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Research Article

Code Design for Multihop Wireless Relay Networks

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We consider a wireless relay network, where a transmitter node communicates with a receiver node with the help of relay nodes. Most coding strategies considered so far assume that the relay nodes are used for one hop. We address the problem of code design when relay nodes may be used for more than one hop. We consider as a protocol a more elaborated version of amplify-and-forward, called *distributed space-time coding*, where the relay nodes multiply their received signal with a unitary matrix, in such a way that the receiver senses a space-time code. We first show that in this scenario, as expected, the so-called *full-diversity* condition holds, namely, the codebook of distributed space-time codewords has to be designed such that the difference of any two distinct codewords is full rank. We then compute the diversity of the channel, and show that it is given by the minimum number of relay nodes among the hops. We finally give a systematic way of building fully diverse codebooks and provide simulation results for their performance.

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1. INTRODUCTION

Cooperative diversity is a popular coding technique for wireless relay networks [1]. When a transmitter node wants to communicate with a receiver node, it uses its neighbor nodes as relays, in order to get the diversity known to be achieved by MIMO systems. Intuitively, one can think of the relay nodes playing the role of multiple antennas. What the relays perform on their received signal depends on the chosen protocol, generally categorized between *amplify-and-forward (AF)* and *decode-and-forward (DF)*. In order to evaluate their proposed cooperative schemes (for either strategy), several authors have adopted the diversity-multiplexing gain tradeoff proposed originally by Zheng and Tse for the MIMO channel, for single or multiple antenna nodes [2–5].

As specified by its name, AF protocols ask the relay nodes to just forward their received signal, possibly scaled by a power factor. Distributed space-time coding [6] can be seen as a sophisticated AF protocol, where the relays perform on their received vector signal a matrix multiplication instead of a scalar multiplication. The receiver thus senses a space-time code, which has been "encoded" by both the transmitter and the relay nodes with their matrix multiplication.

Extensive work has been done on distributed space-time coding since its introduction. Different code designs have been proposed, aiming at improving either the coding gain, the decoding, or the implementation of the scheme [7–10]. Scenarios where different antennas are available have been considered in [11, 12].

Recently, distributed space-time coding has been combined with differential modulation to allow communication over relay channels with no channel information [13–15]. Schemes are also available for multiple antennas [16].

Finally, distributed space-time codes have been considered for asynchronous communication [17].

In this paper, we are interested in considering distributed space-time coding in a multihop setting. The idea is to iterate the original two-step protocol: in a first step, the transmitter broadcasts the signal to the relay nodes. The relays receive the signal, multiply it by a unitary matrix, and send it to a new set of relays, which do the same, and forward the signal to the final receiver. Some multihop protocols have been recently proposed in [18, 19], for the amplify-and-forward protocol. Though we will give in detail most steps with a two-hop protocol for the sake of clarity, we will also emphasize how each step is generalized to more hops.

The paper is organized as follows. In Section 2, we present the channel model, for a two-hop channel. We then derive a Chernoff bound on the pairwise probability of error (Section 3), which allows us to derive the full-diversity condition as a code design criterion. We further compute the diversity of the channel, and show that if we have a two-hop network, with R_1 relay nodes at the first hop, and R_2 relay nodes at the second hop, then the diversity of the network is $\min(R_1, R_2)$. Section 4 is dedicated to the code construction itself, and some examples of proposed codes are simulated in Section 5.

2. A TWO-HOP RELAY NETWORK MODEL

Let us start by describing precisely the three-step transmission protocol, already sketched above, that allows communication for a two-hop wireless relay network. It is based on the two step protocol of [6].

We assume that the power available in the network is, respectively, P_1T, P_2T, and P_3T at the transmitter, at the first hop relays, and at the second hop relays for T-time transmission. We denote by $A_i \in \mathbb{C}^{T\times T}$, $i = 1,\dots,R_1$, the unitary matrices that the first hop relays will use to process their received signal, and by $B_j \in \mathbb{C}^{T\times T}$, $j = 1,\dots,R_2$, those at the second hop relays. Note that the matrices A_i, $i = 1,\dots,R_1$, B_j, $j = 1,\dots,R_2$, are computed beforehand, and given to the relays prior to the beginning of transmission. They are then used for all the transmission time.

Remark 1 (the unitary condition). Note that the assumption that the matrices have to be unitary has been introduced in [6] to ensure equal power among the relays, and to keep the forwarded noise white. It has been relaxed in [4].

The protocol is as follows.

(1) The transmitter sends its signal $\mathbf{s} \in \mathbb{C}^T$ such that

$$E[\mathbf{s}^*\mathbf{s}] = 1. \tag{1}$$

(2) The ith relay during the first hop receives

$$\mathbf{r}_i = \underbrace{\sqrt{P_1T}}_{c_1} f_i\mathbf{s} + \mathbf{v}_i \in \mathbb{C}^T, \quad i = 1,\dots,R_1, \tag{2}$$

where f_i denotes the fading from the transmitter to the ith relay, and $\mathbf{v}_i$ the noise at the ith relay.

(3) The jth relay during the second hop receives

$$\begin{aligned}\mathbf{x}_j &= c_2\sum_{i=1}^{R_1} g_{ij}A_i(c_1f_i\mathbf{s}+\mathbf{v}_i) + \mathbf{w}_j \in \mathbb{C}^T,\\ &= c_1c_2[A_1\mathbf{s},\dots,A_{R_1}\mathbf{s}]\begin{bmatrix} f_1g_{1j}\\ \vdots \\ f_{R_1}g_{R_1j}\end{bmatrix}\\ &\quad + c_2\sum_{i=1}^{R_1} g_{ij}A_i\mathbf{v}_i + \mathbf{w}_j, \quad j = 1,\dots,R_2,\end{aligned} \tag{3}$$

where g_{ij} denotes the fading from the ith relay in the first hop to the jth relay in the second hop. The normalization factor c_2 guarantees that the total energy used at the first hop relays is P_2T (see Lemma 1). The noise at the jth relay is denoted by $\mathbf{w}_j$.

(4) At the receiver, we have

$$\begin{aligned}\mathbf{y} &= c_3\sum_{j=1}^{R_2} h_jB_j\mathbf{x}_j + \mathbf{z} \in \mathbb{C}^T\\ &= c_3c_2c_1\sum_{j=1}^{R_2} h_jB_j[A_1\mathbf{s},\dots,A_{R_1}\mathbf{s}]\begin{bmatrix} f_1g_{1j}\\ \vdots \\ f_{R_1}g_{R_1j}\end{bmatrix}\\ &\quad + c_3\sum_{j=1}^{R_2} h_jB_j\left(c_2\sum_{i=1}^{R_1} g_{ij}A_i\mathbf{v}_i + \mathbf{w}_j\right) + \mathbf{z}\\ &= c_3c_2c_1\underbrace{[B_1A_1\mathbf{s},\dots,B_1A_{R_1}\mathbf{s},\dots,B_{R_2}A_1\mathbf{s},\dots,B_{R_2}A_{R_1}\mathbf{s}]}_{S\in\mathbb{C}^{T\times R_1R_2}}\\ &\quad \times \underbrace{\begin{bmatrix} f_1g_{11}h_1\\ \vdots\\ f_{R_1}g_{R_11}h_1\\ \vdots\\ f_1g_{1R_2}h_{R_2}\\ \vdots\\ f_{R_1}g_{R_1R_2}h_{R_2}\end{bmatrix}}_{H\in\mathbb{C}^{R_1R_2\times 1}}\\ &\quad \underbrace{+ c_3c_2\sum_{i=1}^{R_1}\sum_{j=1}^{R_2} h_jg_{ij}B_jA_i\mathbf{v}_i + c_3\sum_{j=1}^{R_2} h_jB_j\mathbf{w}_j + \mathbf{z},}_{W\in\mathbb{C}^{T\times 1}}\end{aligned} \tag{4}$$

where h_j denotes the fading from the jth relay to the receiver. The normalization factor c_3 (see Lemma 1) guarantees that the total energy used at the first hop relays is P_3T. The noise at the receiver is denoted by $\mathbf{z}$.

In the above protocol, all fadings and noises are assumed to be complex Gaussian random variables, with zero mean and unit variance.

Though relays and transmitters have no knowledge of the channel, we do assume that the channel is known at the receiver. This makes sense when the channel stays roughly the same long enough so that communication starts with a training sequence, which consists of a known code. Thus, instead of decoding the data, the receiver gets knowledge of the channel H, since it does not need to know every fading independently.

