-optimal and superlinear band-Toeplitz preconditioners for asymptotically ill-conditioned positive definite Toeplitz systems," *Mathematics of Computation*, vol. 66, no. 218, pp. 651–665, 1997.

[10] P. Favati, G. Lotti, and O. Menchi, "Preconditioners based on fit techniques for the iterative regularization in the image deconvolution problem," *BIT Numerical Mathematics*, vol. 45, no. 1, pp. 15–35, 2005.

[11] R. H. Chan and K.-P. Ng, "Toeplitz preconditioners for Hermitian Toeplitz systems," *Linear Algebra and Its Applications*, vol. 190, pp. 181–208, 1993.

[12] M. Hanke and J. Nagy, "Restoration of atmospherically blurred images by symmetric indefinite conjugate gradient techniques," *Inverse Problems*, vol. 12, no. 2, pp. 157–173, 1996.

[13] U. Grenander and G. Szegö, *Toeplitz Forms and Their Applications*, Chelsea, New York, NY, USA, 2nd edition, 1984.

[14] P. Tilli, "Asymptotic spectral distribution of Toeplitz-related matrices," in *Fast Reliable Algorithms for Matrices with Structure*, T. Kailath and A. H. Sayed, Eds., pp. 153–187, SIAM, Philadelphia, Pa, USA, 1999.

[15] B. N. Parlett, *The Symmetric Eigenvalue Problem*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1980.

[16] P. Favati, G. Lotti, and O. Menchi, "A polynomial fit preconditioner for band Toeplitz matrices in image reconstruction," *Linear Algebra and Its Applications*, vol. 346, no. 1–3, pp. 177–197, 2002.

[17] S.-L. Lei, K.-I. Kou, and X.-Q. Jin, "Preconditioners for ill-conditioned block Toeplitz systems with application in image restoration," *East-West Journal of Numerical Mathematics*, vol. 7, no. 3, pp. 175–185, 1999.

[18] E. E. Tyrtyshnikov, "Optimal and superoptimal circulant preconditioners," *SIAM Journal on Matrix Analysis and Applications*, vol. 13, no. 2, pp. 459–473, 1992.

[19] G. H. Golub and C. Van Loan, *Matrix Computation*, Academic Press, New York, NY, USA, 1981.

[20] G. Dahlquist and A. Björck, *Numerical Methods*, Prentice-Hall, Englewood Cliffs, NJ, USA, 1974.

[21] D. A. Bini, P. Favati, and O. Menchi, "A family of modified regularizing circulant preconditioners for two-levels Toeplitz systems," *Computers & Mathematics with Applications*, vol. 48, no. 5-6, pp. 755–768, 2004.

[22] F. Di Benedetto and S. Serra Capizzano, "A note on the superoptimal matrix algebra operators," *Linear and Multilinear Algebra*, vol. 50, no. 4, pp. 343–372, 2002.

[23] F. Di Benedetto, C. Estatico, and S. Serra Capizzano, "Superoptimal preconditioned conjugate gradient iteration for image deblurring," *SIAM Journal of Scientific Computing*, vol. 26, no. 3, pp. 1012–1035, 2005.

[24] K. P. Lee, J. Nagy, and L. Perrone, "Iterative methods for image restoration: a Matlab object oriented approach," 2002, http://www.mathcs.emory.edu/~nagy/RestoreTools.

[25] "NRL Monterey Marine Meteorology Division (Code 7500)," http://www.nrlmry.navy.mil/sat_products.html.

P. Favati received her Laurea degree (magna cum laude) in mathematics in the academic year 1981-1982 from the University of Pisa. She is currently a Research Manager at the Institute of Informatics and Telematics of the Italian CNR. Her main research interest is the design and analysis of numerical algorithms. In particular, she got results in the following fields: numerical integration, numerical solution of large linear systems with or without "structure," regularization methods for discrete ill-posed problems, algorithmics in Web search. In these areas, she has published more than 45 journal articles.

G. Lotti is a Professor of computer science at Parma University. She received her Laurea degree (magna cum laude) in computer science from the University of Pisa in the academic year 1973-1974. Her research interests are focused on computational complexity, on the design and analysis of sequential or parallel algorithms, particularly those concerned with problems of linear algebra, and numerical analysis. In these areas, she developed new algorithms for matrix multiplication, for the solution of linear systems, the numerical approximations of integrals, and image reconstruction.

O. Menchi is an Associate Professor at Pisa University, where she received her Laurea degree in mathematics in 1965. For more than 30 years, she has given courses on various areas of numerical calculus to students in mathematics, computer science, and physics. She is coauthor of textbooks and papers on problems and methods in different fields of numerical analysis. Her current research interests include numerical algorithms for the solution of structured and ill-posed problems of linear algebra.