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Research Article

Extended LaSalle's Invariance Principle for Full-Range Cellular Neural Networks

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In several relevant applications to the solution of signal processing tasks in real time, a cellular neural network (CNN) is required to be convergent, that is, each solution should tend toward some equilibrium point. The paper develops a Lyapunov method, which is based on a generalized version of LaSalle's invariance principle, for studying convergence and stability of the differential inclusions modeling the dynamics of the full-range (FR) model of CNNs. The applicability of the method is demonstrated by obtaining a rigorous proof of convergence for symmetric FR-CNNs. The proof, which is a direct consequence of the fact that a symmetric FR-CNN admits a strict Lyapunov function, is much more simple than the corresponding proof of convergence for symmetric standard CNNs.

1. Introduction

The Full-Range (FR) model of cellular neural networks (CNNs) has been introduced in [1] in order to obtain advantages in the VLSI implementation of CNN chips with a large number of neurons. One main feature is the use of hard-limiter nonlinearities that constrain the evolution of the FR-CNN trajectories within a closed hypercube of the state space. This improved range of the trajectories has enabled us to reduce the power consumption and obtain higher cell densities and increased processing speed [1–4] compared to the original standard (S) CNN model by Chua and Yang [5].

In several applications for solving signal processing tasks in real time it is needed that a FR-CNN is convergent (or completely stable), that is, each solution is required to approach some equilibrium point in the long-run behavior [5–7]. For example, given a two-dimensional image, a CNN is able to perform contour extraction and morphological operations, noise filtering, or motion detection, during the transient motion toward an equilibrium point [8]. Other relevant applications of convergent FR-CNN dynamics concern the solution of optimization or identification problems or the implementation of nonlinear electronic devices for pattern formation [9, 10].

An FR-CNN is characterized by ideal hard-limiter nonlinearities with vertical segments in the i-v characteristic, hence its dynamics is mathematically described by a differential inclusion, where a set-valued vector field models the set of feasible velocities for each state of the FR-CNN. A recent paper [11] has been devoted to the rigorous mathematical foundation of the FR model within the framework of the theory of differential inclusions [12]. The goal of this paper is to extend the results in [11] by developing a generalized Lyapunov approach for addressing stability and convergence of FR-CNNs. The approach is based on a suitable notion of derivative of a (candidate) Lyapunov function and a generalized version of LaSalle's invariance principle for the differential inclusions modeling the FR-CNNs.

The Lyapunov method developed in the paper is formulated in a general fashion, which makes it suitable to check if a continuously differentiable (candidate) Lyapunov function is decreasing along the solutions of a FR-CNNs, and to verify if this property in turn implies convergence of each FR-CNN solution. The applicability of the method is demonstrated by obtaining a rigorous convergence proof for the important and widely used class of symmetric FR-CNNs. It is shown that the proof is more simple than the proof of an analogous convergence result in [11], which is not based on an invariance principle for FR-CNNs. The same proof is also much more simple than the proof of convergence for symmetric S-CNNs. We refer the reader to [13] for other applications of the method to classes of FR-CNNs with nonsymmetric interconnection matrices used in the real-time solution of some classes of global optimization problems.

The structure of the paper is briefly outlined as follows. Section 2 introduces the FR-CNN model studied in the paper, whereas Section 3 gives some fundamental properties of the solutions of FR-CNNs. The extended LaSalle's invariance principle for FR-CNNs and the convergence results for FR-CNNs are described in Sections 4 and 5, respectively. Section 6 discusses the significance of the convergence results and, finally, Section 7 draws the main conclusions of the paper.

Notation. Let $\mathbb{R}^n$ be the real n-space. Given matrix $A \in \mathbb{R}^{n\times n}$, by A' we mean the transpose of A. In particular, by E_n we denote the $n \times n$ identity matrix. Given the column vectors $x, y \in \mathbb{R}^n$, we denote by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$ the scalar product of x and y, while $\|x\| = \sqrt{\langle x, x\rangle}$ is the Euclidean norm of x. Sometimes, use is made of the norm $\|x\|_\infty = \max_{i=1,2,\dots,n}|x_i|$. Given a set $D \subset \mathbb{R}^n$, by $\mathrm{cl}(D)$ we denote the closure of D, while $\mathrm{dist}(x, D) = \inf_{y\in D}\|x - y\|$ is the distance of vector $x \in \mathbb{R}^n$ from D. By $B(z, r) = \{y \in \mathbb{R}^n : \|y - z\| < r\}$ we mean an n-dimensional open ball with center $z \in \mathbb{R}^n$ and radius r.