Lemma 1. *The normalization factors c_2 and c_3 are, respectively, given by*

$$\begin{aligned} c_2 &= \sqrt{\frac{P_2}{P_1+1}},\\ c_3 &= \sqrt{\frac{P_3}{P_2R_1+1}}.\end{aligned} \tag{5}$$

Proof. (1) Since $E[\mathbf{r}_i^*\mathbf{r}_i] = (P_1+1)T$, we have that

$$E[c_2^2(A_i\mathbf{r}_i)*A_i\mathbf{r}_i] = P_2T \Longleftrightarrow c_2^2(P_1+1)T = P_2T \Longleftrightarrow c_2 = \sqrt{\frac{P_2}{P_1+1}}. \tag{6}$$

(2) We proceed similarly to compute the power at the second hop. We have

$$\begin{aligned} E[\mathbf{x}_j^*\mathbf{x}_j] &= E\left[c_2^2\left(\sum_{i=1}^{R_1} g_{ij}A_i\mathbf{r}_i\right)^*\left(\sum_{k=1}^{R_1} g_{kj}A_k\mathbf{r}_k\right)\right] + E[\mathbf{w}_j^*\mathbf{w}_j] \\ &= c_2^2\sum_{i=1}^{R_1} E[\mathbf{r}_i^*\mathbf{r}_i] + T = (P_2R_1+1)T, \end{aligned} \tag{7}$$

so that

$$E[c_3^2(B_j\mathbf{x}_j)^*B_j\mathbf{x}_j] = P_3T \Longleftrightarrow c_3^2(P_2R_1+1)T = P_3T \Longleftrightarrow c_3 = \sqrt{\frac{P_3}{P_2R_1+1}}. \tag{8}$$

□

Note that from (4), the channel can be summarized as

$$\mathbf{y} = c_1c_2c_3SH + W, \tag{9}$$

which has the form of a MIMO channel. This explains the terminology *distributed space-time coding*, since the codeword S has been encoded in a distributed manner among the transmitter and the relays.

Remark 2 (generalization to more hops). Note furthermore the shape of the channel matrix H. Each row describes a path from the transmitter to the receiver. More precisely, each row is of the form $f_ig_{ij}h_j$, which gives the path from the transmitter to the ith relay in the first hop, then from the ith relay to the jth relay in the second hop, and finally from the jth relay to the receiver. Thus, though we have given the model for a two-hop network, the generalization to more hops is straightforward.

3. PAIRWISE ERROR PROBABILITY

In this section, we compute a Chernoff bound on the pairwise probability of error of transmitting a signal $\mathbf{s}$, and decoding a wrong signal. The goal is to derive the so-called diversity property as code-design criterion (Section 3.1). We then further elaborate the upper bound given by the Chernoff bound, and prove that the diversity of a two-hop relay network is actually $\min(R_1, R_2)$, where R_1 and R_2 are the number of relay nodes at the first and second hops, respectively, (Section 3.2).

In the following, the matrix $\mathbf{I}$ denotes the identity matrix.

3.1. Chernoff bound on the pairwise error probability

In order to determine the maximum likelihood decoder, we first need to compute

$$P(\mathbf{y} \mid \mathbf{s}, f_i, g_{ij}, h_j). \tag{10}$$

If g_{ij} and h_j are known, then W is Gaussian with zero mean. Thus knowing f_i, g_{ij}, h_j, H and $\mathbf{s}$, we know that $\mathbf{y}$ is Gaussian.

(1) The expectation of $\mathbf{y}$ given $\mathbf{s}$ and H is

$$E[\mathbf{y}] = c_1c_2c_3SH. \tag{11}$$

(2) Thevariance of $\mathbf{y}$ given g_{ij} and h_j is

$$\begin{aligned} &E[(\mathbf{y}-E[\mathbf{y}])(\mathbf{y}-E[\mathbf{y}])^*] \\ &\quad= E[WW^*] \\ &\quad= c_3^2c_2^2E\left[\sum_{i=1}^{R_1}\sum_{j=1}^{R_2} h_jg_{ij}B_jA_i\mathbf{v}_i\sum_{k=1}^{R_1}\sum_{l=1}^{R_2}(h_lg_{kl}B_lA_k\mathbf{v}_k)^*\right] \\ &\qquad+ c_3^2E\left[\sum_{j=1}^{R_2} h_jB_j\mathbf{w}_j\sum_{l=1}^{R_2}(h_lB_l\mathbf{w}_l)^*\right] + E[\mathbf{z}\mathbf{z}^*] \\ &\quad= c_3^2c_2^2\sum_{i=1}^{R_1}\left(\sum_{j=1}^{R_2} g_{ij}h_jB_j\right)\left(\sum_{l=1}^{R_2} g_{il}^*h_l^*B_l^*\right) \\ &\qquad+ c_3^2\sum_{j=1}^{R_2}|h_j|^2\mathbf{I}_T + \mathbf{I}_T =: R_{\mathbf{y}}, \end{aligned} \tag{12}$$

where

$$c_2^2c_3^2 = \frac{P_2P_3}{(P_1+1)(P_2R_1+1)}. \tag{13}$$

Summarizing the above computation, we obtain the obvious following proposition.

Proposition 1.

$$\begin{aligned} &P(\mathbf{y} \mid \mathbf{s}, f_i, g_{ij}, h_j) \\ &= \frac{1}{\pi^T\det(R_{\mathbf{y}})}\exp\left(-(\mathbf{y}-c_1c_2c_3SH)^*\times R_{\mathbf{y}}^{-1}(\mathbf{y}-c_1c_2c_3SH)\right). \end{aligned} \tag{14}$$

Thus the maximum likelihood (ML) decoder of the system is given by

$$\arg\max_{\mathbf{s}} P(\mathbf{y} \mid \mathbf{s}, f_i, g_{ij}, h_j) = \arg\min_{\mathbf{s}} \|\mathbf{y} - c_1c_2c_3SH\|^2. \tag{15}$$

From the ML decoding rule, we can compute the pairwise error probability (PEP).

Lemma 2 (Chernoff bound on the PEP). *The PEP of sending a signal $\mathbf{s}_k$ and decoding another signal $\mathbf{s}_l$ has the following Chernoff bound:*

$$\begin{aligned} &P(\mathbf{s}_k \longrightarrow \mathbf{s}_l) \\ &\le E_{f_i,g_{ij},h_j}\exp\left(-\frac{1}{4}c_1^2c_2^2c_3^2H^*\times(S_k-S_l)^*R_{\mathbf{y}}^{-1}(S_k-S_l)H\right). \end{aligned} \tag{16}$$

Proof. By definition,

$$\begin{aligned}
&P(\mathbf{s}_k \longrightarrow \mathbf{s}_l \mid f_i, g_{ij}, h_j) \\
&\quad = P(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j) > P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j)) \\
&\quad = P(\ln(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j)) \\
&\qquad - \ln(P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j)) > 0) \\
&\quad \leq E_W[\exp\lambda(\ln(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j)) \\
&\qquad - \ln(P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j)))],
\end{aligned} \tag{17}$$

where the last inequality is obtained by applying the Chernoff bound, and $\lambda > 0$. Using **Proposition 1**, we have

$$\begin{aligned}
&\lambda(\ln(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j)) - \ln(P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j))) \\
&\quad = -\lambda[c_1^2 c_2^2 c_3^2 H^* (S_K^* - S_l^*) R_{\mathbf{y}}^{-1} (S_k - S_l) H + c_1 c_2 c_3 H^* \\
&\qquad \times (S_K^* - S_l^*) R_y^{-1} W + c_1 c_2 c_3 W^* R_{\mathbf{y}}^{-1} (S_k - S_l) H] \\
&\quad = -(\lambda c_1 c_2 c_3 (S_k - S_l) H + W)^* \\
&\qquad \times R_{\mathbf{y}}^{-1} (\lambda c_1 c_2 c_3 (S_k - S_l) H + W) \\
&\qquad + (\lambda^2 - \lambda) c_1^2 c_2^2 c_3^2 H^* (S_k - S_l)^* R_{\mathbf{y}}^{-1} (S_k - S_l) H \\
&\qquad + W^* R_{\mathbf{y}}^{-1} W,
\end{aligned} \tag{18}$$

and thus

$$\begin{aligned}
&E_W[\exp\lambda(\ln(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j)) - \ln(P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j)))] \\
&\quad = \int \frac{\exp(-W^* R_W^{-1} W)}{\pi^T \det(R_W^{-1})} \exp\lambda(\ln(P(\mathbf{y} \mid \mathbf{s}_l, f_i, g_{ij}, h_j)) \\
&\qquad - \ln(P(\mathbf{y} \mid \mathbf{s}_k, f_i, g_{ij}, h_j))) dW \\
&\quad = \exp((\lambda^2 - \lambda) c_1^2 c_2^2 c_3^2 H^* (S_k - S_l)^* R_{\mathbf{y}}^{-1} (S_k - S_l) H)
\end{aligned} \tag{19}$$

since $R_{\mathbf{w}} = R_{\mathbf{y}}$ and

$$\begin{aligned}
&\frac{1}{\pi^T \det(R_W^{-1})} \int \exp(-(\lambda c_1 c_2 c_3 (S_k - S_l) H + W)^* \\
&\qquad \times R_{\mathbf{y}}^{-1} (\lambda c_1 c_2 c_3 (S_k - S_l) H + W)) \\
&\qquad \times dW = 1.
\end{aligned} \tag{20}$$

