

Research Article

TSI Finders for Estimation of the Location of an Interference Source Using an Ariborne Array

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An algorithm based on space fast time adaptive processing to estimate the physical location of an interference source closely associated with a physical object and enhancing the detection performance against that object using a phased array radar is presented. Conventional direction finding techniques can estimate all the signals and their associated multipaths usually in a single spectrum. However, none of the techniques are currently able to identify direct path (source direction of interest) and its associated multipath individually. Without this knowledge, we are not in a position to achieve an estimation of the physical location of the interference source via ray tracing. The identification of the physical location of an interference source has become an important issue for some radar applications. The proposed technique identifies all the terrain bounced interference paths associated with the source of interest only (main lobe interferer). This is achieved via the introduction of a postprocessor known as the terrain scattered interference (TSI) finder.

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1. INTRODUCTION

The issue of source localization has been discussed in the literature widely by mainly referring to the estimation of source powers, bearings, and associated multipaths. By sources we mean electromagnetic sources that emit random signals, which can be considered as interferers in communication or radar applications. Some of the conventional techniques that can be used to estimate the signal direction and its associated multipaths include MUSIC [1], spatially smoothed MUSIC [2, 3], maximum likelihood methods (MLM) [4], and estimation of signal parameters via rotational invariance technique (ESPIRIT) [5]. All these techniques use the array's spatial covariance matrix to estimate the direction of arrivals (DOAs), some of which are direct emissions and others are multipath bounces off various objects including the ground or sea surface. For example, the MLM estimator is capable of estimating all the bearings and the associated multipaths. However, none of the techniques are able to identify each source and its associated multipaths when there are multiple sources and multipaths. If we are able to identify each source and its associated multipath, then we will be able to use the

ray tracing to locate the position of each offending source. In many applications, it is sufficient to estimate the direction of an interferer and place a null in the direction of the source to retain the performance of the system; however, there are a number of scenarios where the interference is closely associated with an object that we wish to detect and characterize; in which case, we need to localize and suppress the interference and enhance our ability to detect and characterize the object of interest.

The objective of this study is to present a technique based on the space fast time covariance matrix to locate the multipath arrival or, in radar applications, the terrain scattered interference (TSI) related to each source of interest and to use this information to estimate the location of the offending source. In earlier work [6], a space fast time domain TSI finder was introduced to determine the formation of an efficient space fast time adaptive processor which would efficiently null the main lobe interferer and detect a target which shares the same direction of arrival with the interference source. The TSI finder is able to identify the associated multipath arrivals with each source of interest (once the direction of the source is identified).

In this paper, we briefly discuss the available techniques for identifying the DOA of sources. The main body of the work concentrates on the application of the TSI finders for identifying the physical location of the source of interest. First, we study the TSI finder in detail for its processing gain properties, which has not been discussed earlier [6]. Furthermore, we introduce a new angle domain TSI finder that works in conjunction with the lag domain TSI finder as a postprocessor. These two processors can lead to the physical location of the source of interest.

Section 2 formulates the multichannel radar model with several interference sources and Section 3 briefly discusses some appropriate direction finding techniques including the recently introduced super gain beamformer (SGB) [7]. It is important to note that MUSIC and ESPRIT also present potential processing techniques applicable to this problem, but these methods consume considerably more computation power and require additional processing to extract all of the information of interest. The rest of the paper assumes the receiver processing has clearly identified the direction of arrival of the offending source. Under this assumption, in Sections 4 and 5 we introduce the TSI finder in the lag and angle domains and analyse them in detail. Section 6 introduces the necessary formulas for estimating the location of the interferer source using TSI. Section 7 illustrates some simulated examples.

2. FORMULATION

Suppose an N -channel airborne radar whose $N \times 1$ steering manifold is represented by $\mathbf{s}(\varphi, \theta)$, where φ is the azimuth angle and θ is the elevation angle, transmits a single pulse where $\mathbf{s}(\varphi, \theta)^H \mathbf{s}(\varphi, \theta) = N$, and the superscript H denotes the Hermitian transpose. For the range gate r (r is also the fast time scale or an instant of sampling in fast time), $N \times 1$ measured signal $\mathbf{x}(r)$ can be written as

$$\begin{aligned} \mathbf{x}(r) &= j_1(r)\mathbf{s}(\varphi_1, \theta_1) + j_2(r)\mathbf{s}(\varphi_2, \theta_2) \\ &+ \sum_{m=1}^{a_1} \beta_{1,m} j_1(r - n_{1,m})\mathbf{s}(\varphi_{1,m}, \theta_{1,m}) \\ &+ \sum_{m=1}^{a_2} \beta_{2,m} j_2(r - n_{2,m})\mathbf{s}(\varphi_{2,m}, \theta_{2,m}) + \varepsilon, \end{aligned} \quad (1)$$

where $j_1(r)$, $j_2(r)$ represent a series of complex random amplitudes corresponding to two far field sources, with the directions of arrival pairs, (φ_1, θ_1) and (φ_2, θ_2) , respectively. The third term represents the terrain scattered interference (TSI) paths of the first source with time lags (path lags) $n_{1,1}, n_{1,2}, n_{1,3}, \dots, n_{1,a_1}$, the scattering coefficients $|\beta_{1,m}|^2 < 1$, $m = 1, 2, \dots, a_1$, and the associated direction of arrival pairs $(\varphi_{1,m}, \theta_{1,m})$ ($m = 1, 2, \dots, a_1$). The fourth term is the TSI from the second source with path delays $n_{2,1}, n_{2,2}, n_{2,3}, \dots, n_{2,a_2}$, the scattering coefficients $|\beta_{2,m}|^2 < 1$, $m = 1, 2, \dots, a_2$, and the associated direction of arrivals $(\varphi_{2,m}, \theta_{2,m})$ ($m = 1, 2, \dots, a_2$). More sources and multiple TSI paths from each source are accepted in general, but for the sake of brevity, we are restricting this paper to one of each, and ε represents the $N \times 1$ white noise component.

In this study, we consider the clutter-free case (in practice, this can be achieved in many ways, by exploiting a transmission silence, by using Doppler to suppress the clutter, or by shaping the transmit beam). Furthermore, we assume $\rho_k^2 = E\{|j_k(r)|^2\}$ ($k = 1, 2, \dots$) are the power levels of each source and $|\beta_{k,m}|^2 \rho_k^2$ ($m = 1, 2, \dots$) represent the TSI power levels associated with each TSI path from the k th source, where $E\{\cdot\}$ denotes the expectation operator over the fast time samples. Throughout the analysis we assume that we are interested only in the source powers (as offending sources) that are above the channel noise power, that is, $\mathbb{J}_k = \rho_k^2 / \sigma_n^2 > 1$, $k = 1, 2, \dots$, $E\{\varepsilon\varepsilon^H\} = \sigma_n^2 \mathbf{I}_N$, where \mathbb{J}_k is the interferer source power to noise power ratio per channel, σ_n^2 is the white noise power present in any channel and \mathbf{I}_N is the unit identity matrix. Without loss of generality, we use the notation \mathbf{s}_1 and \mathbf{s}_2 to represent $\mathbf{s}(\varphi_1, \theta_1)$ and $\mathbf{s}(\varphi_2, \theta_2)$, respectively, but the steering vectors associated with TSI arrivals are represented by two subscript notation $\mathbf{s}_{1,m} = \mathbf{s}(\varphi_{1,m}, \theta_{1,m})$ ($m = 1, 2, \dots, a_1$), $\mathbf{s}_{2,m} = \mathbf{s}(\varphi_{2,m}, \theta_{2,m})$ ($m = 1, 2, \dots, a_2$), and so forth. Furthermore, it is assumed that $E\{j_k(r+l)j_k^*(r+m)\} = \rho_k^2 \delta(l-m)$ ($k = 1, 2, \dots$), where $*$ denotes the complex conjugate operation. This last assumption restricts the application of this theory to noise sources that are essentially continuous over the period of examination.

In general, the first objective would be to identify the source directions of high significance to the radar systems performance, which are identified as $\mathbb{J}_k = \rho_k^2 / \sigma_n^2 > 1$. Choices for estimating the direction of arrival using the array's measured spatial covariance matrix are diverse as discussed earlier. The most commonly used beamformer for estimating the number of sources and the power levels in a single spectrum is the MPDR [8]. This approach optimizes the power output of the array subject to a linear constraint and is applicable to arbitrary array geometries and achieves signal to noise gain of N at the output, using N sensors. Other computationally intensive super resolution direction finding techniques such as the MUSIC, ESPRIT, or multidimensional optimization techniques based on the MLM estimator are suitable for locating the direction of arrival of signals, but require further postprocessing to estimate the source power levels.

This study proposes the recently introduced [7] superior version of the MPDR estimator to achieve an upper limit of N^2 processing gain in noise. Furthermore, the new estimator is able to detect extremely weak signals if a large number of samples are available which is particularly applicable to airborne radar.

3. DIRECTION OF ARRIVAL ESTIMATION

3.1. MPDR approach

The MPDR [1] power spectrum obtained by minimizing $\mathbf{w}_1^H \mathbf{R}_x \mathbf{w}_1$ subject to the constraint $\mathbf{w}_1^H \mathbf{s}(\varphi, \theta) = 1$ is given by

$$\begin{aligned} P_m(\varphi, \theta) &= \mathbf{w}_1(\varphi, \theta)^H \mathbf{R}_x \mathbf{w}_1(\varphi, \theta) \\ &= (\mathbf{s}(\varphi, \theta)^H \mathbf{R}_x^{-1} \mathbf{s}(\varphi, \theta))^{-1}, \end{aligned} \quad (2)$$

where

$$\mathbf{w}_1(\varphi, \theta) = \frac{\mathbf{R}_x^{-1} \mathbf{s}(\varphi, \theta)}{\mathbf{s}(\varphi, \theta)^H \mathbf{R}_x^{-1} \mathbf{s}(\varphi, \theta)} \quad (3)$$

and $\mathbf{R}_x = E\{\mathbf{x}(r)\mathbf{x}(r)^H\}$.

To understand the concept of the processing gain in noise, let us assume a single source in the direction (φ, θ) is present. In this case, we have $\mathbf{R}_x = \rho_1^2 \mathbf{s}(\varphi_1, \theta_1) \mathbf{s}(\varphi_1, \theta_1)^H + \sigma_n^2 \mathbf{I}_N$. The inverse of \mathbf{R}_x is

$$\mathbf{R}_x^{-1} = \frac{1}{\sigma_n^2} \left[\mathbf{I}_N - \frac{\mathbf{s}(\varphi_1, \theta_1) \mathbf{s}(\varphi_1, \theta_1)^H}{N + \sigma_n^2 / \rho_1^2} \right]. \quad (4)$$

The MPDR power spectrum is given by

$$P_M(\varphi, \theta) = \frac{\rho_1^2 N + \sigma_n^2}{\rho_1^2 (N^2 - |\mathbf{s}^H \mathbf{s}_1|^2) / \sigma_n^2 + N}, \quad (5)$$

where $\mathbf{s} = \mathbf{s}(\varphi, \theta)$ and $\mathbf{s}_1 = \mathbf{s}(\varphi_1, \theta_1)$. This can be rewritten (noting that for $(\varphi, \theta) \neq (\varphi_1, \theta_1)$, $\mathbf{s}^H \mathbf{s}_1 \approx 0$) as

$$P_m(\varphi, \theta) = \begin{cases} \rho_1^2 + \frac{\sigma_n^2}{N} & \text{for } (\varphi, \theta) = (\varphi_1, \theta_1), \\ \frac{\sigma_n^2}{N} & \text{for } (\varphi, \theta) \neq (\varphi_1, \theta_1). \end{cases} \quad (6)$$

The output signal to residual noise ratio (residual interference in the case of multiple sources) is

$$\frac{P_M(\varphi_1, \theta_1)}{P_M(\varphi, \theta)_{(\varphi, \theta) \neq (\varphi_1, \theta_1)}} = \frac{\rho_1^2 N}{\sigma_n^2} + 1 \approx \frac{N \rho_1^2}{\sigma_n^2} \quad (7)$$

which is approximately N times the input signal to interference plus noise ratio (SINR_{in}). Note, $(\varphi, \theta) \neq (\varphi_1, \theta_1)$ really means that the value of (φ, θ) is not in the vicinity of the point (φ_1, θ_1) or any other source direction. This notation will be used throughout this study as a way of indicating the averaged power output corresponding to a direction with no associated source power. This can be considered as the averaged output power due to the input noise.

This improvement factor (N) can generally be defined as the processing gain factor. In theory, the processing gain can take higher values as the number of sources increases. For example, if P_1 represents the total input power due to other sources, $\text{SINR}_{\text{in}} = \rho_1^2 / (\sigma_n^2 + P_1)$. If all of them are nulled while maintaining $\mathbf{w}^H \mathbf{s} = 1$, then $\text{SINR}_{\text{out}} \approx \rho_1^2 / (\sigma_{\text{out}}^2)$, where σ_{out}^2 is the output noise power. This leads to the processing gain: $\text{SINR}_{\text{out}} / \text{SINR}_{\text{in}} = G \times \text{INR}$, where $G = \sigma_n^2 / \sigma_{\text{out}}^2$ is the processing gain in noise ($\approx N$ when a small number of interfering sources are present), and $\text{INR} = (\sigma_n^2 + P_1) / \sigma_n^2$ is the total interference to noise at the input (≥ 1).