1.1. Preliminaries

1.1.1. Tangent and Normal Cones. This section reports the definitions of tangent and normal comes to a closed convex set and some related properties that are used throughout the paper. The reader is referred to [12, 14, 15] for a more thorough treatment.

Let $Q \subset \mathbb{R}^n$ be a nonempty closed convex set. The tangent cone to Q at $x \in Q$ is given by [14, 15]

$$T_Q(x) = \left\{ v \in \mathbb{R}^n : \liminf_{\rho \to 0^+} \frac{\mathrm{dist}(x + \rho v, Q)}{\rho} = 0 \right\}, \quad (1)$$

while the normal cone to Q at $x \in Q$ is defined as

$$N_Q(x) = \{ p \in \mathbb{R}^n : \langle p, v\rangle \le 0,\ \forall v \in T_Q(x)\}. \quad (2)$$

The orthogonal set to $N_Q(x)$ is given by

$$N_Q^{\perp}(x) = \{ v \in \mathbb{R}^n : \langle p, v\rangle = 0,\ \forall p \in N_Q(x)\}. \quad (3)$$

From a geometrical point of view, the tangent cone is a generalization of the notion of the tangent space to a set, which can be applied when the boundary is not necessarily smooth. In particular, $T_Q(x)$ is the closure of the cone formed by all half lines originating at x and intersecting Q in at least one point y distinct from x. The normal cone is the dual cone of the tangent cone, that is, it is formed by all directions with an angle of at least ninety degrees with any direction belonging to the tangent cone. It is known that $T_Q(x)$ and $N_Q(x)$ are nonempty closed convex cones in $\mathbb{R}^n$, which possibly reduce to the singleton $\{0\}$. Moreover, $N_Q^{\perp}(x)$ is a vector subspace of $\mathbb{R}^n$, and we have $N_Q^{\perp}(x) \subset T_Q(x)$. The next property holds [11].

Property 1. If Q coincides with the hypercube $K = [-1, 1]^n$, then $N_K(x)$, $T_K(x)$, and $N_K^{\perp}(x)$ have the following analytical expressions.

For any $x \in K$ we have

$$N_K(x) = H(x) = (h(x_1), h(x_2), \dots, h(x_n))', \quad (4)$$

where

$$h(\rho) = \begin{cases} (-\infty, 0], & \rho = -1, \\ 0, & \rho \in (-1, 1), \\ [0, +\infty), & \rho = 1, \end{cases} \quad (5)$$

whereas

$$T_K(x) = H_T(x) = (h_T(x_1), h_T(x_2), \dots, h_T(x_n))', \quad (6)$$

with

$$h_T(\rho) = \begin{cases} [0, +\infty), & \rho = -1, \\ (-\infty, +\infty), & \rho \in (-1, 1), \\ (-\infty, 0], & \rho = 1. \end{cases} \quad (7)$$

Finally, for any $x \in K$ we have

$$N_K^{\perp}(x) = H^{\perp}(x) = (h^{\perp}(x_1), h^{\perp}(x_2), \dots, h^{\perp}(x_n))', \quad (8)$$

where

$$h^{\perp}(\rho) = \begin{cases} 0, & \rho = -1, \\ (-\infty, +\infty), & \rho \in (-1, 1), \\ 0, & \rho = 1. \end{cases} \quad (9)$$

The above cones, evaluated at some points of the set $K = [-1, 1]^2$, are reported in Figure 1.

Let $Q \subset \mathbb{R}^n$ be a nonempty closed convex set. The orthogonal projector onto Q is a mathematical operator which associates to any $x \in \mathbb{R}^n$ the set $\mathcal{P}_Q(x)$, composed by the points of Q that are closest to x, namely,

$$\|x - \mathcal{P}_Q(x)\| = \mathrm{dist}(x, Q) = \min_{y\in Q}\|y - x\|. \quad (10)$$

Under the considered assumptions, $\mathcal{P}_Q(x)$ always contains exactly one point. The name derives from the fact that $x - \mathcal{P}_Q(x) \in N_Q(x)$.