To conclude, we choose $\lambda = 1/2$, which maximizes $\lambda^2 - \lambda$, and thus minimizes $-(\lambda - \lambda^2)$. □

We now compute the expectation over f_i. Note that one has to be careful since the coefficients f_i are repeated in the matrix H, due to the second hop.

Lemma 3 (bound by integrating over $\mathbf{f}$). *The following upper bound holds on the PEP:*

$$\begin{aligned}
&P(\mathbf{s}_k \longrightarrow \mathbf{s}_l) \\
&\leq E_{g_{ij}, h_j} \det\left(\mathbf{I}_{R_1} + \frac{1}{4} c_1^2 c_2^2 c_3^2 \mathcal{H}^* (S_k - S_l)^* R_{\mathbf{y}}^{-1} (S_k - S_l) \mathcal{H}\right)^{-1}
\end{aligned} \tag{21}$$

where $\mathcal{H}$ is given in (22).

Proof. We first rewrite the channel matrix H as $H = \mathcal{H}\mathbf{f}$, with

$$\mathbf{f} = \begin{bmatrix} f_1 \\ \vdots \\ f_{R_1} \end{bmatrix} \in \mathbb{C}^{R_1},$$
$$\mathcal{H} = \begin{bmatrix} g_{11} h_1 & & \\ & \ddots & \\ & & g_{R_1 1} h_1 \\ & \vdots & \\ g_{1 R_2} h_{R_2} & & \\ & \ddots & \\ & & g_{R_1 R_2} h_{R_2} \end{bmatrix} \in \mathbb{C}^{R_1 R_2 \times R_1}. \tag{22}$$

Thus we have, since $\mathbf{f}$ is Gaussian with 0 mean and variance $\mathbf{I}_{R_1}$,

$$\begin{aligned}
&E_{f_i} \exp\left(-\frac{1}{4} c_1^2 c_2^2 c_3^2 H^* (S_k - S_l)^* R_{\mathbf{y}}^{-1} (S_k - S_l) H\right) \\
&\quad = \int \frac{\exp(-\mathbf{f}^* \mathbf{f})}{\pi^{R_1}} \exp\Big(-\frac{1}{4} c_1^2 c_2^2 c_3^2 \mathbf{f}^* \mathcal{H}^* (S_k - S_l)^* \\
&\qquad \times R_{\mathbf{y}}^{-1} (S_k - S_l) \mathcal{H} \mathbf{f}\Big) d\mathbf{f} \\
&\quad = \frac{1}{\pi^{R_1}} \int \exp\Big(-\mathbf{f}^* \Big[\mathbf{I}_{R_1} + \frac{1}{4} c_1^2 c_2^2 c_3^2 \mathcal{H}^* (S_k - S_l)^* \\
&\qquad \times R_{\mathbf{y}}^{-1} (S_k - S_l) \mathcal{H}\Big] \mathbf{f}\Big) d\mathbf{f} \\
&\quad = \det\left(\mathbf{I}_{R_1} + \frac{1}{4} c_1^2 c_2^2 c_3^2 \mathcal{H}^* (S_k - S_l)^* \times R_{\mathbf{y}}^{-1} (S_k - S_l) \mathcal{H}\right)^{-1}.
\end{aligned} \tag{23}$$

□

Similarly to the standard MIMO case, and to the previous work on distributed space-time coding [6], the *full-diversity* condition can be deduced from (21). In order to see it, we first need to determine the dominant term as a function of P, the power used for the whole network.

Remark 3 (power allocation). In this paper, we assume that the power P is shared equally among the transmitter and the three hops, namely,

$$P_1 = \frac{P}{3}, \qquad P_2 = \frac{P}{3R_1}, \qquad P_3 = \frac{P}{3R_2}. \tag{24}$$

It is not clear that this strategy is the best, however, it is a priori the most natural one to try. Under this assumption, we have that

$$\begin{aligned}
c_3^2 &= \frac{P}{R_2(P+3)}, \\
c_2^2 c_3^2 &= \frac{P^2}{R_1 R_2 (P+3)^2}, \\
c_1^2 c_2^2 c_3^2 &= \frac{P^3 T}{3 R_1 R_2 (P+3)^2}.
\end{aligned} \tag{25}$$

Thus, when P grows, $c_1^2 c_2^2 c_3^2$ grows like P.

Remark 4 (full diversity). It is now easy to see from (21) that if $S_l - S_k$ drops rank, then the exponent of P increases, so that the diversity decreases. In order to minimize the Chernoff bound, one should then design distributed space-time codes such that $\det(S_k - S_l)^*(S_k - S_l) \neq 0$ (property well known as full diversity). Note that the term $R_{\mathbf{y}}^{-1}$ between $S_k - S_l$ and its conjugate does not interfere with this reasoning, since $R_{\mathbf{y}}$ can be upper bounded by $\operatorname{tr}(R_{\mathbf{y}})\mathbf{I}$ (see also Proposition 2 for more details). Finally, the whole computation that yields to the full-diversity criterion does not depend on H being the channel matrix of a two-hop protocol, since the decomposition of H used in the proof of Lemma 3 could be done similarly if there were three hops or more.

3.2. Diversity analysis

The goal is now to show that the upper bound given in (21) behaves like $P^{\min(R_1,R_2)}$ when we let P grows. To do so, let us start by further bounding the pairwise error probability.

Proposition 2. *Assuming that the code is fully diverse, it holds that the PEP can be upper bounded as follows:*

$$\begin{aligned}
&P(\mathbf{s}_k \longrightarrow \mathbf{s}_l)\\
&\leq E_{g_{ij},h_j}\prod_{i=1}^{R_1}\\
&\quad\times\Bigg(1+\frac{\lambda_{\min}^2 c_1^2c_2^2c_3^2}{4T}\\
&\qquad\times\frac{\sum_{j=1}^{R_2}|h_j|^2|g_{ij}|^2}{c_3^2c_2^2\sum_{k=1}^{R_1}\left|\sum_{j=1}^{R_2}h_jg_{kj}\right|^2+c_3^2\sum_{j=1}^{R_2}|h_j|^2+1}\Bigg)^{-1}\\
&\leq E_{g_{ij},h_j}\prod_{i=1}^{R_1}\\
&\quad\times\Bigg(1+\frac{\lambda_{\min}^2 c_1^2c_2^2c_3^2}{4T}\\
&\qquad\times\frac{\sum_{j=1}^{R_2}|h_jg_{ij}|^2}{c_3^2c_2^2(2R_2-1)\sum_{k=1}^{R_1}\sum_{j=1}^{R_2}|h_jg_{kj}|^2+c_3^2\sum_{j=1}^{R_2}|h_j|^2+1}\Bigg)^{-1}.
\end{aligned}\tag{26}$$

Proof. (1) Note first that

$$\begin{aligned}
R_{\mathbf{y}} &\leq \operatorname{tr}(R_{\mathbf{y}})\mathbf{I}_T\\
&= \Bigg(\underbrace{c_3^2c_2^2\sum_{i=1}^{R_1}\operatorname{tr}\Bigg(\sum_{j=1}^{R_2}g_{ij}h_jB_j\sum_{l=1}^{R_2}g_{il}^*h_l^*B_l^*\Bigg)}_{\alpha}\\
&\quad + T\Bigg(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\Bigg)\Bigg)\mathbf{I}_T,
\end{aligned}\tag{27}$$

so that

$$\begin{aligned}
&P(\mathbf{s}_k \longrightarrow \mathbf{s}_l)\\
&\leq E_{g_{ij},h_j}\det\Bigg(\mathbf{I}_{R_1}+\frac{c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2\alpha+T\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)}\\
&\qquad\qquad\times\mathcal{H}^*(S_k-S_l)^*(S_k-S_l)\mathcal{H}\Bigg)^{-1}\\
&\leq E_{g_{ij},h_j}\det\Bigg(\mathbf{I}_{R_1}+\frac{\lambda_{\min}^2c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2\alpha+T\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)}\mathcal{H}^*\mathcal{H}\Bigg)^{-1},
\end{aligned}\tag{28}$$

where $\lambda_{\min}^2$ denote the smallest eigenvalue of $(S_k - S_l)^*(S_k - S_l)$, which is strictly positive under the assumption that the codebook is fully diverse.