3.2. Super gain beam former (SGB)

Consider the SGB [7] spectrum $|P_s(\varphi, \theta)|$ where

$$P_s(\varphi, \theta) = \frac{1}{N^2} \sum_{k=1}^N \left(\frac{\mathbf{u}_k^H \mathbf{s}(\varphi, \theta)}{\mathbf{r}_k^H \mathbf{s}(\varphi, \theta)} - \frac{1}{\mathbf{u}_k^H \mathbf{r}_k} \right). \quad (8)$$

\mathbf{u}_k is an $N \times 1$ column vector of zeros except unit value at the k th position, and \mathbf{r}_k is the k th column of \mathbf{R}_x^{-1} . For a single source $\mathbf{R}_x = \rho_1^2 \mathbf{s}(\varphi_1, \theta_1) \mathbf{s}(\varphi_1, \theta_1)^H + \mathbf{R}_n$. In order to gain some insight in to the behaviour of (8), we break the uniform noise assumption and assume $\mathbf{R}_n = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_N^2)$ is the noise only spatial covariance matrix. The exact inversion of \mathbf{R}_x is given by $\mathbf{R}_x^{-1} = \mathbf{R}_n^{-1} - \beta \mathbf{R}_n^{-1} \mathbf{s}_1 \mathbf{s}_1^H \mathbf{R}_n^{-1}$, where $\beta = (\Delta + 1/\rho_1^2)^{-1}$ and $\Delta = \sum_{j=1}^N \sigma_j^{-2}$. Furthermore $\mathbf{r}_k = \mathbf{R}_x^{-1} \mathbf{u}_k = \sigma_k^{-2} (\mathbf{I}_N - \beta \mathbf{R}_n^{-1} \mathbf{s}_1 \mathbf{s}_1^H) \mathbf{u}_k$, and for $(\varphi, \theta) = (\varphi_1, \theta_1)$ we have $\mathbf{s}_1^H \mathbf{R}_n^{-1} \mathbf{s} = \Delta$ and $\mathbf{s}_1^H \mathbf{R}_n^{-1} \mathbf{s} \approx 0$ whenever $(\varphi, \theta) \neq (\varphi_1, \theta_1)$ (in fact, when (φ, θ) point is furthest away from (φ_1, θ_1)). Therefore, for a single source, assuming $\rho_1^2 \neq 0$ we have

$$P_s(\varphi, \theta) = \begin{cases} \frac{\rho_1^2}{N^2} \sum_{k=1}^N \frac{(\sigma_k^2 \Delta - 1)(\rho_1^2 \Delta + 1)}{\rho_1^2 \Delta + 1 - \rho_1^2 / \sigma_k^2} & \text{for } (\varphi, \theta) = (\varphi_1, \theta_1), \\ \frac{\rho_1^2}{N^2} \sum_{k=1}^N \frac{-1}{\rho_1^2 \Delta + 1 - \rho_1^2 / \sigma_k^2} & \text{for } (\varphi, \theta) \neq (\varphi_1, \theta_1). \end{cases} \quad (9)$$

Now if we restore the uniform noise assumption, $\sigma_k^2 = \sigma_n^2$ ($k = 1, 2, \dots, N$), we have

$$P_s(\varphi, \theta) = \begin{cases} \rho_1^2 + \frac{\sigma_n^2}{G} & \text{for } (\varphi, \theta) = (\varphi_1, \theta_1), \\ -\frac{\sigma_n^2}{G} & \text{for } (\varphi, \theta) \neq (\varphi_1, \theta_1), \end{cases} \quad (10)$$

where $G = N(N-1) + N\sigma_n^2/\rho_1^2 \approx N^2$ is the processing gain of $|P_s(\varphi, \theta)|$. This also suggests that for extremely weak signals, that is, as $p \rightarrow 0$, the processing gain tends to infinity [7]. In fact this is not the case, and the gain will be determined by the number of samples averaged to produce the covariance matrix. The SGB estimator is clearly able to identify the source signals as well as weak TSI signals in a single spectrum with a very clear margin as discussed in [7]. The price to pay to get a very low output noise level is a large sample support ($>10N$) for SGB. The angular resolution is only slightly better than the MPDR solution. The main advantage of SGB spectrum is its very low output noise floor level which enables us to detect weak signals. Attempting to apply higher processing gain algorithms, such as SGB(N^3) would require more than $100N$ sample support and this would not be practical for radar applications. Hence, direction finding is a matured area and the intention of this section is to highlight the fact that it is not possible to relate each source with its associated TSI path using available techniques. This task will be carried out using the TSI finders.

4. TSI FINDER (LAG DOMAIN)

This section looks at a technique that will identify each source (given the source direction) and its associated TSI arrival (if present). Here we assume that the radar has been able to identify the DOA of an offending source (i.e., $\rho_k^2/\sigma_n^2 > 1$) and we would like to identify all its associated TSI paths. The formal use of the TSI paths or the interference mainlobe multipaths is very well known in the literature under the topic mainlobe jammer nulling, for example [9–11]. However the

use of the TSI path in this study is to locate the noise source. The array's $N \times N$ spatial covariance matrix has the following structure (for the case where two sources and one TSI off each source is present):

$$\mathbf{R}_x = \rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H + \rho_2^2 \mathbf{s}_2 \mathbf{s}_2^H + \rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + \rho_2^2 |\beta_{2,1}|^2 \mathbf{s}_{2,1} \mathbf{s}_{2,1}^H + \sigma_n^2 \mathbf{I}_N. \quad (11)$$

Suppose now we compute the space fast time covariance \mathbf{R}_2 of size $2N \times 2N$ corresponding to an arbitrarily chosen fast time lag n ; then we have

$$\mathbf{R}_2 = E\{\mathbf{X}_n(r)\mathbf{X}_n(r)^H\} = \begin{pmatrix} \mathbf{R}_x & \mathbf{O}_{N \times N} \\ \mathbf{O}_{N \times N} & \mathbf{R}_x \end{pmatrix} \quad (12)$$

for $n \neq n_{1,m}$ or $n_{2,m}$ $m = 1, 2, \dots$,

where $\mathbf{X}_n(r) = (\mathbf{x}(r)^T, \mathbf{x}(r+n)^T)^T$ is termed as the $2N \times 1$ space fast time snapshot for the selected lag n and $\mathbf{O}_{N \times N}$ is the $N \times N$ matrix with zero entries. However, if $n = n_{1,m}$ or $n_{2,m}$ for some m , then we have (say $n = n_{1,1}$ as an example)

$$\begin{aligned} \mathbf{X}_{n_1}(r) &= \begin{pmatrix} \mathbf{x}(r) \\ \mathbf{x}(r+n_{1,1}) \end{pmatrix} \\ &= j_1(r) \begin{pmatrix} \mathbf{s}_1 \\ \beta_{1,1} \mathbf{s}_{1,1} \end{pmatrix} + j_2(r) \begin{pmatrix} \mathbf{s}_2 \\ \mathbf{o}_{N \times 1} \end{pmatrix} \\ &\quad + \beta_{1,1} j_1(r-n_{1,1}) \begin{pmatrix} \mathbf{s}_{1,1} \\ \mathbf{o}_{N \times 1} \end{pmatrix} \\ &\quad + \beta_{2,1} j_2(r-n_{2,1}) \begin{pmatrix} \mathbf{s}_{2,1} \\ \mathbf{o}_{N \times 1} \end{pmatrix} \\ &\quad + j_1(r+n_{1,1}) \begin{pmatrix} \mathbf{o}_{N \times 1} \\ \mathbf{s}_1 \end{pmatrix} + j_2(r+n_{1,1}) \begin{pmatrix} \mathbf{o}_{N \times 1} \\ \mathbf{s}_2 \end{pmatrix} \\ &\quad + \beta_{2,1} j_2(r-n_{2,1}+n_{1,1}) \begin{pmatrix} \mathbf{o}_{N \times 1} \\ \mathbf{s}_{2,1} \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}, \end{aligned} \quad (13)$$

where ε_1 and ε_2 represent two independent measurements of the white noise component, and $\mathbf{o}_{N \times 1}$ is the $N \times 1$ column of zeros. In this case, the space fast time covariance matrix is given by

$$\mathbf{R}_2 = \begin{pmatrix} \mathbf{R}_x & \mathbf{Q}^H \\ \mathbf{Q} & \mathbf{R}_x \end{pmatrix}, \quad (14)$$

where $\mathbf{Q} = \rho^2 \beta_{1,1} \mathbf{s}_{1,1} \mathbf{s}_1^H$.

It is important to note that we assume $n_{1,m}$ ($m = 1, 2, \dots$) represent digitized sample values of the fast time variable r and the reflected path is an integer valued delay of the direct path. If this assumption is not satisfied, one would not achieve a perfect decorrelation, resulting in a nonzero off diagonal term in (12). In other words, a clear distinction between (12) and (14) will not be possible. The existence of the delayed value of the term \mathbf{Q} can be made equal to zero or not be suitably choosing a delay value for $n_{1,1}$ when forming the space time covariance matrix. However, \mathbf{Q} is a matrix and, as a result one may tend to consider its determinant value in order to differentiate the two cases in (12) and (14). After extensive analysis, one may find the signal processing gain

is not acceptable for this choice. More physically meaningful measure would be to consider its contribution to the overall processor output power (when minimized with respect to the look direction constraint). Depending on whether the power contribution is zero or not, we have the situation described in (12) or (14) clearly identified under the above assumptions. Therefore, the scaled measure was introduced as the TSI finder [6], which is a function of the chosen delay value, n must represent the scaled version of the contribution due to the presence of \mathbf{Q} at the total output power. Even though one can come up with many variations of the TSI finder based on the same principle, one expressed in this study is tested and verified to have high signal processing gain as seen later. Now suppose the direction of arrival of the interference source to be (φ_1, θ_1) , the first objective is to find all its associated path delays, which may be of low power and may not have been identified by the usual direction finders. This is carried out by the TSI finder in the lag domain by searching over all possible lag values while the look direction is fixed at the desired source direction (φ_1, θ_1) . This is given by the spectrum

$$T_s(n) = \left\{ \frac{1}{P_{\text{out}}(\mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1)} - 1 \right\}, \quad (15)$$

where $P_{\text{out}} = \mathbf{w}^H \mathbf{R}_2 \mathbf{w}$, \mathbf{w} is the $2N \times 1$ space fast time weights vector which minimizes the power while looking into the direction of the source of interest subject to the constraints: $\mathbf{w}^H \mathbf{s}_A = 1$ and $\mathbf{w}^H \mathbf{s}_B = 0$, where $\mathbf{s}_A = (\mathbf{s}_1^T, \mathbf{o}_{N \times 1}^T)^T$, $\mathbf{s}_B = (\mathbf{o}_{N \times 1}^T, \mathbf{s}_1^T)^T$. The solution \mathbf{w} for each lag is given by $\mathbf{w} = \lambda \mathbf{R}_2^{-1} \mathbf{s}_A + \mu \mathbf{R}_2^{-1} \mathbf{s}_B$ where the parameters λ and μ are given by (one may apply Lagrange multiplier technique and optimize the function $\Phi(\mathbf{w}) = \mathbf{w}^H \mathbf{R}_2 \mathbf{w} + \beta(\mathbf{w}^H \mathbf{s}_A - 1) + \rho \mathbf{w}^H \mathbf{s}_B$ with respect to \mathbf{w} , where β, ρ are arbitrary parameters, as a result, $\partial \Phi / \partial \mathbf{w} = 0$ gives us $\mathbf{w} = \lambda \mathbf{R}_2^{-1} \mathbf{s}_A + \mu \mathbf{R}_2^{-1} \mathbf{s}_B$)

$$\begin{pmatrix} \mathbf{s}_A^H \mathbf{R}_2^{-1} \mathbf{s}_A & \mathbf{s}_B^H \mathbf{R}_2^{-1} \mathbf{s}_A \\ \mathbf{s}_A^H \mathbf{R}_2^{-1} \mathbf{s}_B & \mathbf{s}_B^H \mathbf{R}_2^{-1} \mathbf{s}_B \end{pmatrix} \begin{pmatrix} \lambda^* \\ \mu^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (16)$$

As the search function $T_s(n)$ scans through all potential lag values, one is able to identify the points at which a corresponding delayed version of the look direction signal is encountered as seen in the next section.

Denoting $\mathbf{R}_x = \rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H + \mathbf{R}_1$, we have

$$\mathbf{R}_1 = \rho_2^2 \mathbf{s}_2 \mathbf{s}_2^H + |\beta_{1,1}|^2 \rho_1^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + |\beta_{2,1}|^2 \rho_2^2 \mathbf{s}_{2,1} \mathbf{s}_{2,1}^H + \sigma_n^2 \mathbf{I}_N. \quad (17)$$

4.1. Analysis of the TSI finder

Now, for the sake of convenience, we represent the $2N \times 1$ space fast time weights vector as $\mathbf{w}^T = (\mathbf{w}_1^T, \mathbf{w}_2^T)^T$, where the $N \times 1$ vector \mathbf{w}_1 refers to the first N components of \mathbf{w} and the rest is represented by the $N \times 1$ vector \mathbf{w}_2 . First suppose the chosen lag n is not equal to any of the values $n_{1,j}$ ($j = 1, 2, \dots$). In this case, substituting (12) and (17) in $P_{\text{out}} = \mathbf{w}^H \mathbf{R}_2 \mathbf{w}$ we have

$$P_{\text{out}} = \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + \rho_1^2 \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1 + \rho_1^2 \mathbf{w}_2^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_2. \quad (18)$$

The minimization of power subject to the same constraints, $\mathbf{w}^H \mathbf{s}_A = 1$ and $\mathbf{w}^H \mathbf{s}_B = 0$ (i.e., $\mathbf{w}_1^H \mathbf{s}_1 = 1$ and $\mathbf{w}_2^H \mathbf{s}_1 = 0$), leads to the following solution:

$$\mathbf{w}_1 = \frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1}, \quad \mathbf{w}_2 = \mathbf{0}_{N \times 1}. \quad (19)$$

In this case, we have the following expression for the space fast time processor output power:

$$P_{\text{out}} = \mathbf{w}^H \mathbf{R}_2 \mathbf{w} = \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \rho_1^2 \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1 = (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} + \rho_1^2. \quad (20)$$

Substituting this expression into (15) leads to

$$T_s(n)_{n \neq n_k} = \frac{(\mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1)^{-1}}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} + \rho_1^2} - 1 \equiv 0 \quad (21)$$

(see Appendix A for a proof of the result $(\mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1)^{-1} = (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} + \rho_1^2$). It was noticed that $\mathbf{w}_2 = \mathbf{0}_{N \times 1}$ if and only if $\mathbf{Q} = \mathbf{0}_{N \times N}$. As a result, we would consider the scaled quantity $(P_{\text{out}} - \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 - \rho_1^2 \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1) / P_{\text{out}}$ which is a function of \mathbf{w}_2 only as a suitable TSI finder. Simplification of this quantity using the look direction constraints together with the results in Appendix A and (19) leads to (15).

The most important fact here is that we do not have to assume the simple case of a main lobe interferer and one TSI path to prove that this quantity is zero. The TSI finder spectrum has the following properties, as we look into the direction (φ_1, θ_1) :

$$T_s(n) \approx \begin{cases} P_{\text{out}}^{-1} (\mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1)^{-1} - 1, & n = n_{1,j} \text{ for some } j, \\ 0, & n \neq n_{1,j}. \end{cases} \quad (22)$$

The TSI estimator indicates infinite processing gain when inverted (at least in theory), and is able to detect extremely small TSI power. In the next section we would like to further investigate the properties of the estimator and its processing gain.