2. CNN Models and Motivating Results

The dynamics of the S-CNNs, introduced by Chua and Yang in the fundamental paper [5], can be described by the differential equations:

$$\dot{x}(t) = -x(t) + AG(x(t)) + I, \quad (\mathrm{S})$$

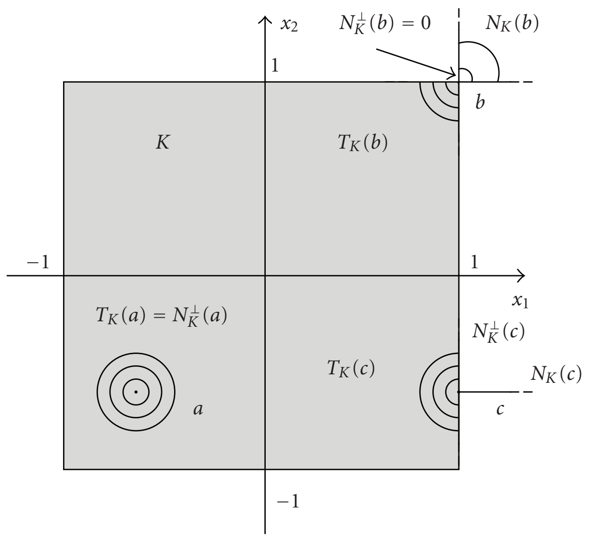

Figure 1: Set $K = [-1,1]^2$ and cones T_K, N_K, and $N_K^\perp$ at points a, b, and c of K (the cones are shown translated into the corresponding points of K). Point a belongs to the interior of K, and hence $T_K(a)$ is the whole space $\mathbb{R}^2$, while $N_Q(a)$ reduces to $\{0\}$. Point b coincides with a vertex of K, and so $T_K(b)$ corresponds to the third quadrant of $\mathbb{R}^2$, while $N_K(b)$ corresponds to the first quadrant of $\mathbb{R}^2$. Finally, point c belongs to the right edge of the square and, consequently, $T_K(c)$ coincides to the left half plane of $\mathbb{R}^2$, while $N_K(c)$ coincides with the nonnegative part of x_1 axis.

where $x \in \mathbb{R}^n$ is the vector of neuron state variables; $A \in \mathbb{R}^{n\times n}$ is the neuron interconnection matrix; $I \in \mathbb{R}^n$ is the constant input; $G(x) = (g(x_1), g(x_2), \ldots, g(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$, where the piecewise-linear neuron activation g is given by

$$g(\rho) = \frac{1}{2}(|\rho+1| - |\rho-1|) = \begin{cases} 1, & \rho > 1, \\ \rho, & -1 \le \rho \le 1, \\ -1, & \rho < -1, \end{cases} \tag{11}$$

see Figure 2(a). It is convenient to define

$$\mathcal{A} = A - E_n, \tag{12}$$

which is the matrix of the affine system satisfied by (S) in the linear region $|x_i| < 1$, $i = 1, 2, \ldots, n$.

The improved signal range (ISR) model of CNNs has been introduced in [1, 2] with the goal to obtain advantages in the electronic implementation of CNN chips. The dynamics of an ISR-CNN can be described by the differential equations:

$$\dot{x}(t) = -x(t) + AG(x(t)) + I - mL(x(t)), \tag{I}$$

where $m \ge 0$, $L(x) = (\ell(x_1), \ell(x_2), \ldots, \ell(x_n))' : \mathbb{R}^n \to \mathbb{R}^n$ and

$$\ell(\rho) = \begin{cases} \rho - 1, & \rho \ge 1, \\ 0, & -1 < \rho < 1, \\ \rho + 1, & \rho \le -1, \end{cases} \tag{13}$$

see Figure 2(b). When the slope m of the nonlinearity $m\ell(\cdot)$ is large, $m\ell(\cdot)$ plays the role of a limiter device that prevents the state variables x_i of (I) from exceedingly enter the saturation regions where $|x_i(t)| > 1$. The larger m, the smaller the neighborhood of the hypercube:

$$K = [-1, 1]^n, \tag{14}$$

where the state variables x_i are constrained to evolve for all large t.

A particularly interesting limiting situation is that where $m \to +\infty$, in which case $m\ell(\cdot)$ approaches the ideal hard-limiter nonlinearity $h(\cdot)$ given in (5); see Figure 2(c). The hard-limiter $h(\cdot)$ now constrains the state variables of (F) to evolve within K, that is, we have $|x_i(t)| \le 1$ for all t and for all $i = 1, 2, \ldots, n$. Since for $x \in K$ we have $x = G(x)$, (I) becomes the FR model of CNNs [1, 2, 11]:

$$\dot{x}(t) \in -x(t) + Ax(t) + I - H(x(t)), \tag{F}$$

where $H(x) = (h(x_1), h(x_2), \ldots, h(x_n))'$, and h is given in (5).