Furthermore, we have that

$$\begin{aligned}
\mathcal{H}^*\mathcal{H} &= \sum_{j=1}^{R_2}\begin{pmatrix}|h_j|^2|g_{1j}|^2 & & \\ & \ddots & \\ & & |h_j|^2|g_{R_1j}|^2\end{pmatrix}\\
&= \begin{pmatrix}\sum_{j=1}^{R_2}|h_j|^2|g_{1j}|^2 & & \\ & \ddots & \\ & & \sum_{j=1}^{R_2}|h_j|^2|g_{R_1j}|^2\end{pmatrix},
\end{aligned}\tag{29}$$

which yields

$$\begin{aligned}
&\det\Bigg(\mathbf{I}_{R_1}+\frac{\lambda_{\min}^2c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2\alpha+T\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)}\mathcal{H}^*\mathcal{H}\Bigg)^{-1}\\
&=\prod_{i=1}^{R_1}\Bigg(1+\frac{\lambda_{\min}^2c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2\alpha+T\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)}\sum_{j=1}^{R_2}|h_j|^2|g_{ij}|^2\Bigg)^{-1},
\end{aligned}\tag{30}$$

where

$$\begin{aligned}
\alpha &\leq |\alpha|\\
&= \left|\sum_{k=1}^{R_1}\operatorname{tr}\Bigg(\sum_{j=1}^{R_2}g_{kj}h_jB_j\sum_{l=1}^{R_2}g_{kl}^*h_l^*B_l^*\Bigg)\right|\\
&\leq \sum_{k=1}^{R_1}\left|\operatorname{tr}\Bigg(\sum_{j=1}^{R_2}g_{kj}h_jB_j\sum_{l=1}^{R_2}g_{kl}^*h_l^*B_l^*\Bigg)\right|\\
&\leq \sum_{k=1}^{R_1}\sqrt{\operatorname{tr}\Bigg(\sum_{j,j'=1}^{R_2}g_{kj}g_{kj'}^*h_jh_{j'}^*B_jB_{j'}^*\Bigg)\operatorname{tr}\Bigg(\sum_{l,l'=1}^{R_2}g_{kl}g_{kl'}^*h_lh_{l'}^*B_lB_{l'}^*\Bigg)},
\end{aligned}\tag{31}$$

where the last inequality uses Cauchy-Schwartz inequality. Now recall that B_j, $j = 1,\ldots,R_2$, are unitary, thus $B_jB_{j'}^*$ and $B_lB_{l'}^*$ are unitary matrices, and

$$\operatorname{tr}(B_kB_{k'}^*) \leq T \quad \forall k, k'. \tag{32}$$

Thus

$$\alpha \le T\sum_{k=1}^{R_1}\sqrt{\left(\sum_{j,j'=1}^{R_2} g_{kj}g_{kj'}^* h_j h_{j'}^*\right)\left(\sum_{l,l'=1}^{R_2} g_{kl}g_{kl'}^* h_l h_{l'}^*\right)}$$
$$= T\sum_{k=1}^{R_1}\sqrt{\left|\sum_{j=1}^{R_2} h_j g_{kj}\right|^2\left|\sum_{l=1}^{R_2} h_l g_{kl}\right|^2} \tag{33}$$
$$= T\sum_{k=1}^{R_1}\left|\sum_{j=1}^{R_2} h_j g_{kj}\right|^2.$$

We can now rewrite

$$P(\mathbf{s}_k \longrightarrow \mathbf{s}_l)$$
$$\le E_{g_{ij},h_j}\prod_{i=1}^{R_1}\left(1+\frac{\lambda_{\min}^2 c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2\alpha+T\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)}\times\sum_{j=1}^{R_2}|h_j|^2|g_{ij}|^2\right)^{-1}$$
$$\le E_{g_{ij},h_j}\prod_{i=1}^{R_1}\times\left(1+\frac{\lambda_{\min}^2 c_1^2c_2^2c_3^2}{4\left(c_3^2c_2^2T\sum_{k=1}^{R_1}\left|\sum_{j=1}^{R_2}h_jgk_j\right|^2+T\left(c_3^2\left(c_3^2\sum_{j=1}^{R_2}|h_j|^2+1\right)\right)\right.}\times\sum_{j=1}^{R_2}|h_j|^2|g_{ij}|^2\right)^{-1}, \tag{34}$$

which proves the first bound.

(2) To get the second bound, we need to prove that

$$\left|\sum_{j=1}^{R_2} h_j g_{kj}\right|^2 \le (2R_2-1)\sum_{j=1}^{R_2}|h_jg_{kj}|^2. \tag{35}$$

By the triangle inequality, we have that

$$\left|\sum_{j=1}^{R_2} h_j g_{kj}\right|^2 \le \left(\sum_{j=1}^{R_2}|h_jg_{kj}|\right)^2$$
$$= \sum_{j=1}^{R_2}|h_jg_{kj}|^2+\sum_{j=1}^{R_2}|h_jg_{kj}|\sum_{l=1,l\neq j}^{R_2}|h_lg_{kl}|. \tag{36}$$

Using the inequality of arithmetic and geometric means, we get

$$|h_jg_{kj}|\,|h_lg_{kl}| = \sqrt{|h_jg_{kj}|^2|h_lg_{kl}|^2} \le |h_jg_{kj}|^2+|h_lg_{kl}|^2, \tag{37}$$

so that

$$\left|\sum_{j=1}^{R_2} h_j g_{kj}\right|^2 \le \sum_{j=1}^{R_2}|h_jg_{kj}|^2+\sum_{j=1}^{R_2}\sum_{l=1,l\neq j}^{R_2}\left(|h_jg_{kj}|^2+|h_lg_{kl}|^2\right)$$
$$= R_2\sum_{j=1}^{R_2}|h_jg_{kj}|^2+\sum_{j=1}^{R_2}\sum_{l=1,l\neq j}^{R_2}|h_lg_{kl}|^2$$
$$= (2R_2-1)\sum_{j=1}^{R_2}|h_jg_{kj}|^2, \tag{38}$$

which concludes the proof. □

We now set $x_i := \sum_{j=1}^{R_2}|h_jg_{ij}|^2$, so that the bound

$$E_{g_{ij},h_j}\prod_{i=1}^{R_1}\times\left(1+\underbrace{\frac{\lambda_{\min}^2c_1^2c_2^2c_3^2}{4T}}_{\gamma_1}\times\frac{\sum_{j=1}^{R_2}|h_jg_{ij}|^2}{\underbrace{c_2^2c_3^2(2R_2-1)}_{\gamma_2}\sum_{k=1}^{R_1}\sum_{j=1}^{R_2}|h_jg_{kj}|^2+c_3^2\sum_{j=1}^{R_2}|h_j|^2+1}\right)^{-1} \tag{39}$$

can be rewritten as

$$E_{g_{ij},h_j}\prod_{i=1}^{R_1}\left(1+\gamma_1\frac{x_i}{\gamma_2\sum_{k=1}^{R_1}x_k+c_3^2\sum_{j=1}^{R_2}|h_j|^2+1}\right)^{-1}. \tag{40}$$

Note here that by choice of power allocation (see Remark 3),

$$\gamma_1 = \frac{\lambda_{\min}^2P^3T}{4T3R_1R_2(P+3)^2} = \frac{\lambda_{\min}^2P^3}{12R_1R_2(P+3)^2},$$
$$\gamma_2 = \frac{(2R_2-1)P^2}{R_1R_2(P+3)^2}, \tag{41}$$
$$c_3^2 = \frac{P}{R_2(P+3)}.$$

In order to compute the diversity of the channel, we will consider the asymptotic regime in which $P\to\infty$. We will thus use the notation

$$x \doteq y \Longleftrightarrow \lim_{P\to\infty}\frac{x}{\log P} = \lim_{P\to\infty}\frac{y}{\log P}. \tag{42}$$

With this notation, we have that

$$\gamma_1 \doteq P, \qquad \gamma_2 \doteq P^0 = 1, \qquad c_3^2 \doteq P^0 = 1. \tag{43}$$

In other words, the coefficients γ_2 and c_3 are constants and can be neglected, while γ_1 grows with P.