The output power at the processor P_{out} (for $n = n_{1,1}$) is given by (using (14))

$$\begin{aligned} P_{\text{out}} &= \mathbf{w}^H \mathbf{R}_2 \mathbf{w} = \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 \\ &+ \rho^2 \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1 + \rho_1^2 \mathbf{w}_2^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_2 \\ &+ \rho_1^2 \beta_{1,1}^* \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_2 + \rho_1^2 \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1. \end{aligned} \quad (23)$$

When the constraints $\mathbf{w}_1^H \mathbf{s}_1 = 1$ and $\mathbf{w}_2^H \mathbf{s}_1 = 0$ are imposed, we have

$$\begin{aligned} P_{\text{out}} &= \mathbf{w}^H \mathbf{R}_2 \mathbf{w} = \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 \\ &+ \rho_1^2 + \rho_1^2 (\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_1). \end{aligned} \quad (24)$$

The original power minimization problem can now be broken into two independent minimization problems as fol-

lows.

- (1) Minimize $\mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1$ subject to the constraint $\mathbf{w}_1^H \mathbf{s}_1 = 1$.
- (2) Minimize $\mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + \rho_1^2 + \rho_1^2 (\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_1)$ subject to $\mathbf{w}_2^H \mathbf{s}_1 = 0$.

The solution can be expressed as

$$\mathbf{w}_1 = \frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1}, \quad (25)$$

$$\mathbf{w}_2 = -\beta_{1,1} \rho_1^2 \mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \beta_{1,1} \rho_1^2 \left(\frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \mathbf{R}_1^{-1} \mathbf{s}_1. \quad (26)$$

The above representation of the solution cannot be used to compute the space-time weights vector \mathbf{w} due to the fact that the quantities involved are not measurable. Instead, the result in (16) is implemented to evaluate \mathbf{w} as described in the previous section.

Substituting $\mathbf{R}_1 = \rho_2^2 \mathbf{s}_2 \mathbf{s}_2^H + \rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + \rho_2^2 |\beta_{2,1}|^2 \mathbf{s}_{2,1} \mathbf{s}_{2,1}^H + \sigma_n^2 \mathbf{I}_N$ into (24) and noting that $\rho_1^2 |\beta_{1,1}|^2 \mathbf{w}_2^H \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{w}_2 + \rho_1^2 + \rho_1^2 (\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_1) = \rho_1^2 |1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2$, we have the following expression for the output power:

$$\begin{aligned} P_{\text{out}} &= \rho_1^2 |\beta_{1,1}|^2 |\mathbf{w}_1^H \mathbf{s}_{1,1}|^2 + \rho_1^2 |1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2 \\ &+ \mathbf{w}_1^H \mathbf{R}_0 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_0 \mathbf{w}_2 + \sigma_n^2 (\mathbf{w}_1^H \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{w}_2), \end{aligned} \quad (27)$$

where $\mathbf{R}_0 = \rho_2^2 \mathbf{s}_2 \mathbf{s}_2^H + |\beta_{2,1}|^2 \rho_2^2 \mathbf{s}_{2,1} \mathbf{s}_{2,1}^H$ is the output energy due to any second source and associated multipaths present at the input. It should be noted that this component of the output also contains any output energy due to any second (unmatched) multipath of the look direction source (e.g., $|\beta_{1,2}|^2 \mathbf{s}_{1,2} \mathbf{s}_{1,2}^H$ terms). The most general form would be

$$\begin{aligned} \mathbf{R}_0 &= \sum_{j=2}^{a_1} \rho_1^2 |\beta_{1,j}|^2 \mathbf{s}_{1,j} \mathbf{s}_{1,j}^H + \sum_{k=2}^q \rho_k^2 \mathbf{s}_k \mathbf{s}_k^H \\ &+ \sum_{k=2}^q \sum_{j=1}^{a_k} \rho_k^2 |\beta_{k,j}|^2 \mathbf{s}_{k,j} \mathbf{s}_{k,j}^H, \end{aligned} \quad (28)$$

where q is the number of sources, and a_k is the number of TSI paths available for the k th source. The expression for P_{out} in (27) clearly indicates that the best \mathbf{w}_1 that (which has N degrees of freedom) would minimize P_{out} is likely to be orthogonal to $\mathbf{s}_{1,1}$, that is, $|\mathbf{w}_1^H \mathbf{s}_{1,1}| \approx 0$, and furthermore it would be attempting to satisfy $|1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2 \approx 0$ while being orthogonal to all other signals present in \mathbf{R}_0 .

Note that

$$\begin{aligned} \mathbf{w}_1^H \mathbf{R}_0 \mathbf{w}_1 &= \sum_{j=2}^{a_1} \rho_1^2 |\beta_{1,j}|^2 |\mathbf{w}_1^H \mathbf{s}_{1,j}|^2 + \sum_{k=2}^q \rho_k^2 |\mathbf{w}_1^H \mathbf{s}_k|^2 \\ &+ \sum_{k=2}^q \sum_{j=1}^{a_k} \rho_k^2 |\beta_{k,j}|^2 |\mathbf{w}_1^H \mathbf{s}_{k,j}|^2 \end{aligned} \quad (29)$$

and a similar expression holds for $\mathbf{w}_2^H \mathbf{R}_0 \mathbf{w}_2$.

Any remaining degrees of freedom would be used to minimize the contribution due to the white noise component. In order to investigate the properties of the solution for \mathbf{w} , let us

assume we have only a look direction signal and its TSI path, in which case we have $\mathbf{R}_0 = \mathbf{O}_{N \times N}$ and

$$P_{\text{out}} = \rho_1^2 |\beta_{1,1}|^2 |\mathbf{w}_1^H \mathbf{s}_{1,1}|^2 + \rho_1^2 |1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2 + \sigma_n^2 (\mathbf{w}_1^H \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{w}_2). \quad (30)$$

In this case, $\mathbf{R}_1 = |\beta_{1,1}|^2 \rho_1^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + \sigma_n^2 \mathbf{I}_N$ and the inverse of which is given by

$$\mathbf{R}_1^{-1} = \frac{1}{\sigma_n^2} \left[\mathbf{I}_N - \frac{(\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H)}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right]. \quad (31)$$

As a result, we have

$$\mathbf{R}_1^{-1} \mathbf{s}_1 = \frac{1}{\sigma_n^2} \left[\mathbf{s}_1 - \frac{(\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{s}_1)}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right], \quad (32)$$

$$\mathbf{R}_1^{-1} \mathbf{s}_{1,1} = \frac{\mathbf{s}_{1,1}}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)}, \quad (33)$$

$$\mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1} = \frac{N}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)}, \quad (34)$$

$$\mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_1 = \frac{\mathbf{s}_{1,1}^H \mathbf{s}_1}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)}. \quad (35)$$

Furthermore, we adopt the notation $\mathbb{J}_1 = \mathbb{J}$ for the look direction interferer to noise power and (for $N |\beta_{1,1}|^2 \mathbb{J} \gg 1$)

$$\begin{aligned} \mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_1 &= \frac{1}{\sigma_n^2} \left[N - \frac{(\rho_1^2 |\beta_{1,1}|^2 |\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2)}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right] \\ &= \frac{N}{\sigma_n^2} \left[1 - \frac{|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 |\beta_{1,1}|^2 \mathbb{J}}{N(1 + N |\beta_{1,1}|^2 \mathbb{J})} \right] \\ &\approx \frac{N}{\sigma_n^2} \left(1 - \frac{|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2}{N^2} \right) \approx \frac{N}{\sigma_n^2}. \end{aligned} \quad (36)$$

The assumption made in the last expression (i.e., $|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 / N^2 \approx 0$) is very accurate when the signals are not closely spaced. This assumption cannot be verified analytically, it depends on the structure of the array, however, it can be numerically verified for a commonly used linear equispaced array with half wavelength spacing. The other assumption made throughout this study is that the look direction interferer is above the noise floor (i.e., $\mathbb{J} > 1$). In this case, we need at least $|\beta_{1,1}|^2 \gg 1/N$ (or equivalently $N |\beta_{1,1}|^2 \mathbb{J} \gg 1$) in order to detect any TSI power as seen later. We will also see that when $|\beta_{1,1}|^2$ is closer to the lower bound of $1/N$ we do not achieve good processing gain to detect TSI unless \mathbb{J} is extremely large (but this case is not presented here).

Now, we would like to investigate the two cases $|\beta_{1,1}|^2 \gg 1/N$ and $|\beta_{1,1}|^2 \ll 1/N$ simultaneously.

The value of the expression (36) for $|\beta_{1,1}|^2 \ll 1/N$ can be simplified as follows:

$$\begin{aligned} \mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_1 &\approx \frac{N}{\sigma_n^2} \left[1 - \frac{|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 |\beta_{1,1}|^2 \mathbb{J}}{N} \right] \\ &\approx \frac{N}{\sigma_n^2} \left[1 - \frac{|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 (N |\beta_{1,1}|^2 \mathbb{J})}{N^2} \right] \approx \frac{N}{\sigma_n^2}. \end{aligned} \quad (37)$$

Throughout the study, this case is taken to be equivalent to $N |\beta_{1,1}|^2 \mathbb{J} \ll 1$ as well, because \mathbb{J} is not assumed to take excessively large values for $|\beta_{1,1}|^2 \ll 1/N$. The investigation of the signal processing gain for the case where $|\beta_{1,1}|^2 \ll 1/N$ and at the same time \mathbb{J} is very large is outside of the scope of this study.

Furthermore, applying the above formula and (33) in (25), we can see that

$$\begin{aligned} |\mathbf{w}_1^H \mathbf{s}_{1,1}|^2 &= \left| \frac{\mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right|^2 \\ &= \left| \frac{\mathbf{s}_{1,1}^H}{\mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_1} \cdot \frac{\mathbf{s}_{1,1}}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right|^2 \\ &\approx \frac{(|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 / N^2)}{(1 + N |\beta_{1,1}|^2 \mathbb{J})^2} \approx 0. \end{aligned} \quad (38)$$

This expression shows how closely we have achieved the orthogonality requirement expected above. It is reasonable to assume that $\mathbf{w}_1^H \mathbf{s}_{1,1} \approx 0$ (or equivalently $|\mathbf{s}_{1,1}^H \mathbf{s}_{1,1}|^2 / N^2 \approx 0$) for all possible positive values of $N |\beta_{1,1}|^2$. We may now investigate the second and third terms as the dominant terms at the processor output in (30). The approximate expressions for these two terms can be derived using (32)–(36) (see Appendix B) as

$$\begin{aligned} |1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2 &\approx \begin{cases} \frac{1}{(N |\beta_{1,1}|^2 \mathbb{J})^2} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ 1 - 2N |\beta_{1,1}|^2 \mathbb{J} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1, \end{cases} \\ \sigma_n^2 \|\mathbf{w}\|^2 &= \begin{cases} \sigma_n^2 \left(\frac{1}{N} + \frac{1}{N |\beta_{1,1}|^2} \right) & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ \sigma_n^2 \left(\frac{1}{N} + N |\beta_{1,1}|^2 \mathbb{J} \right) & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{cases} \end{aligned} \quad (39)$$

Substituting (39) in (30), we can evaluate $P_{\text{out}} / \sigma_n^2$ as

$$\frac{P_{\text{out}}}{\sigma_n^2} = \begin{cases} \left(\frac{1}{N} + \frac{1}{N |\beta_{1,1}|^2} \right) + \frac{1}{N^2 |\beta_{1,1}|^4 \mathbb{J}} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ \frac{1}{N} + \mathbb{J} - N |\beta_{1,1}|^2 \mathbb{J}^2 & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1 \end{cases} \quad (40)$$

which becomes

$$\frac{P_{\text{out}}}{\sigma_n^2} \begin{cases} = \frac{N |\beta_{1,1}|^4 \mathbb{J} + N |\beta_{1,1}|^2 \mathbb{J} + 1}{N^2 |\beta_{1,1}|^4 \mathbb{J}} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ \approx \frac{1}{N} + \mathbb{J} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{cases} \quad (41)$$

After substituting $N|\beta_{1,1}|^4\mathbb{J} + N|\beta_{1,1}|^2\mathbb{J} + 1 \approx N|\beta_{1,1}|^4\mathbb{J} + N|\beta_{1,1}|^2\mathbb{J}$ in the above expression for the $N|\beta_{1,1}|^2\mathbb{J} \gg 1$ case, we have

$$\frac{\sigma_n^2}{P_{\text{out}}} \approx \begin{cases} \frac{N|\beta_{1,1}|^2}{1 + |\beta_{1,1}|^2} & \text{for } N|\beta_{1,1}|^2\mathbb{J} \gg 1, \\ \frac{N}{1 + N\mathbb{J} - N^2|\beta_{1,1}|^2\mathbb{J}^2} & \text{for } N|\beta_{1,1}|^2\mathbb{J} \ll 1. \end{cases} \quad (42)$$

As seen later in the simulation section, the conclusions drawn here do not change significantly when one or two sidelobe interferers (other sources) are considered. The only difference is that (30) will have additional terms due to side lobe interferers and other TSI paths. The added terms in (30) are of the form $\rho_k^2 |\mathbf{w}_1^H \mathbf{s}_k|^2$ ($k = 1, 2, \dots$) and they should satisfy the orthogonality requirement in a very similar manner. By denoting the value of $T_s(n)$ for $n = n_{1,1}$ by $T_s(n)_{n=n_{1,1}}$ we can use the result in (42) and the identity obtained in Appendix A to further simplify (22) to show that

$$T_s(n)_{n=n_{1,1}} = \begin{cases} (\rho_1^2 + (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1}) \frac{N|\beta_{1,1}|^2}{\sigma_n^2(1 + |\beta_{1,1}|^2)} - 1, & N|\beta_{1,1}|^2\mathbb{J} \gg 1, \\ (\rho_1^2 + (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1}) \frac{N}{\sigma_n^2(1 + N\mathbb{J} - N^2|\beta_{1,1}|^2\mathbb{J}^2)} - 1 & N|\beta_{1,1}|^2\mathbb{J} \ll 1. \end{cases} \quad (43)$$

For the case of a small number of jammers and TSI paths, we have shown that $(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1) \approx N/\sigma_n^2$ for $N|\beta_{1,1}|^2\mathbb{J} \ll 1$ and $N|\beta_{1,1}|^2\mathbb{J} \gg 1$. As a result, we have for $N|\beta_{1,1}|^2\mathbb{J} \gg 1$ that

$$\begin{aligned} T_s(n)_{n=n_{1,1}} &= \left(\rho_1^2 + \frac{\sigma_n^2}{N} \right) \frac{N|\beta_{1,1}|^2}{\sigma_n^2(1 + |\beta_{1,1}|^2)} - 1 \\ &\approx \frac{N|\beta_{1,1}|^2\mathbb{J} - 1}{(1 + |\beta_{1,1}|^2)} \approx N|\beta_{1,1}|^2\mathbb{J} \end{aligned} \quad (44)$$

and for $N|\beta_{1,1}|^2\mathbb{J} \ll 1$ that

$$\begin{aligned} T_s(n)_{n=n_{1,1}} &= \frac{(\rho_1^2 + \sigma_n^2/N)}{P_{\text{out}}} - 1 \\ &= \frac{N(\rho_1^2 + \sigma_n^2/N)}{\sigma_n^2(1 + N\mathbb{J} - N^2|\beta_{1,1}|^2\mathbb{J}^2)} - 1 \\ &\approx \frac{N^2|\beta_{1,1}|^2\mathbb{J}^2}{(1 + N\mathbb{J})(1 - N|\beta_{1,1}|^2\mathbb{J})} \\ &\approx \frac{N^2|\beta_{1,1}|^2\mathbb{J}^2}{1 + N\mathbb{J}} \approx N|\beta_{1,1}|^2\mathbb{J}. \end{aligned} \quad (45)$$

The TSI finder spectrum has the following properties:

$$T_s(n) = \begin{cases} N|\beta_{1,1}|^2\mathbb{J} & \text{for } n = n_{1,1}, \\ 0 & \text{for } n \neq n_{1,1}. \end{cases} \quad (46)$$

In order to quantify the processing gain of this spectrum one has to replace the zero figure with a quantity which would represent the average output interference level present in the spectrum whenever a lag mismatch occurs. Replacing $\mathbf{Q}^H = \mathbf{O}_{N \times N}$ in (12) by an approximate figure (when $n \neq n_{1,1}$) would give rise to a small nonzero value. This figure can be shown to be of the order $N/M\mathbb{J}$ (written as $O(N/M\mathbb{J})$) where M is the number of samples used in covariance averaging. As a result we can establish processing gain as

$$\frac{T_s(n)_{n=n_{1,1}}}{T_s(n)_{n \neq n_{1,1}}} \approx \frac{N|\beta_{1,1}|^2\mathbb{J}}{O(N/M\mathbb{J})} \approx O(M|\beta_{1,1}|^2\mathbb{J}^2) \quad (47)$$

(see Appendix C for the proof). This equation allows us to establish the following lemma.

Lemma 1. *In order to detect very small TSI power level of the order $1/N$ (i.e., $|\beta_{1,1}|^2 \approx 1/N$ while satisfying $\mathbb{J} > 1$), with a processing gain of approximately 10 dB (value at peak point when a match occurs/the average output level when a mismatch occurs), one needs to average about $10N$ ($= M$) samples at the covariance matrix. However, if \mathbb{J} is very large (i.e., $\gg 1$) we can use fewer samples.*

For example if $\mathbb{J} = 10$ dB, then any value of $N(> M)$ can produce 20 dB processing gain at the spectrum for TSI signals of order $|\beta_{1,1}|^2 \approx 1/N$. In fact, simulations generally show much better processing gains in the TSI estimator as discussed later.

5. TSI FINDER IN ANGLE DOMAIN

The fundamental assumption we make here is that given the interferer direction of arrival (φ_1, θ_1) , one is able to accurately identify at least one TSI path and its associated fast time lag ($= n_{1,1}$). The remaining issue we need to resolve here is to estimate the direction of arrival of this particular TSI path (i.e., $(\varphi_{1,1}, \theta_{1,1})$) in the azimuth/elevation plane. This is carried out by the search function $\|\mathbf{F}(\varphi, \theta)\|^{-2} = 1/\mathbf{F}(\varphi, \theta)^H \mathbf{F}(\varphi, \theta)$, where

$$\mathbf{F}(\varphi, \theta) = \frac{N\mathbf{Q}\mathbf{s}_1(\varphi_1, \theta_1)}{\mathbf{s}(\varphi, \theta)^H \mathbf{Q}\mathbf{s}_1(\varphi_1, \theta_1)} - \mathbf{s}(\varphi, \theta), \quad (48)$$

$$\begin{aligned} \mathbf{Q}^H &= E\{\mathbf{x}(r)\mathbf{x}(r + n_{1,1})^H\} \\ &= \rho_1^2 \beta_{1,1} \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \quad (\text{or } \mathbf{Q} = \rho_1^2 \beta_{1,1} \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H). \end{aligned} \quad (49)$$

At this stage of postprocessing, the existence of $\beta_{1,1} (\neq 0)$ has been guaranteed, but the value of this reflectivity constant maybe anywhere between zero and 1 (or higher).

The reason for choosing (48) as a suitable spectrum is as follows: if we manipulate the value of \mathbf{Q} to avoid the unknown quantities $\beta_{1,1}$ and ρ_1^2 we can see the fact that $N\mathbf{Q}\mathbf{s}_1/\mathbf{s}^H \mathbf{Q}\mathbf{s}_1 = N\mathbf{s}_{1,1}/(\mathbf{s}_{1,1}^H \mathbf{s}_{1,1})$ where \mathbf{s} is a general search vector to represent the array manifold. This quantify approaches $\mathbf{s}_{1,1}$ as \mathbf{s} approaches $\mathbf{s}_{1,1}$. Since \mathbf{Q} is guaranteed to be nonzero due to the known presence of the TSI path, we can now estimate the steering value of the TSI direction. The best way to achieve the desired result is to set up a search function by

inverting the difference function ($N\mathbf{Q}\mathbf{s}_1/\mathbf{s}^H\mathbf{Q}\mathbf{s}_1 - \mathbf{s}$). Such a search function will face a singularity at the point of interest which will generally result in a good signal processing gain as seen later.

Now we would like to include the next highest order term for \mathbf{Q} as (using (C.3))

$$\mathbf{Q}^H = \rho_1^2 \beta_{1,1} \mathbf{s}_1 \mathbf{s}_1^H + O(\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H / \sqrt{M}) \quad (50)$$

or we may write this as

$$\mathbf{Q} = \rho_1^2 \beta_{1,1} \mathbf{s}_1 \mathbf{s}_1^H + A(\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H / \sqrt{M}), \quad (51)$$

where A is a small scalar which is only used in identifying the nature of the spectrum in (48) whenever $\beta_{1,1}$ is close to zero (i.e., $N|\beta_{1,1}|^2 \ll 1$) and at all other times we ignore its presence.

Now we have

$$\frac{N\mathbf{Q}\mathbf{s}_1}{\mathbf{s}^H\mathbf{Q}\mathbf{s}_1} = \frac{N^2 \beta_{1,1} \rho_1^2 \mathbf{s}_1 + AN^2 \rho_1^2 \mathbf{s}_1 / \sqrt{M}}{N \beta_{1,1} \rho_1^2 \mathbf{s}_1^H \mathbf{s}_1 + AN \rho_1^2 \mathbf{s}_1^H \mathbf{s}_1 / \sqrt{M}}, \quad (52)$$

$$\begin{aligned} & \|\mathbf{F}(\varphi, \theta)\|^{-2} \\ &= \frac{|\beta_{1,1} \mathbf{s}_1^H \mathbf{s}_1 + A \mathbf{s}_1^H \mathbf{s}_1 / \sqrt{M}|^2}{\|\beta_{1,1} [N \mathbf{s}_1 - (\mathbf{s}_1^H \mathbf{s}_1) \mathbf{s}] + [(A/\sqrt{M})(N \mathbf{s}_1 - (\mathbf{s}_1^H \mathbf{s}_1) \mathbf{s})]\|^2}. \end{aligned} \quad (53)$$

For $N|\beta_{1,1}|^2 \gg 1$, we have the following generic pattern:

$$\|\mathbf{F}(\varphi, \theta)\|^2 = \frac{|\mathbf{s}_1^H \mathbf{s}_1|^2}{\|N \mathbf{s}_1 - (\mathbf{s}_1^H \mathbf{s}_1) \mathbf{s}\|^2} \quad (54)$$

which has a singularity when $(\varphi, \theta) = (\varphi_{1,1}, \theta_{1,1})$ (i.e., $\mathbf{s} = \mathbf{s}_{1,1}$) which is the direction of arrival of the TSI path. This pattern is independent of the radar parameters and its first side lobe occurs below -35 dB (for 16×16 planer array) the pattern is illustrated in Figure 1. Here in general we are interested only in the case $N|\beta_{1,1}|^2 \gg 1$, but if $\beta_{1,1}$ is negligibly small and A is of dominant value then, we would obtain the spectrum (letting $\beta_{1,1} \rightarrow 0$ in (53))

$$\|\mathbf{F}(\varphi, \theta)\|^2 = \frac{|\mathbf{s}_1^H \mathbf{s}_1|^2}{\|N \mathbf{s}_1 - \mathbf{s}(\mathbf{s}_1^H \mathbf{s}_1)\|^2}. \quad (55)$$

This is the same pattern as before but the peak is at $(\varphi, \theta) = (\varphi_1, \theta_1)$ (corresponds to the look direction of the interferer). This sudden shift of the peak (singularity) occurs when $\beta_{1,1}$ is incredibly small. We may now represent the two cases as

$$\|\mathbf{F}(\varphi, \theta)\|^2 = \begin{cases} \frac{|\mathbf{s}_1^H \mathbf{s}_1|^2}{\|N \mathbf{s}_1 - (\mathbf{s}_1^H \mathbf{s}_1) \mathbf{s}\|^2} & \text{for } \beta_{1,1} \neq 0, \\ \frac{|\mathbf{s}_1^H \mathbf{s}_1|^2}{\|N \mathbf{s}_1 - \mathbf{s}(\mathbf{s}_1^H \mathbf{s}_1)\|^2} & \text{for } \beta_{1,1} = 0 \text{ or } n \neq n_{1,1}. \end{cases} \quad (56)$$

The case for $\beta_{1,1} = 0$ is not relevant at this stage of post-processing since this is the case where TSI was nonexistent

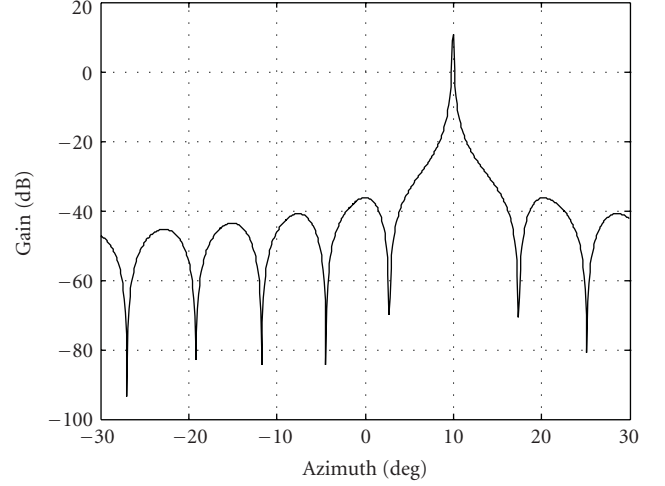


FIGURE 1: Theoretical pattern for the TSI finder (elevation, $\theta = -5^\circ$) in the angle domain where the angle of arrival is 10° in azimuth (16×16 planer equispaced array with half wavelength element spacing in azimuth and elevation).

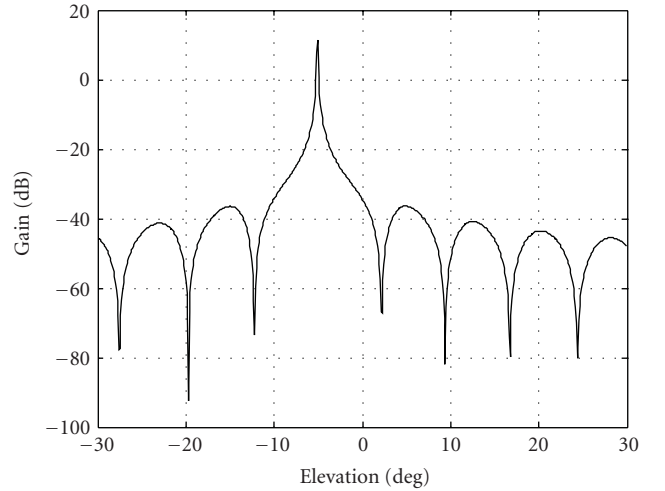


FIGURE 2: Theoretical pattern for the TSI finder (azimuth, $\varphi = 10^\circ$) in the angle domain where the angle of arrival is -5° in azimuth. (16×16 planer equispaced array with half wavelength spacing).

or negligible, but it can be shown that any lag mismatch is equivalent to the case $\beta_{1,1} = 0$ as well. Figures 1 and 2 illustrate horizontal and vertical cuts of the pattern in (55) for a two dimensional 16×16 element linear equispaced rectangular array with half wavelength spacing when the angle of arrival of the TSI path is $(\varphi_{1,1}, \theta_{1,1}) = (10^\circ, -5^\circ)$. The signal processing gain of this processor approaches infinity due to the fact that the peak point is a singularity. In practice, this is not the case. In order to get a feel for the value of the peak point, we may use the following argument. Suppose (φ, θ)

is approaching $(\varphi_{1,1}, \theta_{1,1})$, and using $(N\mathbf{s}_{1,1} - (\mathbf{s}^H \mathbf{s}_{1,1})\mathbf{s}) \rightarrow 0$, then in (53) we have

$$\begin{aligned} & \|\mathbf{F}(\varphi, \theta)\|^2 \\ &= \frac{|\beta_{1,1} \mathbf{s}^H \mathbf{s}_{1,1}|^2}{\|[(A/\sqrt{M})(N\mathbf{s}_1 - (\mathbf{s}^H \mathbf{s}_1)\mathbf{s})]\|^2} \approx \frac{|\beta_{1,1} \mathbf{s}^H \mathbf{s}_{1,1}|^2}{A^2(1/M)\|(N\mathbf{s}_1 - (\mathbf{s}^H \mathbf{s}_1)\mathbf{s})\|^2} \\ &\approx \frac{M|\beta_{1,1}|^2}{A^2\|(\mathbf{s}_1 - (\mathbf{s}^H \mathbf{s}_1)\mathbf{s}/N)\|^2} \approx \frac{M|\beta_{1,1}|^2}{A^2\|\mathbf{s}_1\|^2} \approx \frac{M|\beta_{1,1}|^2}{A^2N} \end{aligned} \quad (57)$$

(we have replaced $\mathbf{s}^H \mathbf{s}_{1,1}$ by N (as $\mathbf{s}^H \rightarrow \mathbf{s}_{1,1}$) to obtain the first term in the second row of the above equation and further assumed that $(\mathbf{s}^H \mathbf{s}_1)\mathbf{s}/N \rightarrow (\mathbf{s}_{1,1}^H \mathbf{s}_1)\mathbf{s}_{1,1}/N$ which is a very small contribution compared to \mathbf{s}_1). Consider the case where $10N$ or more data points are averaged in forming the covariance matrix; we have a rough figure of $10|\beta_{1,1}|^2/A^2$. When we assume the smallest expected value of detecting the TSI as indicated earlier, as $|\beta_{1,1}|^2 \approx 1/N$, then the peak value of the spectrum is of the order $10/(NA^2)$. Now for an array of around 10 elements or more we have $10/A^2$ which is still expected to be of greater than unity since A is generally expected to be between 0 and 1. The peak point occurs at a much higher point than at 0 dB point on the pattern, while the first side lobe occurs below -35 dB thus producing a very good ability to detect the presence of the signal.