Theorem 1. *It holds that*

$$E_{g_{ij},h_j}\prod_{i=1}^{R_1}\left(1+P\frac{x_i}{\sum_{k=1}^{R_2}x_k+\sum_{j=1}^{R_2}|h_j|^2+1}\right)^{-1} \doteq P^{-\min\{R_1,R_2\}}, \tag{44}$$

where $x_i := \sum_{j=1}^{R_2} |h_j g_{ij}|^2$. *In other words, the diversity of the two-hop wireless relay network is* $\min(R_1, R_2)$.

Proof. Since we are interested in the asymptotic regime in which $P \to \infty$, we define the random variables α_j, β_{ij}, so that

$$|h_j|^2 = P^{-\alpha_j}, \quad |g_{ij}|^2 = P^{-\beta_{ij}}, \quad i = 1, \dots, R_1, \ j = 1, \dots, R_2. \tag{45}$$

We thus have that

$$\begin{aligned} x_i &= \sum_{j=1}^{R_2} |h_j g_{ij}|^2 = \sum_{j=1}^{R_2} P^{-(\alpha_j + \beta_{ij})} \\ &\doteq P^{\max_j\{-(\alpha_j+\beta_{ij})\}} = P^{-\min_j\{\alpha_j+\beta_{ij}\}}, \end{aligned} \tag{46}$$

where the third equality comes from the fact that $P^a + P^b \doteq P^{\max\{a,b\}}$.

Similarly (and using the same fact), we have that

$$\begin{aligned} \sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1 &\doteq \sum_{k=1}^{R_2} P^{-\min_j\{\alpha_j+\beta_{kj}\}} + \sum_{j=1}^{R_2} P^{-\alpha_j} + 1 \\ &\doteq P^{\max_k(-\min_j(\alpha_j+\beta_{kj}))} + P^{\max_j(-\alpha_j)} + 1 \\ &\doteq P^{\max(-\min_{jk}(\alpha_j+\beta_{kj}), -\min_j \alpha_j)} + 1. \end{aligned} \tag{47}$$

The above change of variable implies that

$$d|h_j|^2 = (\log P) P^{-\alpha_j}\, d\alpha_j, \qquad d|g_{ij}|^2 = (\log P) P^{-\beta_{ij}}\, d\beta_{ij}, \tag{48}$$

and recalling that $|h_j|^2$ and $|g_{ij}^2|$ are independent, exponentially distributed, random variables with mean 1, we get

$$\begin{aligned} &E_{g_{ij},h_j} \prod_{i=1}^{R_1} \left(1 + P \frac{x_i}{\sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1}\right)^{-1} \\ &= \int_0^\infty \prod_{i=1}^{R_1} \left(1 + P \frac{x_i}{\sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1}\right)^{-1} \\ &\qquad \times \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} \exp(-|g_{ij}|^2)\, d|g_{ij}|^2 \\ &\qquad \times \prod_{j=1}^{R_2} \exp(-|h_j|^2)\, d|h_j|^2 \\ &= \int_{-\infty}^\infty \prod_{i=1}^{R_1} \left(1 + P \frac{P^{-\min_j\{\alpha_j+\beta_{ij}\}}}{P^{-\min(\min_{jk}(\alpha_j+\beta_{kj}), \min_j \alpha_j)} + 1}\right)^{-1} \\ &\qquad \times \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} \exp(-P^{-\beta_{ij}}) (\log P) P^{-\beta_{ij}}\, d\beta_{ij} \\ &\qquad \times \prod_{j=1}^{R_2} \exp(-P^{-\alpha_j}) (\log P)\, P^{-\alpha_j}\, d\alpha_j. \end{aligned} \tag{49}$$

Note that to lighten the notation by a single integral, we mean that this integral applies to all the variables. Now recall that

$$\exp(-P^{-a}) \doteq 0, \quad a < 0, \qquad \exp(-P^{-a}) \doteq 1, \quad a > 0, \tag{50}$$

and that

$$\begin{aligned} &\exp(-P^{-a}) \exp(-P^{-b}) \\ &\quad = \exp(-(P^{-a} + P^{-b})) \doteq \exp(-P^{-\min(a,b)}) \end{aligned} \tag{51}$$

meaning that in a product of exponentials, if at least one of the variables is negative, then the whole product tends to zero. Thus, only the integral where all the variables are positive does not tend to zero exponentially, and we are left with integrating over the range for which $\alpha_j \ge 0$, $\beta_{ij} \ge 0$, $i = 1, \dots, R_1$, $j = 1, \dots, R_2$. This implies in particular that

$$P^{-\min(\min_{jk}(\alpha_j+\beta_{kj}), \min_j \alpha_j)} + 1 \doteq P^{-c} + 1 \doteq P^{\max(-c,0)} \doteq 1 \tag{52}$$

since $c > 0$. This means that the denominator does not contribute in P. Note also that the $(\log P)$ factors do not contribute to the exponential order.

Hence

$$\begin{aligned} &E_{g_{ij},h_j} \prod_{i=1}^{R_1} \left(1 + P \frac{x_i}{\sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1}\right)^{-1} \\ &\doteq \int_0^\infty \prod_{i=1}^{R_1} \left(1 + P^{1-\min_j\{\alpha_j+\beta_{ij}\}}\right)^{-1} \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} P^{-\beta_{ij}}\, d\beta_{ij} \prod_{j=1}^{R_2} P^{-\alpha_j}\, d\alpha_j \\ &\doteq \int_0^\infty \prod_{i=1}^{R_1} \left(P^{(1-\min_j\{\alpha_j+\beta_{ij}\})^+}\right)^{-1} \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} P^{-\beta_{ij}}\, d\beta_{ij} \prod_{j=1}^{R_2} P^{-\alpha_j}\, d\alpha_j \\ &= \int_0^\infty \prod_{i=1}^{R_1} P^{-(1-\min_j\{\alpha_j+\beta_{ij}\})^+} \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} P^{-\beta_{ij}}\, d\beta_{ij} \prod_{j=1}^{R_2} P^{-\alpha_j}\, d\alpha_j, \end{aligned} \tag{53}$$

where $(\cdot)^+$ denotes $\max\{\cdot, 0\}$ and the second equality is obtained by writing $1 = P^0$.

By Laplace's method [20, page 50], [21], this expectation is equal in order to the dominant exponent of the integrand

$$\begin{aligned} &E_{g_{ij},h_j} \prod_{i=1}^{R_1} \left(1 + P \frac{x_i}{\sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1}\right)^{-1} \\ &\quad \doteq \int_0^\infty P^{-f(\alpha_j,\beta_{ij})} \prod_{i=1}^{R_1} \prod_{j=1}^{R_2} d\beta_{ij} \prod_{j=1}^{R_2} d\alpha_j \\ &\quad \doteq P^{-\inf f(\alpha_j,\beta_{ij})}, \end{aligned} \tag{54}$$

where

$$f(\alpha_j, \beta_{ij}) = \sum_{i=1}^{R_1} \left(1 - \min_j\{\alpha_j + \beta_{ij}\}\right)^+ + \sum_{i=1}^{R_1} \sum_{j=1}^{R_2} \beta_{ij} + \sum_{j=1}^{R_2} \alpha_j. \tag{55}$$

In order to conclude the proof, we are left to show that

$$\inf_{\alpha_j, \beta_{ij}} f(\alpha_j, \beta_{ij}) = \min\{R_1, R_2\}. \tag{56}$$

(i) First note that if $R_1 < R_2$, R_1 is achieved when $\alpha_j = 0$, $\beta_{ij} = 0$ and if $R_1 > R_2$, R_2 is achieved when $\alpha_j = 1$, $\beta_{ij} = 0$.