6. SOURCE LOCATION

The diagram in Figure 3 illustrates the geometry of the scenario related to the selected TSI path of the mainlobe interference only. The unit vector pointing from the array to source is denoted by \mathbf{k}_s and the unit vector representing the TSI path is \mathbf{k}_t (only the section of the path from array to reflection point on the ground). The unit vector \mathbf{k}_h points towards the ground vertically below the platform. The distance from source to array is D , the distance from ground reflection point (of the TSI path) to the source is d_1 , the distance from the source to the ground reflection point is d_2 , and the array height is h . Now assuming an xyz right-handed coordinate system where the y axis is pointed upwards positive, the x axis points to the direction of travel (array is assumed to be in the xy plane), we have the following data:

$$\begin{aligned} \mathbf{k}_h &= (0, -h, 0), \\ \mathbf{k}_s &= (\cos \theta_1 \sin \varphi_1, \sin \theta_1, \cos \theta_1 \cos \varphi_1), \\ \mathbf{k}_t &= (\cos \theta_{1,1} \sin \varphi_{1,1}, \sin \theta_{1,1}, \cos \theta_{1,1} \cos \varphi_{1,1}). \end{aligned} \quad (58)$$

The angles φ_0 and φ_1 (as seen in Figure 3) maybe computed from

$$\begin{aligned} \varphi_0 &= \cos^{-1} \left(\frac{\mathbf{k}_s \cdot \mathbf{k}_t}{\|\mathbf{k}_s\| \cdot \|\mathbf{k}_t\|} \right), \\ \varphi_1 &= \cos^{-1} \left(\frac{\mathbf{k}_t \cdot \mathbf{k}_h}{\|\mathbf{k}_t\| \cdot \|\mathbf{k}_h\|} \right). \end{aligned} \quad (59)$$

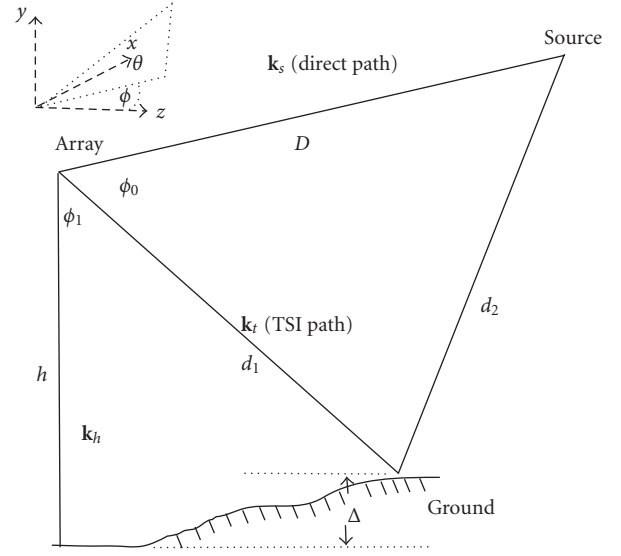


FIGURE 3: TSI scenario and associated parameters.

Furthermore, we have

$$\begin{aligned} d_1 + d_2 &= D + m\delta R, \\ d_2^2 &= (D \sin \varphi_0)^2 + (d_1 - D \cos \varphi_0)^2, \\ d_1 &= \frac{h}{\cos \varphi_1}. \end{aligned} \quad (60)$$

The integer value m is the estimated TSI lag and δR is the radar range resolution which is the fast time sampling interval multiplied by the speed of light. The assumption that the path difference is an integer multiple of the range resolution is only an approximation. This is reasonable for high-resolution radar. If this figure is not an integer value it can cause some error in the estimate of D . It should be noted that h is only a very rough value to represent the height of the platform, since the terrain below is not generally flat and may not lie in the same horizontal plane as the ground reflection point as shown in Figure 3. We have represented this difference by the symbol Δ which will not be directly measurable. It is also possible to use all the TSI paths available (of the interference source, identified by the TSI finder) to be used in making multiple estimates of the same parameter D . Multiple ground reflections are a real possibility in many environments.

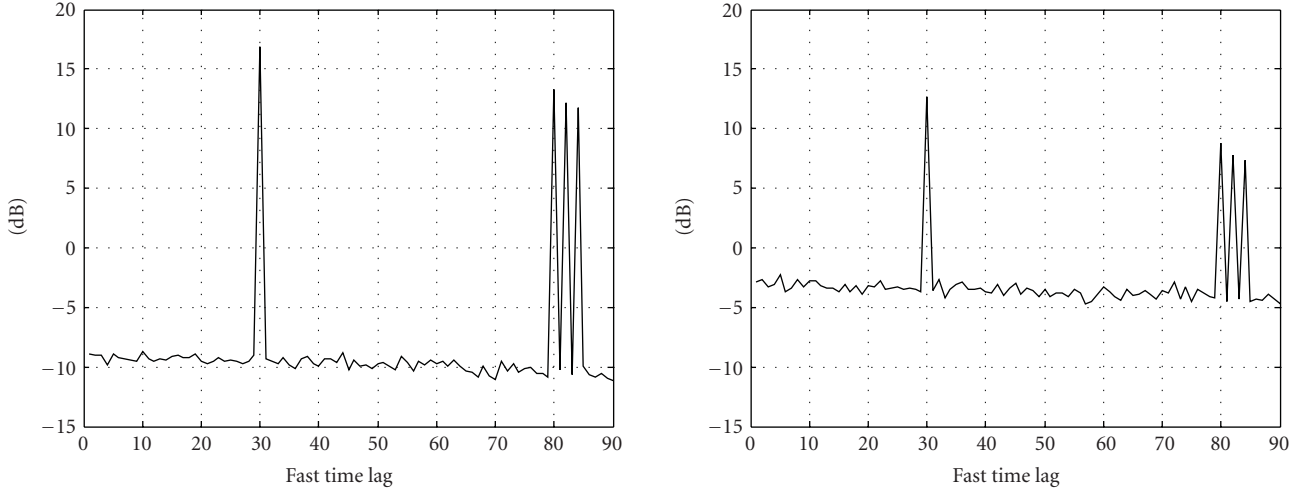
Assuming $\Delta = 0$ and eliminating d_2 from the above three equations, we arrive at

$$\begin{aligned} (D + m\delta R - d_1)^2 &= D^2 - 2Dd_1 \cos \varphi_0 + d_1^2, \\ D &= \frac{m\delta R(2h - m\delta R \cos \varphi_1)}{2(m\delta R \cos \varphi_1 - h + h \cos \varphi_0)} \end{aligned} \quad (61)$$

which is a function of φ_1 , θ_1 , $\varphi_{1,1}$, $\theta_{1,1}$, and h only.

7. SIMULATION

In the simulated example, we have considered a 16×16 ($N = 256$) planar array, with a first interferer arriving from the



(a) Scenario 1: all TSI paths have lags which are integer multiples of the range resolution, the interference power in the look direction is 10 dB and consists of four TSI arrivals corresponding to lags 30, 80, 82, and 85

(b) Scenario 2: as in scenario 1, except that the noise floor has been increased by a factor of four and the first TSI path is now at a lag of 30.5 units

FIGURE 4: The output of the TSI finder in the lag domain.

array broadside, $(\varphi_1, \theta_1) = (0^\circ, 0^\circ)$, with an interferer to noise ratio of 10 dB ($= \sigma_j^2$), where $\sigma_n^2 = 1$ is set without loss of generality. Four TSI paths are simulated with power levels $\beta_1^2 = 1/20$, $\beta_2^2 = 1/40$, $\beta_3^2 = 1/80$ and $\beta_4^2 = 1/90$. The corresponding TSI angles of arrival pairs are given by $(\varphi_{1,1}, \theta_{1,1}) = (10^\circ, -3^\circ)$, $(\varphi_{1,2}, \theta_{1,2}) = (0^\circ, -20^\circ)$, $(\varphi_{1,3}, \theta_{1,3}) = (0^\circ, -30^\circ)$ and, $(\varphi_{1,4}, \theta_{1,4}) = (0^\circ, -35^\circ)$. The corresponding fast time lags are 30, 80, 82, and 84, respectively. The second independent interferer is 10 dB above noise and has an angle of arrival pair $(\varphi_2, \theta_2) = (20^\circ, -40^\circ)$. For the second interference source we have added one multipath with parameters $(\varphi_{2,1}, \theta_{2,1}) = (30^\circ, -15^\circ)$, $\beta^2 = 1/20$. In this study we will assume the direction of arrival of the interferer has been identified to be the array's broadside and do not display the direction of arrival spectrum (a well-established capability). Figure 4(a) shows the output of the lag domain TSI finder as defined in (15). The postprocessor identifies all the TSI arrivals and their associated fast time lags very clearly in the simulation. The presence of the second interference source and its multipath does not visibly affect the processing gain of the TSI spectrum (spectrum is almost the same, with or without the second interference and its multipaths for the array of 16×16 elements, this is due to the high degree of freedom available in a 256 element array). TSI spectrum is designed to pick up every delayed version of the look direction interferer only (by an integer multiple of the range resolution, as we can scan through all possible fast time lag values). Also, the TSI spectrum excludes the look direction of the interferer itself. In other words, the spectrum contains only the multipaths of the look direction interferer. Further it was noticed that if the number of sidelobe interference sources and their multipaths increases, then the TSI spectrums which is related to the mainlobe interferer gradually loses its processing gain by increasing the noise floor. This is expected in any array processor due to its degree of freedom limitations.

This effect is really not significant until 6 or more interferes are introduced for this simulated example with 16×16 elements. For this simulation, we have generated 2000 range samples ($\approx 4 \times 2N$, where $2N \times 2N$ is the size of the space time covariance matrix). Processing gain is much better than the theoretically expected values. For the smallest peak, that is, for $\beta_4^2 = 1/90$, the theoretical expectation of the processing gain is around $O(4 \times 256 \times (1/90) \times 10) \approx 20$ dB whereas in the simulation this peak rises more than 20 dB above the average output noise floor level in Figure 4.

The simulation study has shown that the usual $3 \times 2N$ (= number of samples) rule seems to be sufficient in averaging the covariance matrix in order to obtain better than the theoretical predicted processing gain levels. The computational complexity of the TSI finder is of the order of N^3 (for an N element array) which is expected as it requires to invert the covariance matrix in (15). Figure 4(b) illustrates the results when the noise floor is increased by a factor 4 ($\sigma^2 = 4$). The raise can be continued until the mainlobe interferer is reasonably above the noise floor (at least 3 dB for this array). Due to high signal processing gain of the TSI finder, one is able to detect very weak TSI signals ($\beta^2 < 1/40$) of the main lobe interferer provided the direct interferer power is 3 or 4 dB above the noise level. A large number of simulation runs have confirmed that when the TSI path is not an integer multiple of the range resolution, the performance degradation in the TSI spectrum is less than 1 or 2 dB at most. This was carried out by linearly interpolating generated TSI path data and shifting it by a fraction of the fast time lag.

The input to the second processor can be selected as any one those lag values selected from Figure 4. In our example, the lag = 30 as the input to the angle domain finder is used, the output of which is illustrated in Figure 5. The TSI finder in the angle domain has shown more robustness in all above cases discussed.

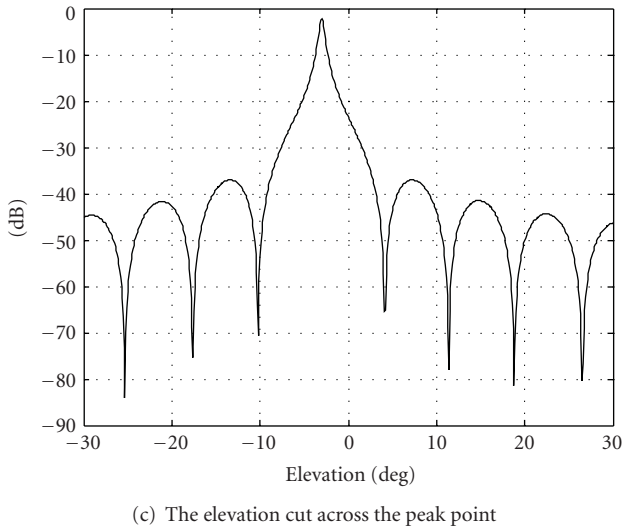
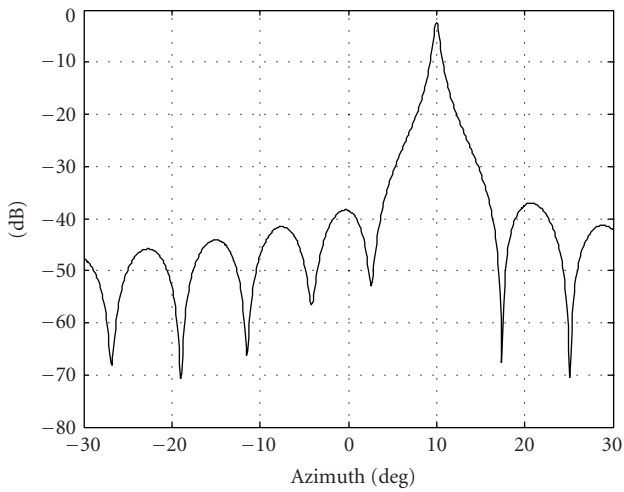
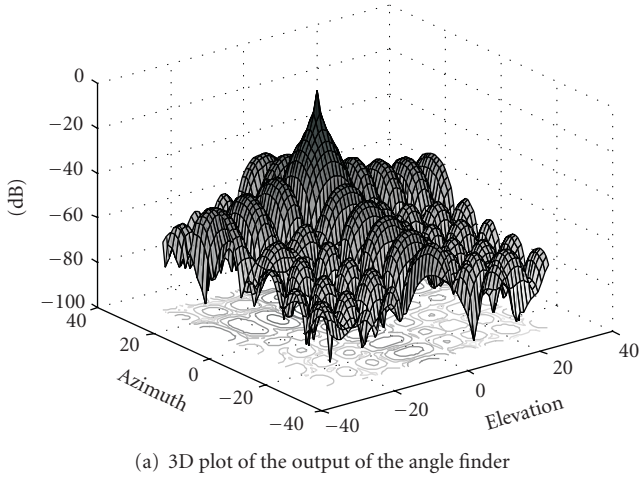


FIGURE 5: Output of the angle domain TSI finder where the desired lag is fixed at 30. Peak occurs at $\varphi = 10^\circ$, $\theta = -3^\circ$.