(ii) We now look at optimizing over β_{ij}. Note that one cannot optimize the terms of the sum separately. Indeed, if

β_{ij} are reduced to make $\sum_{i=1}^{R_1}\sum_{j=1}^{R_2}\beta_{ij}$ smaller, then the first term increases, and vice versa. One can actually see that we may set all $\beta_{ij} = 0$ since increasing any β_{ij} from zero does not decrease the sum.

(iii) Then the optimization becomes one over the α_j:

$$\inf_{\alpha_j \geq 0} \sum_{i=1}^{R_1} \left(1 - \min_j\{\alpha_j\}\right)^+ + \sum_{j=1}^{R_2} \alpha_j. \tag{57}$$

Using a similar argument as above, note that if α_j are taken greater than 1, then the first term cancels, but then the second term grows. Thus the minimum is given by considering $\alpha_j \in [0,1]$ which means that we can rewrite the optimization problem as

$$\inf_{\alpha_j \in [0,1]} \sum_{i=1}^{R_1} \left(1 - \min_j\{\alpha_j\}\right)^+ + \sum_{j=1}^{R_2} \alpha_j. \tag{58}$$

Now we have that

$$\begin{aligned}
&\sum_{i=1}^{R_1} \left(1 - \min_j\{\alpha_j\}\right) + \sum_{j=1}^{R_2} \alpha_j \\
&\quad = R_1\left(1 - \min_j\{\alpha_j\}\right) + \sum_{j=1}^{R_2} \alpha_j \\
&\quad \geq R_1\left(1 - \min_j\{\alpha_j\}\right) + R_2 \min_j\{\alpha_j\} \\
&\quad = R_1 + (R_2 - R_1)\min_j\{\alpha_j\}.
\end{aligned} \tag{59}$$

(iv) This final expression is minimized when $\alpha_j = 0$, $j = 1, \ldots, R_2$ for $R_1 < R_2$ and $\alpha_j = 1$, $j = 1, \ldots, R_2$ for $R_1 > R_2$, since if $R_2 - R_1 < 0$, one will try to remove as much as possible from R_1. Since $\alpha_j \leq 1$, the optimal is to take $\alpha_j = 1$. Thus if $R_1 < R_2$, the minimum is given by R_1, while it is given by $R_1 + R_2 - R_1 = R_2$ if $R_2 < R_1$, which yields $\min\{R_1, R_2\}$.

Hence $\inf_{\alpha_j, \beta_{ij}} f(\alpha_j, \beta_{ij}) = \min\{R_1, R_2\}$ and we conclude that

$$E_{g_{ij}, h_j} \prod_{i=1}^{R_1} \left(1 + P \frac{x_i}{\sum_{k=1}^{R_2} x_k + \sum_{j=1}^{R_2} |h_j|^2 + 1}\right)^{-1} \doteq P^{-\min\{R_1, R_2\}}. \tag{60}$$

□

Let us now comment the interpretation of this result. Since the diversity is also interpreted as the number of independent paths from transmitter to receiver, one intuitively expects the diversity to behave as the minimum between R_1 and R_2, since the bottleneck in determining the number of independent paths is clearly $\min(R_1, R_2)$.

4. CODING STRATEGY

We now discuss the design of the distributed space-time code

$$S = [B_1A_1\mathbf{s}, \ldots, B_1A_{R_1}\mathbf{s}, \ldots, B_{R_2}A_1\mathbf{s}, \ldots, B_{R_2}A_{R_1}\mathbf{s}] \in \mathbb{C}^{T \times R_1R_2}. \tag{61}$$

For the code design purpose, we assume that $T = R_1R_2$.

Remark 5. There is no loss in generality in assuming that the distributed space-time code is square. Indeed, if one needs a rectangular space-time code, one can always pick some columns (or rows) of a square code. If the codebook satisfies that $(S_k - S_l)^*(S_k - S_l)$ is fully diverse, then the codebook obtained by removing columns will be fully diverse too (see, e.g., [12] where this phenomenon has been considered in the context of node failures). This will be further illustrated in Section 5.

The coding problem consists of designing unitary matrices A_i, $i = 1, \ldots, R_1$, B_j, $j = 1, \ldots, R_2$, such that S as given in (61) is full rank, as explained in the previous section (see Remark 4). We will show in this section how such matrices can be obtained algebraically.

Recall that given a monic polynomial

$$p(x) = p_0 + p_1x + \cdots + p_{n-1}x^{n-1} + x^n \in \mathbb{C}[x], \tag{62}$$

its companion matrix is defined by

$$C(p) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & & 0 & -p_1 \\ 0 & 1 & & 0 & -p_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & & 1 & -p_{n-1} \end{pmatrix}. \tag{63}$$

Set $\mathbb{Q}(i) := \{a + ib,\ a, b \in \mathbb{Q}\}$, which is a subfield of the complex numbers.

Proposition 3. *Let $p(x)$ be a monic irreducible polynomial of degree n in $\mathbb{Q}(i)[x]$, and denote by θ one of its roots. Consider the vector space K of degree n over $\mathbb{Q}(i)$ with basis $\{1, \theta, \ldots, \theta^{n-1}\}$.*

(1) *The matrix M_s of multiplication by*

$$s = s_0 + s_1\theta + \cdots + s_{n-1}\theta^{n-1} \in K \tag{64}$$

is of the form

$$M_s = [\mathbf{s}, C(p)\mathbf{s}, \ldots, C(p)^{n-1}\mathbf{s}], \tag{65}$$

where $\mathbf{s} = [s_0, s_1, \ldots, s_{n-1}]^T$ and $C(p)$ is the companion matrix of $p(x)$.

(2) *Furthermore,*

$$\det(M_s) = 0 \Longleftrightarrow s = 0. \tag{66}$$

Proof. (1) By definition, M_s satisfies

$$(1, \theta, \ldots, \theta^{n-1})M_s = s(1, \theta, \ldots, \theta^{n-1}). \tag{67}$$

Thus the first column of M_s is clearly $\mathbf{s}$, since

$$(1, \theta, \ldots, \theta^{n-1})\mathbf{s} = s. \tag{68}$$

Now, we have that

$$\begin{aligned}
s\theta &= s_0\theta + s_1\theta^2 + \cdots + s_{n-2}\theta^{n-1} + s_{n-1}\theta^n \\
&= -p_0s_{n-1} + \theta(s_0 - p_1s_{n-1}) + \cdots \\
&\quad + \theta^{n-1}(s_{n-2} - p_{n-1}s_{n-1})
\end{aligned} \tag{69}$$

since $\theta^n = -p_0 - p_1\theta - \cdots - p_{n-1}\theta^{n-1}$. Thus the second column of M_s is clearly

$$\begin{pmatrix} -p_0 s_{n-1} \\ s_0 - p_1 s_{n-1} \\ \vdots \\ s_{n-2} - p_{n-1}s_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -p_0 \\ 1 & 0 & & 0 & -p_1 \\ 0 & 1 & & 0 & -p_2 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & 0 & & 1 & -p_{n-1} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}. \tag{70}$$

We have thus shown that for any $s \in K$, $s\theta = C(p)\mathbf{s}$. By iterating this processing, we have that

$$s\theta^2 = (s\theta)\theta = C(p)\mathbf{s}\theta = C(p)^2\mathbf{s}, \tag{71}$$

and thus $s\theta^j = C(p)^j\mathbf{s}$ is the $j+1$ column of M_s, $j = 1,\dots,n-1$.

(2) Denote by $\theta_1,\dots,\theta_n$ the n roots of p. Let θ be any of them. Denote by σ_j the following $\mathbb{Q}(i)$-linear map:

$$\sigma_j(\theta) = \theta_j, \quad j = 1,\dots,n. \tag{72}$$

Now, it is clear, by definition of M_s, namely,

$$(1,\theta,\dots,\theta^{n-1})M_s = s(1,\theta,\dots,\theta^{n-1}), \tag{73}$$

that s is an eigenvalue of M_s associated to the eigenvector $(1,\theta,\dots,\theta^{n-1})$. By applying σ_j to the above equation, we have, by $\mathbb{Q}(i)$-linearity, that

$$(1,\sigma_j(\theta),\dots,\sigma_j(\theta^{n-1}))M_s = \sigma_j(s)(1,\sigma_j(\theta),\dots,\sigma_j(\theta^{n-1})). \tag{74}$$

Thus $\sigma_j(s)$ is an eigenvalue of M_s, $j = 1,\dots,n$, and

$$\det(M_s) = \prod_{j=1}^{n}\sigma_j(s), \tag{75}$$

which concludes the proof. □

The matrix M_s, as described in the above proposition, is a natural candidate to design a distributed space-time code, since it has the right structure, and is proven to be fully diverse. However, in this setting, $C(p)$ and its powers correspond to products of A_iB_j, which are unitary. Thus, $C(p)$ has to be unitary. A straightforward computation shows the following.