Suppose the angle of arrival of one of the TSI paths is used to estimate the distance to the interferer by setting

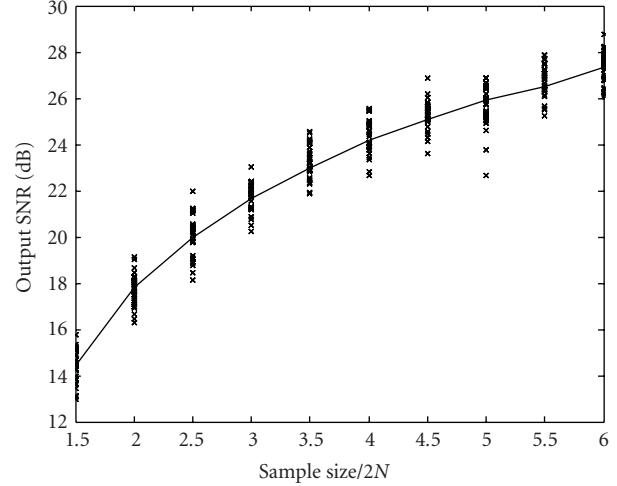


FIGURE 6: Estimated SNR for each iteration (data points) and mean estimated SNR (the line) at lag 30 in scenario 1, plotted against the number of samples used in the space time covariance matrix ($2N \times 2N$) expressed as a multiple of $2N$.

$\Delta = 0$. For example if $h = 4000$ meters, $m = 5600$, $\varphi_0 = \varphi_1 = 45^\circ$, $\delta R = 1$ meter, the estimate for D is approximately 4057 meters. However, if h is changed to $0.9h$ (which is equivalent to having $\Delta = 0.1h$) we have $D = 3122$ meters. This implies a sensitivity of around 20% due to the change of 10% in the input parameter h for the chosen set of input values. One can always estimate a reasonable confidence level of each estimation for the interfer distance when more than one multipath is being used. This of course depends on the ability to estimate a reasonable mean value of the parameter Δ . Other cases which we are not able to simulate or predict is the fact that the naturally scattering coefficients fluctuate. Another potential problem is the rays from the direct path and those from the TSI path may not intersect always. Hence, in practice we may be limited to making only a very crude assumption as the distance of the interference source.

In order to provide a wider investigation of the performance of the TSI finder as a function of the sample size used, we have included (Figure 6) a Monte Carlo simulation of scenario 1. Ninety simulations were run, and an estimate of the output SNR was generated at a lag of 30 (see Figure 4(a)) as a function of the number of samples used in estimating the covariance. In Figure 6, the number of samples used has been expressed as a multiple of the dimension of the size of the space time covariance matrix ($2N \times 2N$).

8. CONCLUDING REMARKS

Generally, the space fast time adaptive processor is employed to null main lobe interference and detect the target using a TSI finder as discussed in [6]. In addition to the usual lag domain TSI finder which we use for main lobe interferer nulling, we have introduced the angle domain TSI finder. As a result, this research extends its application to locating airborne interferers via the TSI arrivals, mainly using the reflection off the ground or ocean. Furthermore formulas were

established for processing gains of both TSI finders. These can be very helpful indicators in predetermining some of the radar parameters in order to achieve a desired performance level. The technique uses any one of the TSI rays to identify its angle of arrival in azimuth plane as well as in elevation plane and then locates the position of the transmitter. However, in general one maybe able to use more than one TSI path of the same interferer as illustrated in the simulated example. This will allow us to further refine the solution at least in theory. However, there are a number of hurdles to overcome in getting an estimate of the location of an interference source. The approach of ray tracing is known to encounter variety of problems particularly when the paths do not intersect. The aim of this research is to highlight the importance of having a procedure to get a crude estimate of the location of an interference source.

APPENDICES

A. MATRIX LEMMA

Lemma 2. Suppose the square matrix \mathbf{A} is added to an additional Dyad term $\mathbf{u}\mathbf{u}^H$, where \mathbf{u} is a column vector, then the inversion of the new matrix is given by (e.g., Van Trees [8, page 1348])

$$(\mathbf{A} + \mathbf{u}\mathbf{u}^H)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{u}\mathbf{u}^H\mathbf{A}^{-1}}{1 + \mathbf{u}^H\mathbf{A}^{-1}\mathbf{u}}. \quad (\text{A.1})$$

By definition, one has $\mathbf{R}_x = \rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H + \mathbf{R}_1$. Applying the above lemma, one has the following identity:

$$\mathbf{R}_x^{-1} = \mathbf{R}_1^{-1} - \frac{\rho_1^2 (\mathbf{R}_1^{-1} \mathbf{s}_1 \mathbf{s}_1^H \mathbf{R}_1^{-1})}{1 + \rho_1^2 (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}. \quad (\text{A.2})$$

This leads to the expression

$$\begin{aligned} \mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1 &= \mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1 - \frac{\rho_1^2 (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1 \mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}{1 + \rho_1^2 (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)} \\ &= \frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1}{1 + \rho_1^2 (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}, \end{aligned} \quad (\text{A.3})$$

$$(\mathbf{s}_1^H \mathbf{R}_x^{-1} \mathbf{s}_1)^{-1} = \rho_1^2 + (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1}. \quad (\text{A.4})$$

B. SPECTRUM DERIVATION

Using \mathbf{w}_2 from (26), we have

$$\begin{aligned} \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} &= \beta_{1,1} \left[-\beta_{1,1} \rho_1^2 \mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \beta_{1,1} \rho_1^2 \left(\frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \mathbf{R}_1^{-1} \mathbf{s}_1 \right]^H \mathbf{s}_{1,1} \\ &= -|\beta_{1,1}|^2 \rho_1^2 \mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1}. \end{aligned} \quad (\text{B.1})$$

Simplification of (B.1) using (34) leads to

$$\begin{aligned} 1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} &= 1 - \frac{|\beta_{1,1}|^2 \rho_1^2 N}{\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2} + \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \\ &= \frac{\sigma_n^2}{\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2} + \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1}, \end{aligned} \quad (\text{B.2})$$

where the second term on the right-hand side can be simplified using (35), (36) and finally assuming $N |\beta_{1,1}|^2 \gg 1$ (i.e., $1 + N |\beta_{1,1}|^2 \approx N |\beta_{1,1}|^2$) as follows:

$$\begin{aligned} \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} &= \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_{1,1}^H \mathbf{s}_1|^2}{(N/\sigma_n^2) (\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)^2} = \frac{|\beta_{1,1}|^2 |\mathbf{s}_{1,1}^H \mathbf{s}_1|^2 \mathbb{J}}{N (1 + N |\beta_{1,1}|^2 \mathbb{J})^2} \\ &\approx \frac{(|\mathbf{s}_{1,1}^H \mathbf{s}_1|^2 / N^2)}{(N |\beta_{1,1}|^2 \mathbb{J})} \approx 0 \quad \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \frac{|\beta_{1,1}|^2 \rho_1^2 |\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} &= \frac{|\beta_{1,1}|^2 |\mathbf{s}_{1,1}^H \mathbf{s}_1|^2 \mathbb{J}}{N (1 + N |\beta_{1,1}|^2 \mathbb{J})^2} \approx \frac{|\beta_{1,1}|^2 |\mathbf{s}_{1,1}^H \mathbf{s}_1|^2 \mathbb{J}}{N} \\ &= (N |\beta_{1,1}|^2 \mathbb{J}) \frac{|\mathbf{s}_{1,1}^H \mathbf{s}_1|^2}{N^2} \approx 0 \quad \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{aligned} \quad (\text{B.4})$$

As a result, we have

$$|1 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}|^2 \approx \begin{cases} \frac{1}{|1 + N |\beta_{1,1}|^2 \mathbb{J}|^2} \approx \frac{1}{(N |\beta_{1,1}|^2 \mathbb{J})^2} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ 1 - 2N |\beta_{1,1}|^2 \mathbb{J} & \text{for } N |\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{cases} \quad (\text{B.5})$$

The final term of the power output at the processor, that is, $\sigma_n^2 (\mathbf{w}_1^H \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{w}_2) = \sigma_2^2 \|\mathbf{w}\|^2$, can be approximated as follows.

Using (25) and (36), we have

$$\mathbf{w}_1^H \mathbf{w}_1 = \left(\frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right)^H \left(\frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \approx \frac{\sigma_n^4}{N^2} (\mathbf{R}_1^{-1} \mathbf{s}_1)^H (\mathbf{R}_1^{-1} \mathbf{s}_1). \quad (\text{B.6})$$

Substituting (32) and $\mathbf{s}_1^H \mathbf{s}_1 = N$ in the above expression and noting if $N|\beta_{1,1}|^2 \gg 1$ that $1 + N|\beta_{1,1}|^2 \approx N|\beta_{1,1}|^2$, we get

$$\begin{aligned} \mathbf{w}_1^H \mathbf{w}_1 &\approx \frac{\sigma_n^4}{N^2} \cdot \frac{1}{\sigma_n^4} \left(N - \frac{2\rho_1^2 |\beta_{1,1}|^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2} + \frac{\rho_1^4 |\beta_{1,1}|^4 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2 N}{(\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2)^2} \right) \\ &= \frac{1}{N} - \frac{(2|\beta_{1,1}|^2 \mathbb{J} |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2 / N^2)}{(1 + N|\beta_{1,1}|^2 \mathbb{J})} + \frac{(|\beta_{1,1}|^4 \mathbb{J}^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2 / N)}{(1 + N|\beta_{1,1}|^2 \mathbb{J})^2} \\ &\approx \frac{1}{N} - \frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^3} \approx \frac{1}{N} \quad \text{for } N|\beta_{1,1}|^2 \mathbb{J} \gg 1. \end{aligned} \quad (\text{B.7})$$

For $N|\beta_{1,1}|^2 \mathbb{J} \ll 1$, we have $1 + N|\beta_{1,1}|^2 \mathbb{J} \approx 1$ and

$$\begin{aligned} \mathbf{w}_1^H \mathbf{w}_1 &\approx \left(\frac{1}{N} - \frac{2|\beta_{1,1}|^2 \mathbb{J} |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^2} + \frac{|\beta_{1,1}|^4 \mathbb{J}^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N} \right) \\ &= \left(\frac{1}{N} - \frac{2N|\beta_{1,1}|^2 \mathbb{J} |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^3} + \frac{(N|\beta_{1,1}|^2 \mathbb{J})^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^3} \right) \approx \frac{1}{N}. \end{aligned} \quad (\text{B.8})$$

From (26), we have

$$\begin{aligned} \mathbf{w}_2^H \mathbf{w}_2 &= |\beta_{1,1}|^2 \rho_1^4 \left[-\mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \left(\frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \mathbf{R}_1^{-1} \mathbf{s}_1 \right]^H \\ &\quad \times \left[-\mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \left(\frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \mathbf{R}_1^{-1} \mathbf{s}_1 \right]. \end{aligned} \quad (\text{B.9})$$

The dominant term in the expression for $\mathbf{w}_2^H \mathbf{w}_2$ is given by the first term inside the bracket involving $\mathbf{R}_1^{-1} \mathbf{s}_{1,1}$, which can be simplified using (33) as

$$\begin{aligned} \mathbf{w}_2^H \mathbf{w}_2 &\approx |\beta_{1,1}|^2 \rho_1^4 (\mathbf{R}_1^{-1} \mathbf{s}_{1,1})^H (\mathbf{R}_1^{-1} \mathbf{s}_{1,1}) = \frac{|\beta_{1,1}|^2 \rho_1^4 N}{(\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2)^2} \\ &= \frac{|\beta_{1,1}|^2 N \mathbb{J}^2}{(1 + N|\beta_{1,1}|^2 \mathbb{J})^2} \approx \frac{1}{N|\beta_{1,1}|^2} \quad \text{for } N|\beta_{1,1}|^2 \mathbb{J} \gg 1. \end{aligned} \quad (\text{B.10})$$

The final expression is

$$\mathbf{w}_2^H \mathbf{w}_2 \approx \begin{cases} \frac{1}{N|\beta_{1,1}|^2} & \text{for } N|\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ N|\beta_{1,1}|^2 \mathbb{J}^2 & \text{for } N|\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{cases} \quad (\text{B.11})$$

We can show that the contributions arising from the three other terms in (B.9) are negligible as follows. The second term in the brackets of (B.9) contains the term

$(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1} / \mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1) \mathbf{R}_1^{-1} \mathbf{s}_1$, the square of which after substituting (35) and (36) takes the following form:

$$\begin{aligned} &|\beta_{1,1}|^2 \rho_1^4 \left| \frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right|^2 \|\mathbf{R}_1^{-1} \mathbf{s}_1\|^2 \\ &= \frac{|\beta_{1,1}|^2 \sigma_n^4 \rho_1^4 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2 / N^2}{(\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2)^2} \|\mathbf{R}_1^{-1} \mathbf{s}_1\|^2, \end{aligned} \quad (\text{B.12})$$

where from (32),

$$\begin{aligned} &\|\mathbf{R}_1^{-1} \mathbf{s}_1\|^2 \\ &= \frac{1}{\sigma_n^4} \left(\mathbf{s}_1 - \frac{\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2} \right)^H \left(\mathbf{s}_1 - \frac{\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{\sigma_n^2 + N|\beta_{1,1}|^2 \rho_1^2} \right) \\ &= \frac{1}{\sigma_n^4} \left(N - \frac{2|\beta_{1,1}|^2 \mathbb{J} |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{1 + N|\beta_{1,1}|^2 \mathbb{J}} + \frac{|\beta_{1,1}|^4 N \mathbb{J}^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{(1 + N|\beta_{1,1}|^2 \mathbb{J})^2} \right). \end{aligned} \quad (\text{B.13})$$