Lemma 4. *One has that $C(p)$ is unitary if and only if*

$$p_1 = \cdots = p_{n-1} = 0, \qquad |p_0|^2 = 1. \tag{76}$$

The family of codes proposed in [10] is a particular case, when p_0 is a root of unity.

The distributed space-time code design can be summarized as follow.

(1) Choose $p(x)$ such that $|p_0|^2 = 1$ and $p(x)$ is irreducible over $\mathbb{Q}(i)$.

(2) Define

$$\begin{aligned} A_i &= C(p)^{i-1}, \quad i = 1,\dots,R_1, \\ B_j &= C(p)^{R_1(j-1)}, \quad j = 1,\dots,R_2. \end{aligned} \tag{77}$$

Example 5 ($R_1 = R_2 = 2$). We need a monic polynomial of degree 4 of the form

$$p(x) = x^4 - p_0, \qquad |p_0|^2 = 1. \tag{78}$$

For example, one can take

$$p(x) = x^4 - \frac{i+2}{i-2}, \tag{79}$$

which are irreducible over $\mathbb{Q}(i)$. Its companion matrix is given by

$$\begin{pmatrix} 0 & 0 & 0 & \frac{i+2}{i-2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{80}$$

The matrices $A_1,\ A_2,\ B_1,\ B_2$ are given explicitly in next section.

Example 6 ($R_1 = R_2 = 3$). We need now a monic polynomial of degree 9. For example,

$$p(x) = x^9 - \frac{i+2}{i-2}, \tag{81}$$

is irreducible over $\mathbb{Q}(i)$, with companion matrix

$$\begin{pmatrix} 0&0&0&0&0&0&0&0&\frac{i+2}{i-2} \\ 1&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&0&1&0 \end{pmatrix}. \tag{82}$$

5. SIMULATION RESULTS

In this section, we present simulation results for different scenarios. For all plots, the x-axis represents the power (in dB) of the whole network, and the y-axis gives the block error rate (BLER).

Diversity discussion

In order to evaluate the simulation results, we refer to Theorem 1. Since the diversity is interpreted both as the slope of the error probability in log-log scale as well as the exponent of P in the upper bound on the pairwise error probability, one intuitively expects the slope to behave as the minimum between R_1 and R_2.

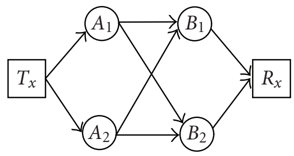

FIGURE 1: On the left, a two-hop network with two nodes at each hop. On the right, a one-hop network with two nodes.

We first consider a simple network with two hops and two nodes at each hop, as shown in the left of Figure 1. The coding strategy (see Example 5) is given by

$$A_1 = \mathbf{I}_4, \qquad A_2 = \begin{pmatrix} 0 & 0 & 0 & \dfrac{i+2}{i-2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$
$$B_1 = \mathbf{I}_4, \qquad B_2 = \begin{pmatrix} 0 & 0 & \dfrac{i+2}{i-2} & 0 \\ 0 & 0 & 0 & \dfrac{i+2}{i-2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{83}$$

We have simulated the BLER of the transmitter sending a signal to the receiver through the two hops. The results are shown in Figure 2, given by the dashed curve. Following the above discussion, we expect a diversity of two. In order to have a comparison, we also plot the BLER of sending a message through a one-hop network with also two relay nodes, as shown on the right of Figure 1. This plot comes from [10], where it has been shown that with one hop and two relays, the diversity is two. The two slopes are clearly parallel, showing that the two-hop network with two relay nodes at each hop has indeed diversity of two. There is no interpretation in the coding gain here, since in the one-hop relay case, the power allocated at the relays is more important (half of the total power, while one third only in the two-hop case), and the noise forwarded is much bigger in the two-hop case. Furthermore, the coding strategies are different.

We also emphasize the importance of performing coding at the relays. Still on Figure 1, we show the performance of doing coding either only at the first hop, or only at the second hop. It is clear that this yields no diversity.

We now consider more in details a two-hop network with three relay nodes at each hop, as show in Figure 3. Transmitter and receiver for a two-hop communication are indicated and are plotted as boxes, while the second hop also contains a box, indicating that this relay is also able to be a transmitter/receiver. We will thus consider both cases, when it is either a relay node or a receiver node. Nodes that serve as relays are all endowed with a unitary matrix, denoted by either A_i at the first hop, or B_j for the second hop, as explained in Section 4.

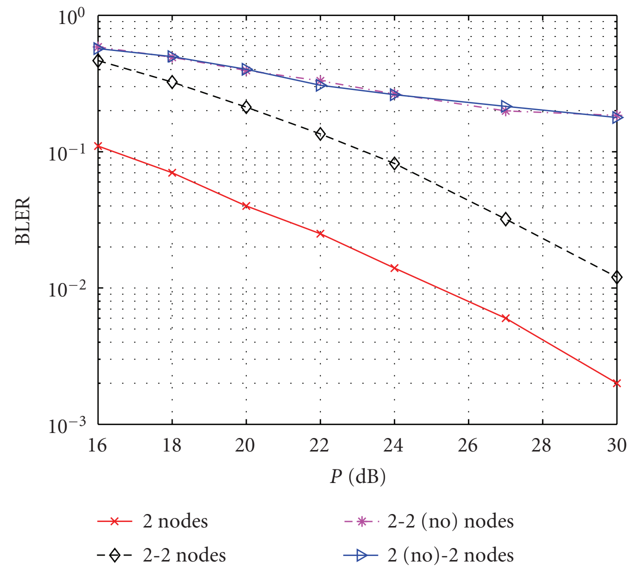

FIGURE 2: Comparison between a one-hop network with two relay nodes and a two-hop network with two relay nodes at each hop, "(no)" means that no coding has been done either at the first or second hop.

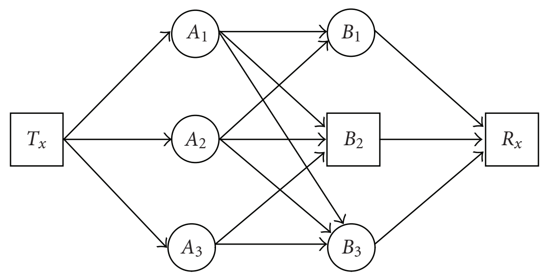

FIGURE 3: A two-hop network with three nodes at each hop. Nodes able to be transmitter/receiver are shown as boxes.

For the upcoming simulations, we have used the following coding strategy (see Example 6). Set

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dfrac{i+2}{i-2} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$
$$A_1 = \mathbf{I}_9, \qquad A_2 = \Gamma, \qquad A_3 = \Gamma^2,$$
$$B_1 = \mathbf{I}_9, \qquad B_2 = \Gamma^3, \qquad B_3 = \Gamma^6. \tag{84}$$

In Figure 4, the BLER of communicating through the two-hop network is shown. The diversity is expected to be three. In order to get a comparison, we reproduce here the performance of the two-hop network with two relay nodes already shown in the previous figure. There is a clear gain in diversity

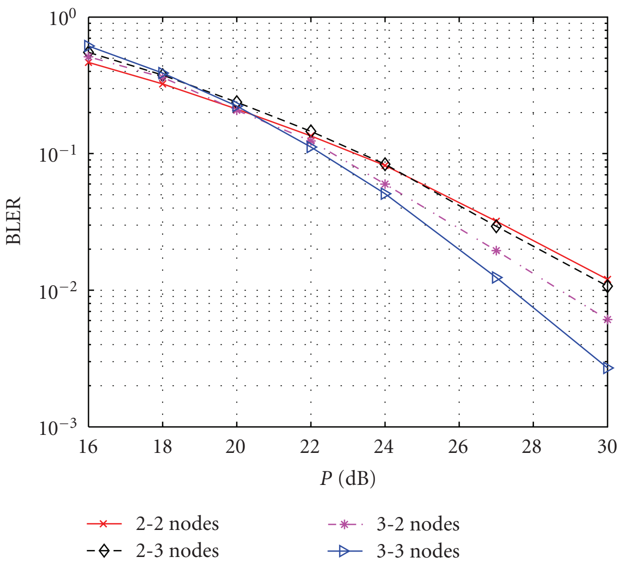

Figure 4: Comparison among different uses of either two or three nodes at, respectively, the first and second hops.