Simplifying the above expression and considering the extreme cases as before, we have

$$\begin{aligned} &\|\mathbf{R}_1^{-1} \mathbf{s}_1\|^2 \\ &\begin{cases} \frac{N}{\sigma_n^4} \left(1 - \frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^2} \right) \approx \frac{N}{\sigma_n^4} & \text{for } N|\beta_{1,1}|^2 \mathbb{J} \gg 1, \\ \frac{N}{\sigma_n^4} \left(1 - (2N|\beta_{1,1}|^2 \mathbb{J}) \frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^2} \right. \\ \quad \left. + (N|\beta_{1,1}|^2 \mathbb{J})^2 \frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^2} \right) \approx \frac{N}{\sigma_n^4} & \text{for } N|\beta_{1,1}|^2 \mathbb{J} \ll 1. \end{cases} \end{aligned} \quad (\text{B.14})$$

Back substitution of these expressions in (B.12) and the use of $1 + N|\beta_{1,1}|^2 \mathbb{J} \approx N|\beta_{1,1}|^2 \mathbb{J}$ lead to the expression

$$\begin{aligned} &|\beta_{1,1}|^2 \rho_1^4 \left| \frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right|^2 \|\mathbf{R}_1^{-1} \mathbf{s}_1\|^2 \\ &= \frac{|\beta_{1,1}|^2 \mathbb{J}^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N(1 + N|\beta_{1,1}|^2 \mathbb{J})^2} \\ &\approx \frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2 / N^2}{N|\beta_{1,1}|^2} \approx 0 \quad \text{for } N|\beta_{1,1}|^2 \mathbb{J} \gg 1, \end{aligned} \quad (\text{B.15})$$

and for $N|\beta_{1,1}|^2 \mathbb{J} \ll 1$ we have

$$\begin{aligned} &\frac{|\beta_{1,1}|^2 \mathbb{J} |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N(1 + N|\beta_{1,1}|^2 \mathbb{J})} \\ &\approx \frac{|\beta_{1,1}|^2 \mathbb{J}^2 |\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N} \approx (N|\beta_{1,1}|^2 \mathbb{J}^2) \left(\frac{|\mathbf{s}_1^H \mathbf{s}_{1,1}|^2}{N^2} \right) \approx 0. \end{aligned} \quad (\text{B.16})$$

The third contribution in (B.9) is given by (sum of two terms)

$$-2\Re\left\{\frac{\rho_1^4 |\beta_{1,1}|^2 (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1) (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{R}_1^{-1} \mathbf{s}_{1,1})}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}\right\} \quad (\text{B.17})$$

$$= \frac{2 |\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\}^2}{N(1 + N |\beta_{1,1}|^2)^3}.$$

(Note. Replacing $(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)$ by the approximation N/σ_n^2 and using of (32), (33), and (35) in (B.17), we arrive at the expression in the right-hand side of (B.17).)

After applying the approximation $1 + N|\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\} \approx N|\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\}$ or $1 + N|\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\} \approx 1$, we can conclude that the right-hand side of (B.17) is approximately equal to zero. From (B.7) and (B.11), the final expression for $\sigma_n^2 \|\mathbf{w}\|^2$ is obtained by combining (B.7) and (B.11):

$$\sigma_n^2 \|\mathbf{w}\|^2 = \begin{cases} \sigma_n^2 \left(\frac{1}{N} + \frac{1}{N |\beta_{1,1}|^2} \right) & \text{for } N |\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\} \gg 1, \\ \sigma_n^2 \left(\frac{1}{N} + N |\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\} \right) & \text{for } N |\beta_{1,1}|^2 \Re\{\mathbf{s}_1^H \mathbf{s}_1\} \ll 1. \end{cases} \quad (\text{B.18})$$

C. PROCESSING GAINS

Consider the case when $n \neq n_{1,1}$, but $\beta_{1,1} \approx 0$. In this case we have

$$\mathbf{X}_n(r) = \begin{pmatrix} \mathbf{x}(r) \\ \mathbf{x}(r+n) \end{pmatrix} = \begin{pmatrix} j_1(r) \mathbf{s}_1 + \epsilon_1 \\ j_1(r+n) \mathbf{s}_1 + \epsilon_2 \end{pmatrix}, \quad (\text{C.1})$$

$$\mathbf{R} = \begin{pmatrix} \mathbf{R}_x & \mathbf{Q} \\ \mathbf{Q} & \mathbf{R}_x \end{pmatrix},$$

where

$$\mathbf{Q}^H = E\{j_1(r) j_1(r+n)^*\} \mathbf{s}_1 \mathbf{s}_1^H + E\{j_1(r) \mathbf{s}_1 \epsilon_2^H\} + E\{j_1(r+n)^* \epsilon_1 \mathbf{s}_1^H\} + E\{\epsilon_1 \epsilon_2^H\}. \quad (\text{C.2})$$

Generally, this term is zero when a large sample support is available for estimating the covariance matrix. However, we would like to estimate the order of the next term as a function of M (number of samples) for large M . Suppose X and Y are two independent complex random variables with zero mean and Gaussian distribution, then $E\{XY^*\} = 0$, but the estimator would be $Z = (1/M) \sum_{i=1}^M x_i y_i^*$, where x_i and y_i are the measured sample values. The variance of the estimator is given by $\text{Var}\{Z\} = E\{|Z|^2\} = (1/M) \sigma_x^2 \sigma_y^2$, where σ_x^2 and σ_y^2 are the respective individual variances. As a result we may approximately take the order of the error term to be in the order $\sigma_x \sigma_y / \sqrt{M}$ (one standard deviation of the mean value), or this will be represented by $O(\sigma_x \sigma_y / \sqrt{M})$. Now we may consider the following approximate representations:

$$E\{j_1(r) j_1(r+n)^* \mathbf{s}_1 \mathbf{s}_1^H\} \approx O(\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H / \sqrt{M}),$$

$$E\{j_1(r) \mathbf{s}_1 \epsilon_2^H\} \approx O(\rho_1 \sigma_n \mathbf{s}_1 \mathbf{u}^H / \sqrt{M}), \quad (\text{C.3})$$

$$E\{j_1(r) \epsilon_1 \mathbf{s}_1^H\} \approx O(\rho_1 \sigma_n \mathbf{u} \mathbf{s}_1^H / \sqrt{M}),$$

$$E\{\epsilon_1 \epsilon_2^H\} \approx O(\sigma_n^2 \mathbf{u} \mathbf{u}^H / \sqrt{M}), \quad (\text{C.4})$$

where $\mathbf{u} = (1, 1, \dots, 1)^T$. The term for $E\{\epsilon_1 \epsilon_2^H\}$ will be ignored as a lower order term when $\rho_1^2 > \sigma_n^2$. Noting that $\mathbf{R}_x = \mathbf{R}_1 + \rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H$ and $\mathbf{R}_1 = \sigma_n^2 \mathbf{I}_N$ (for $\beta_{1,1} = 0$), we have

$$P_{\text{out}} = \mathbf{w}^H E\{\mathbf{X}_n(r) \mathbf{X}_n(r)^H\} \mathbf{w}$$

$$= \mathbf{w}_1^H \mathbf{R}_x \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_x \mathbf{w}_2$$

$$+ O(\mathbf{w}_1^H [\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H + \rho_1 \sigma_n \mathbf{s}_1 \mathbf{u}^H + \rho_1 \sigma_n \mathbf{u} \mathbf{s}_1^H] \mathbf{w}_2) / \sqrt{M}$$

$$+ O(\mathbf{w}_2^H [\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H + \rho_1 \sigma_n \mathbf{u} \mathbf{s}_1^H + \rho_1 \sigma_n \mathbf{s}_1 \mathbf{u}^H] \mathbf{w}_1) / \sqrt{M}. \quad (\text{C.5})$$

Now, considering the requirements in the minimization problem (i.e., $\mathbf{w}_1^H \mathbf{s}_1 = 1$ and $\mathbf{w}_2^H \mathbf{s}_1 = 0$), we have to minimize

$$P_{\text{out}} = \rho_1^2 + \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + O(\rho_1 \sigma_n [\mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u}] / \sqrt{M}). \quad (\text{C.6})$$

The solution for \mathbf{w}_1 (which minimizes P_{out} subject to $\mathbf{w}_1^H \mathbf{s}_1 = 1$) is given by

$$\mathbf{w}_1 = \frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)} = \frac{\mathbf{s}_1}{N} \quad \text{for } \mathbf{R}_1 = \sigma_n^2 \mathbf{I}_N \quad (\text{C.7})$$

and the solution for \mathbf{w}_2 is given by minimizing $\mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + O(\rho_1 \sigma_n [\mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u}] / \sqrt{M})$ subject to $\mathbf{w}_2^H \mathbf{s}_1 = 0$. This leads to

$$\mathbf{w}_2 = -O(\rho_1 \sigma_n \mathbf{R}_1^{-1} \mathbf{u} / \sqrt{M}) + \mu \mathbf{R}_1^{-1} \mathbf{s}_1, \quad (\text{C.8})$$

where μ is a constant.

Now substituting $\mathbf{w}_2^H \mathbf{s}_1 = 0$, we have

$$\mu^* = O\left(\frac{\rho_1 \sigma_n [\mathbf{u}^H \mathbf{R}_1^{-1} \mathbf{s}_1]}{([\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1] / \sqrt{M})}\right). \quad (\text{C.9})$$

As a result, we have

$$\mathbf{w}_2 = O\left(-\left(\frac{\rho_1 \sigma_n}{\sqrt{M}}\right) \mathbf{R}_1^{-1} \mathbf{u} + \left(\frac{\rho_1 \sigma_n}{\sqrt{M}}\right) \left(\frac{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{u})}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}\right) \mathbf{R}_1^{-1} \mathbf{s}_1\right) \quad (\text{C.10})$$

and for $\mathbf{R}_1 = \sigma_n^2 \mathbf{I}_N$ this reduces to

$$\mathbf{w}_2 = O\left(\frac{\rho_1}{\sigma_n} \frac{\mathbf{u}}{\sqrt{M}} + \frac{\rho_1}{\sigma_n} \frac{\mathbf{s}_1^H \mathbf{u}}{N} \frac{\mathbf{s}_1}{\sqrt{M}}\right). \quad (\text{C.11})$$

Since $(\mathbf{s}_1^H \mathbf{u} / N) < 1$, it is reasonable to ignore the low-order term in \mathbf{w}_2 to take the dominant term only and write (ignoring the -ve sign)

$$\mathbf{w}_2 \approx O\left(\frac{\rho_1}{\sigma_n} \frac{\mathbf{u}}{\sqrt{M}}\right) \quad (\text{C.12})$$

and (substituting $\mathbf{R}_1 = \sigma_n^2 \mathbf{I}_N$ as well as $\mathbf{u}^H \mathbf{u} = N$ in P_{out})

$$P_{\text{out}} \approx \rho_1^2 + \frac{\sigma_n^2}{N} + O\left(\frac{\rho_1^2 N}{M}\right). \quad (\text{C.13})$$

Therefore, when $\beta_{1,1} \approx 0$ (i.e., no significant multipath energy is available at the lag of interest) we may use (15) and (A.4) to approximate the following:

$$\begin{aligned} T(n)_{n \neq n_{1,1}} &\approx \frac{\rho_1^2 + (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1}}{\rho_1^2 + \sigma_n^2/N + O(\rho_1^2 N/M)} - 1 \\ &\approx \frac{\rho_1^2 + \sigma_n^2/N - (\rho_1^2 + \sigma_n^2/N + O(\sigma_1^2 N/M))}{\rho_1^2 + \sigma_n^2/N + O(\rho_1^2 N/M)} \\ &\approx \frac{O(\sigma_1^2 N/M)}{\rho_1^2} \approx O\left(\frac{N}{M}\right). \end{aligned} \quad (\text{C.14})$$

Now, we investigate the case $n \neq n_{1,1}$ with TSI energy present ($|\beta_{1,1}|^2 \gg 1/N$). Terms involved in \mathbf{Q}^H are given by

$$\begin{aligned} \mathbf{Q}^H &= \{(j_1(r) \mathbf{s}_1 + \beta_{1,1} j_1(r - n_{1,1}) \mathbf{s}_{1,1} + \varepsilon_1) \\ &\quad \times (j_1(r+n) \mathbf{s}_1^H + \beta_{1,1}^* j_1(r - n_{1,1} + n) \mathbf{s}_{1,1}^H + \varepsilon_1^H)\}. \end{aligned} \quad (\text{C.15})$$

This can be represented by

$$\begin{aligned} \mathbf{Q}^H &\approx O\left(\frac{\rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H}{\sqrt{M}}, \frac{|\beta_{1,1}|^2 \rho_1^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{\sqrt{M}}, \frac{\sigma_n^2 \mathbf{u} \mathbf{u}^H}{\sqrt{M}}, \right. \\ &\quad \frac{\beta_{1,1}^* \rho_1^2 \mathbf{s}_1 \mathbf{s}_{1,1}^H}{\sqrt{M}}, \frac{\rho_1 \sigma_n \mathbf{s}_1 \mathbf{u}^H}{\sqrt{M}}, \frac{\beta_{1,1} \rho_1^2 \mathbf{s}_{1,1} \mathbf{s}_1^H}{\sqrt{M}}, \\ &\quad \left. \frac{\beta_{1,1} \rho_1 \sigma_n \mathbf{s}_{1,1} \mathbf{u}^H}{\sqrt{M}}, \frac{\rho_1 \sigma_n \mathbf{u} \mathbf{s}_1^H}{\sqrt{M}}, \frac{\beta_{1,1}^* \rho_1 \sigma_n \mathbf{u} \mathbf{s}_{1,1}^H}{\sqrt{M}}\right). \end{aligned} \quad (\text{C.16})$$

In $P_{\text{out}} = \mathbf{w}^H \mathbf{R}_2 \mathbf{w}$, the contribution due to the presence of nonzero \mathbf{Q} is given by the term $\mathbf{w}_1^H \mathbf{Q}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{Q} \mathbf{w}_1$. This is equivalent to the terms (all positive contributions)

$$\begin{aligned} &\frac{\rho_1^2 (\mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{s}_1 \mathbf{s}_1^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{|\beta_{1,1}|^2 \rho_1^2 (\mathbf{w}_1^H \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\sigma_n^2 (\mathbf{w}_1^H \mathbf{u} \mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u} \mathbf{u}^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1^2 (\beta_{1,1}^* \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_{1,1}^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} \mathbf{s}_1^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1 \sigma_n (\mathbf{w}_1^H \mathbf{s}_1 \mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u} \mathbf{s}_1^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1^2 (\beta_{1,1} \mathbf{w}_1^H \mathbf{s}_{1,1} \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1}^* \mathbf{w}_2^H \mathbf{s}_1 \mathbf{s}_{1,1}^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1 \sigma_n (\beta_{1,1} \mathbf{w}_1^H \mathbf{s}_{1,1} \mathbf{u}^H \mathbf{w}_2 + \beta_{1,1}^* \mathbf{w}_2^H \mathbf{u} \mathbf{s}_{1,1}^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1 \sigma_n (\mathbf{w}_1^H \mathbf{u} \mathbf{s}_1^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{s}_1 \mathbf{u}^H \mathbf{w}_1)}{\sqrt{M}}, \\ &\frac{\rho_1 \sigma_n (\beta_{1,1}^* \mathbf{w}_1^H \mathbf{u} \mathbf{s}_{1,1}^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} \mathbf{u}^H \mathbf{w}_1)}{\sqrt{M}}. \end{aligned} \quad (\text{C.17})$$