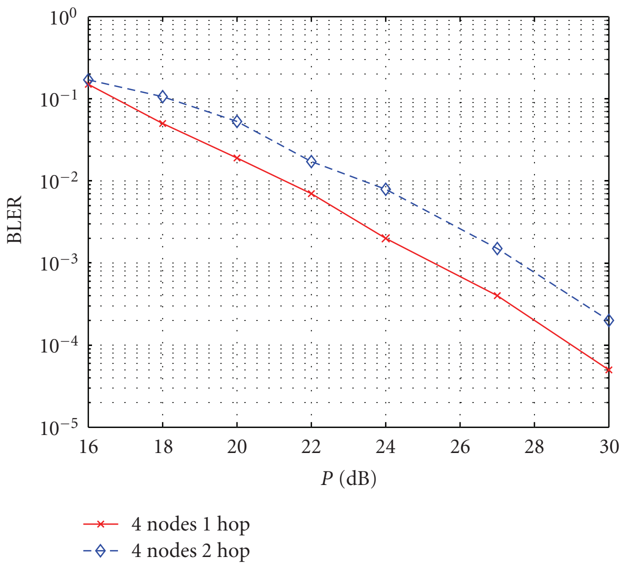

Figure 5: One hop in a one-hop network versus one hop in a two-hop network.

obtained by increasing the number of relay nodes. We now illustrate that the diversity actually depends on $\min\{R_1, R_2\}$, that is, the minimum number relays between the first and the second hops. We assume now that one node in the first hop is not communicating (it may be down, or too far away). We keep the same coding strategy, and thus simulate communication with a first hop that has two relay nodes, and a second hop that has three relay nodes. We see that the diversity immediately drops to the one of a network with two nodes at each hop. There is no gain in having a third relay participating in the second hop. This is true vice versa, if the first hop uses three relays while the second hop uses only two. Though the performance is better, the diversity is two.

Finally, we would like to mention that the scheme proposed does not restrict to the case where communication requires exactly two hops. In order to do so, we assume that one node among those at the second hop can actually be a receiver itself (see Figure 3). We keep the coding strategy described above and simulate a one-hop communication between the transmitter and this new receiver. The performance is shown in Figure 5, where it is compared with a one-hop network (as in [10]). Both strategies have now noise forwarded from only one hop. However, the difference of coding gain is easily explained by the fact that we did not change the power allocation, and thus the best curve corresponds to having half of the power at the first hop relays, while the second curve corresponds to a use of only one third of the power. Diversity is of course similar. The main point here is to notice that the coding strategy does not need to change. Thus the unitary matrices can be allotted before the start of communication, and used for either one or two hops communication.

Decoding issues

All the simulations presented in this paper have been done using a standard sphere decoder algorithm [22, 23].

6. CONCLUSION

In this paper, we considered a wireless relay network with multihops. We first showed that when considering distributed space-time coding, the diversity of such channels is determined by the hop whose number of relays is minimal. We then provided a technique to design systematically distributed space-time codes that are fully diverse for that scenario. Simulation results confirmed the use of doing coding at the relays, in order to get cooperative diversity. Further work now involves studying the power allocation. In order to get diversity results, power is considered in an asymptotic regime. In doing distributed space-time coding for multihop, one drawback is that noise is forwarded from one hop to the other. This will not influence the diversity behavior since the power can grow to infinity. However, for more realistic scenarios where the power is limited, it does matter. In this case, one may need a more elaborated power allocation than just sharing equally the power among the transmitter and relays at all hops.

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REFERENCES

[1] J. N. Laneman and G. W. Wornell, “Distributed space-time-coded protocols for exploiting cooperative diversity in wireless

networks," *IEEE Transactions on Information Theory*, vol. 49, no. 10, pp. 2415–2425, 2003.

[2] K. Azarian, H. El Gamal, and P. Schniter, "On the achievable diversity-multiplexing tradeoff in halfduplex cooperative channels," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4152–4172, 2005.

[3] P. Elia, K. Vinodh, M. Anand, and P. V. Kumar, "D-MG tradeoff and optimal codes for a class of AF and DF cooperative communication protocols," to appear in *IEEE Transactions on Information Theory*.

[4] G. Susinder Rajan and B. Sundar Rajan, "A non-orthogonal distributed space-time coded protocol part I: signal model and design criteria ," in *Proceedings of the IEEE Information Theory Workshop (ITW '06)*, pp. 385–389, Chengdu, China, October 2008.

[5] S. Yang and J.-C. Belfiore, "Optimal space-time codes for the amplify-and-forward cooperative channel," *IEEE Transactions on Information Theory*, vol. 53, no. 2, pp. 647–663, 2007.

[6] Y. Jing and B. Hassibi, "Distributed space-time coding in wireless relay networks," *IEEE Transactions on Wireless Communications*, vol. 5, no. 12, pp. 3524–3536, 2006.

[7] P. Dayal and M. K. Varanasi, "Distributed QAM-based space-time block codes for efficient cooperative multiple-access communication," to appear in *IEEE Transactions on Information Theory*.

[8] Y. Jing and H. Jafarkhani, "CTH17-1: using orthogonal and quasi-orthogonal designs in wireless relay networks," in *Proceedings of the IEEE Global Telecommunications Conference (GLOBECOM '07)*, pp. 1–5, San Francisco, Calif, USA, November 2007.

[9] T. Kiran and B. S. Rajan, "Distributed space-time codes with reduced decoding complexity," in *Proceedings of the IEEE International Symposium on Information Theory (ISIT '06)*, pp. 542–546, Seattle, Wash, USA, September 2006.

[10] F. Oggier and B. Hassibi, "An algebraic family of distributed space-time codes for wireless relay networks," in *Proceedings of the IEEE International Symposium on Information Theory (ISIT '06)*, pp. 538–541, Seattle, Wash, USA, July 2006.

[11] Y. Jing and B. Hassibi, "Cooperative diversity in wireless relay networks with multiple-antenna nodes," to appear in *IEEE Transactions on Signal Processing*.

[12] F. Oggier and B. Hassibi, "An algebraic coding scheme for wireless relay networks with multiple-antenna nodes," to appear in *IEEE Transactions on Signal Processing*.

[13] Y. Jing and H. Jafarkhani, "Distributed differential space-time coding for wireless relay networks," to appear in *IEEE Transactions on Communications*.

[14] T. Kiran and B. S. Rajan, "Partially-coherent distributed space-time codes with differential encoder and decoder," in *Proceedings of the IEEE International Symposium on Information Theory (ISIT '06)*, pp. 547–551, Seattle, Wash, USA, September 2006.

[15] F. Oggier and B. Hassibi, "A coding strategy for wireless networks with no channel information," in *Proceedings of 44th Annual Allerton Conference on Communication, Control, and Computing*, Monticello, Ill, USA, September 2006.

[16] F. Oggier and B. Hassibi, "A coding scheme for wireless networks with multiple antenna nodes and no channel information," in *Proceedings of the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP '07)*, vol. 3, pp. 413–416, Honolulu, Hawaii, USA, April 2007.

[17] X. Guo and X.-G. Xia, "A distributed space-time coding in asynchronous wireless relay networks," to appear in *IEEE Transactions on Wireless Communications*.

[18] S. Yang and J.-C. Belfiore, "Distributed space-time codes for the multi-hop channel," in *Proceedings of International Workshop on Wireless Networks: Communication, Cooperation and Competition (WNC3 '07)*, Limassol, Cyprus, April 2007.

[19] S. Yang and J.-C. Belfiore, "Diversity of MIMO multihop relay channels-part I: amplify-and-forward," to appear in *IEEE Transactions on Information Theory*.

[20] C. Rao, "Asymptotics analysis of wireless systems with rayleigh fading," Ph.D. Thesis, 2007.

[21] D. Zwillinger, *Handbook of Integration*, Jones and Bartlett, Boston, Mass, USA, 1992.

[22] B. Hassibi and H. Vikalo, "On the sphere-decoding algorithm I. Expected complexity," *IEEE Transactions on Signal Processing*, vol. 53, no. 8, pp. 2806–2818, 2005.

[23] E. Viterbo and J. Boutros, "A universal lattice code decoder for fading channels," *IEEE Transactions on Information Theory*, vol. 45, no. 5, pp. 1639–1642, 1999.