As we minimize the power $\mathbf{w}^H \mathbf{R} \mathbf{w}$ subject to $\mathbf{w}_1^H \mathbf{s}_1 = 1$ and $\mathbf{w}_2^H \mathbf{s}_1 = 0$, the natural selection is that \mathbf{w}_1 be almost orthogonal to all the signals including \mathbf{u} (except of course $\mathbf{w}_1^H \mathbf{s}_1 = 1$). As a result, the order of \mathbf{w}_1 will not change and $\mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 = (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} \approx \sigma_n^2/N$ still holds. After assuming the orthogonality and substituting the above two constraints as well, we are left with the contributions $O(\rho_1 \sigma_n (\mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u})/\sqrt{M})$, $O(\rho_1^2 (\beta_{1,1}^* \mathbf{s}_{1,1}^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1})/\sqrt{M})$, and ρ_1^2/\sqrt{M} . Now ignoring the constant terms, our minimization problem for obtaining an approximate highest order for \mathbf{w}_2 is equivalent to minimizing $\mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + O(\rho_1 \sigma_n (\mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u})/\sqrt{M})$ subject to $\mathbf{w}_2^H \mathbf{s}_1 = 0$ or minimize $\mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + O(\rho_1^2 (\beta_{1,1}^* \mathbf{s}_{1,1}^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1})/\sqrt{M})$ subject to the same constraint. If the dominant term out of the last two terms is $O(\rho_1 \sigma_n (\mathbf{u}^H \mathbf{w}_2 + \mathbf{w}_2^H \mathbf{u})/\sqrt{M})$, then we have the same case as before but with $\mathbf{R}_1 = \rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + \sigma_n^2 \mathbf{I}$. However, in this case, using a similar argument and using (32)–(36) we can prove that $T(n)_{n \neq n_{1,1}} \approx O(N/M)$ as follows.

The solution for this case would be

$$\mathbf{w}_2 = O\left(-\frac{\rho_1 \sigma_n}{\sqrt{M}}\right) \mathbf{R}_1^{-1} \mathbf{u} + \frac{\rho_1 \sigma_n}{\sqrt{M}} \frac{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{u})}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)} \mathbf{R}_1^{-1} \mathbf{s}_1. \quad (\text{C.18})$$

The first part of the above expression is simplified as follows (expanding $\mathbf{R}_1^{-1} \mathbf{u}$):

$$\begin{aligned} &O\left(\frac{\rho_1 \sigma_n}{\sqrt{M}}\right) \mathbf{R}_1^{-1} \mathbf{u} \\ &\approx O\left(\frac{\rho_1 \sigma_n}{\sqrt{M}}\right) \cdot \frac{1}{\sigma_n^2} \left(\mathbf{I}_N - \frac{\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{\sigma_n^2 + N \rho_1^2 |\beta_{1,1}|^2}\right) \mathbf{u} \\ &\approx O\left(\frac{\rho_1}{\sqrt{M}}\right) \cdot \frac{1}{\sigma_n} \left(\mathbf{I}_N - \frac{\mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{N}\right) \mathbf{u} \quad \text{for } N |\beta_{1,1}|^2 \gg 1 \\ &\approx O\left(\frac{\rho_1 \mathbf{u}}{\sigma_n \sqrt{M}}\right) - O\left(\frac{\rho_1}{\sigma_n} \frac{\mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{u}}{\sqrt{M} N}\right). \end{aligned} \quad (\text{C.19})$$

The second part of the expression is expanded (using $\mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 = (\mathbf{s}_1 \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} \approx \sigma_n^2/N$) as

$$\begin{aligned} &\frac{\rho_1 \sigma_n}{\sqrt{M}} \frac{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{u})}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)} \mathbf{R}_1^{-1} \mathbf{s}_1 \\ &\approx \frac{\rho_1 \sigma_n}{\sqrt{M}} \left[\frac{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{u}) \sigma_n^2}{N} \right] \mathbf{R}_1^{-1} \mathbf{s}_1 \\ &\approx \frac{\rho_1 \sigma_n^3}{N \sqrt{M}} \left[\frac{\mathbf{s}_1^H}{\sigma_n^2} \left(\mathbf{I}_N - \frac{\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right) \mathbf{u} \right] \\ &\quad \times \left[\frac{1}{\sigma_n^2} \left(\mathbf{s}_1 - \frac{\rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{s}_1}{(\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right) \right] \end{aligned} \quad (\text{C.20})$$

using (32). Now applying $N|\beta_{1,1}|^2 \gg 1$ (i.e., $\sigma_n^2 + N|\beta_{1,1}|^2\rho_1^2 \approx N|\beta_{1,1}|^2\rho_1^2$), we have

$$\begin{aligned} & \frac{\rho_1\sigma_n}{\sqrt{M}} \frac{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{u})}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)} \mathbf{R}_1^{-1} \mathbf{s}_1 \\ & \approx \frac{\rho_1}{\sigma_n N \sqrt{M}} \left[\mathbf{s}_1^H \left(\mathbf{I}_N - \frac{\mathbf{s}_{1,1} \mathbf{s}_{1,1}^H}{N} \right) \mathbf{u} \left(\mathbf{s}_1 - \frac{\mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{s}_1}{N} \right) \right] \\ & \approx \frac{\rho_1}{\sigma_n N \sqrt{M}} \left[\left(\mathbf{s}_1^H \mathbf{u} - \frac{\mathbf{s}_{1,1}^H \mathbf{s}_{1,1} \mathbf{s}_1^H \mathbf{u}}{N} \right) \left(\mathbf{s}_1 - \frac{\mathbf{s}_{1,1} \mathbf{s}_{1,1}^H \mathbf{s}_1}{N} \right) \right] \\ & \approx O \left(\frac{\rho_1}{\sigma_n} \left[\frac{(\mathbf{s}_1^H \mathbf{u}) \mathbf{s}_1}{N \sqrt{M}} - \frac{(\mathbf{s}_1^H \mathbf{u})}{N} \frac{\mathbf{s}_{1,1}}{\sqrt{M}} \frac{(\mathbf{s}_{1,1}^H \mathbf{s}_1)}{N} \right. \right. \\ & \quad \left. \left. - \frac{(\mathbf{s}_1^H \mathbf{s}_{1,1})}{N} \frac{(\mathbf{s}_{1,1}^H \mathbf{u})}{N} \frac{\mathbf{s}_1}{\sqrt{M}} \right. \right. \\ & \quad \left. \left. + \frac{(\mathbf{s}_1^H \mathbf{s}_{1,1})}{N} \frac{(\mathbf{s}_{1,1}^H \mathbf{u})}{N} \frac{\mathbf{s}_{1,1}}{\sqrt{M}} \frac{(\mathbf{s}_{1,1}^H \mathbf{s}_1)}{N} \right] \right). \end{aligned} \quad (\text{C.21})$$

When the two expressions (C.19) and (C.21) are combined to estimate (C.18), we can conclude that the dominant order term is $\approx O(\rho_1 \mathbf{u} / \sigma_n \sqrt{M})$.

If instead the dominant contribution is the $O(\rho_1^2 (\beta_{1,1}^* \mathbf{w}_1^H \mathbf{s}_1 \mathbf{s}_{1,1}^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} \mathbf{s}_1^H \mathbf{w}_1) / \sqrt{M})$ term, then we have to minimize $P_{\text{out}} = \mathbf{w}_1^H \mathbf{R}_x \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_x \mathbf{w}_2 + \mathbf{w}_1^H \mathbf{Q}^H \mathbf{w}_2 + \mathbf{w}_2 \mathbf{Q} \mathbf{w}_1$, which after substituting the constraints and noting that $\mathbf{R}_1 = \rho_1^2 |\beta_{1,1}|^2 \mathbf{s}_{1,1} \mathbf{s}_{1,1}^H + \sigma_n^2 \mathbf{I}_N$ and $\mathbf{R}_x = \mathbf{R}_1 + \rho_1^2 \mathbf{s}_1 \mathbf{s}_1^H$ is reduced to minimize $P_{\text{out}} = \rho_1^2 + \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \mathbf{w}_2^H \mathbf{R}_1 \mathbf{w}_2 + O(\rho_1^2 (\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}) / \sqrt{M})$ subject to the constraints $\mathbf{w}_1^H \mathbf{s}_1 = 1$ and $\mathbf{w}_2^H \mathbf{s}_1 = 0$. This problem has been solved earlier without the factor $O(1/\sqrt{M})$, and as a result, we can write (using (26))

$$\begin{aligned} \mathbf{w}_2 &= O \left(-\frac{\beta_{1,1} \rho_1^2}{\sqrt{M}} \mathbf{R}_1^{-1} \mathbf{s}_{1,1} + O \left(\frac{\beta_{1,1} \rho_1^2}{\sqrt{M}} \right) \left(\frac{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right) \mathbf{R}_1^{-1} \mathbf{s}_1 \right), \\ \mathbf{w}_1 &= \frac{\mathbf{R}_1^{-1} \mathbf{s}_1}{(\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)}. \end{aligned} \quad (\text{C.22})$$

Now, applying earlier results ((B.7) and (B.11)), we can show that $\|\mathbf{w}_1\|^2 \approx 1/N$ and $\|\mathbf{w}_2\|^2 \approx O(1/(NM|\beta_{1,1}|^2))$ for $N|\beta_{1,1}|^2 \gg 1$, and furthermore,

$$\begin{aligned} P_{\text{out}} &\approx \rho_1^2 + \mathbf{w}_1^H \mathbf{R}_1 \mathbf{w}_1 + \sigma_n^2 \|\mathbf{w}_2\|^2 + \rho_1^2 |\beta_{1,1}|^2 |\mathbf{w}_2^H \mathbf{s}_{1,1}|^2 \\ &\quad + O \left(\rho_1^2 \frac{[\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}]}{\sqrt{M}} \right) \\ &\approx \sigma_n^2 \|\mathbf{w}\|^2 + \rho_1^2 + \rho_1^2 |\beta_{1,1}|^2 |\mathbf{w}_2 \mathbf{s}_{1,1}|^2 \\ &\quad + O \left(\rho_1^2 \frac{[\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}]}{\sqrt{M}} \right), \end{aligned} \quad (\text{C.23})$$

where $\mathbf{w}_1^H \mathbf{s}_{1,1} \approx 0$ has been assumed.

Now we have

$$\begin{aligned} & \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} \\ &= O \left(-\frac{|\beta_{1,1}|^2}{\sqrt{M}} \rho_1^2 \mathbf{s}_{1,1}^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1} + \frac{|\beta_{1,1}|^2}{\sqrt{M}} \rho_1^2 \frac{|\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_{1,1}|^2}{\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1} \right); \end{aligned} \quad (\text{C.24})$$

using the results from (B.3) and (34), we have

$$\begin{aligned} & \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1} \\ &\approx O \left(-\frac{|\beta_{1,1}|^2 \rho_1^2 N}{\sqrt{M} (\sigma_n^2 + N |\beta_{1,1}|^2 \rho_1^2)} \right) \\ &= O \left(-\frac{|\beta_{1,1}|^2 N \mathbb{J}}{\sqrt{M} (1 + N |\beta_{1,1}|^2 \mathbb{J})} \right) \approx O \left(-\frac{1}{\sqrt{M}} \right) \approx \beta_{1,1}^* \mathbf{s}_{1,1}^H \mathbf{w}_2 \end{aligned} \quad (\text{C.25})$$

for $N|\beta_{1,1}|^2 \gg 1$.

Now, substituting

$$\begin{aligned} & O \left(\rho_1^2 \frac{[\beta_{1,1}^* \mathbf{s}_1^H \mathbf{w}_2 + \beta_{1,1} \mathbf{w}_2^H \mathbf{s}_{1,1}]}{\sqrt{M}} \right) \approx O \left(\frac{\rho_1^2}{M} \right), \\ & \mathbf{w}_1^H \mathbf{R}_1^{-1} \mathbf{w}_1 = (\mathbf{s}_1^H \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} \approx \frac{\sigma_n^2}{N} \end{aligned} \quad (\text{C.26})$$

$$\|\mathbf{w}_2\|^2 \approx O \left(\frac{1}{NM |\beta_{1,1}|^2} \right)$$

and the above result into the above expression for P_{out} and simplifying we arrive at

$$\begin{aligned} & P_{\text{out}} \\ &\approx \rho_1^2 + \sigma_n^2 \left(\frac{1}{N} \right) + O \left(\frac{\sigma_n^2}{MN |\beta_{1,1}|^2} \right) + O \left(\frac{\rho_1^2}{M} \right) \quad \text{for } N|\beta_{1,1}|^2 \gg 1. \end{aligned} \quad (\text{C.27})$$

Since the term $O(1/MN|\beta_{1,1}|^2)$ is very small compared to the orders of all the other terms, we end up with

$$P_{\text{out}} \approx \rho_1^2 + \frac{\sigma_n^2}{N} + O \left(\frac{\rho_1^2}{M} \right) \quad \text{for } N|\beta_{1,1}|^2 \gg 1. \quad (\text{C.28})$$

Now substituting this result in (15), we have

$$T_s(n)_{n \neq n_{1,1}} \approx \frac{(\mathbf{s}_1 \mathbf{R}_1^{-1} \mathbf{s}_1)^{-1} + \rho_1^2}{\rho_1^2 + \sigma_n^2/N + O(\rho_1^2/M)} - 1. \quad (\text{C.29})$$

This is equivalent to

$$T_s(n)_{n \neq n_{1,1}} \approx O \left(\frac{1}{M} \right), \quad (\text{C.30})$$

which produces a small noise floor, and hence this option is discarded in estimating the signal processing gain.

The final expression for the signal processing gain of the TSI finder is obtained by the use of the earlier result as

$$\frac{T_s(n)_{n=n_{1,1}}}{T_s(n)_{n \neq n_{1,1}}} \approx O \left(\frac{N |\beta_{1,1}|^2 \mathbb{J}}{N/M \mathbb{J}} \right) \approx M |\beta_{1,1}|^2 \mathbb{J}^2. \quad (\text{C.31})$$

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