From a mathematical viewpoint, $h(\rho)$ is a set-valued map assuming the entire interval of values $[0, +\infty)$ (resp., $(-\infty, 0]$) at $\rho = 1$ (resp., $\rho = -1$). As a consequence, the vector field defining the dynamics of (F), $-x+Ax+I-H(x)$, is a set-valued map assuming multiple values when some state variable x_i is saturated at $x_i = \pm 1$, which represent the set of feasible velocities for (F) at point x. An FR-CNN is thus described by a differential inclusion as in (F) [11, 12] and not by an ordinary differential equation.

In [16], Corinto and Gilli have compared the dynamical behavior of (S) ($m = 0$), with that of (I) ($m \gg 0$) and (F) ($m \to +\infty$), under the assumption that the three models are characterized by the same set of parameters (interconnections and inputs). It is shown in [16] that there are cases where the global behavior of (S) and (I) is *not* qualitatively similar for the same set of parameters, due to bifurcations in model (I) occurring for some positive values of m. In particular, a class of completely stable, second-order S-CNNs (S) has been considered, and it has been shown that, for the same parameters, (I) displays a heteroclinic bifurcation at some $m = m_\beta > 0$, which leads to the birth of a stable limit cycle for any $m > m_\beta$. In other words, (I) is not completely stable for $m > m_\beta$, and the same holds for (F), which is the limit of (I) as $m \to +\infty$.

The result in [16] has the important consequence that in the general case the stability of model (F) cannot be deduced from existing results on stability of (S). Hence, it is needed to develop suitable tools, which are based on the theory of differential inclusions, for studying in a rigorous way the stability and convergence of FR-CNNs.

The goal of this paper is to develop an extended Lyapunov approach for addressing stability and convergence of FR-CNNs. The approach is based on a suitable notion of derivative and an extended version of LaSalle's invariance principle for the differential inclusion (F) modeling a FR-CNN.

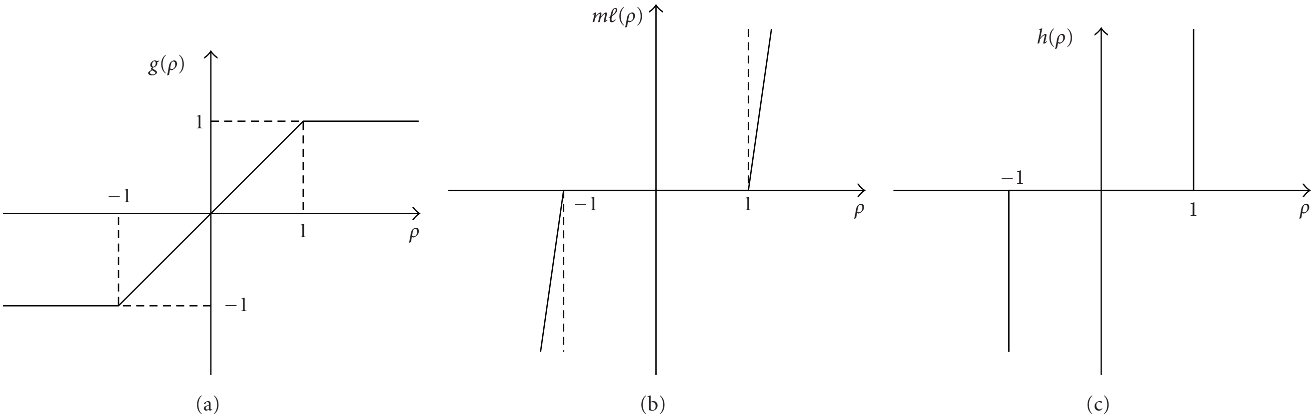

Figure 2: Nonlinearities used in the CNN models (S), (I), and (F).

3. Solutions of FR-CNNs

To the authors knowledge, [11] has been the first paper giving a foundation with the theory of differential inclusions of the FR model of CNNs. One main property noted in [11] is that we have

$$H(x) = N_K(x), \tag{15}$$

for all $x \in K$, that is, $H(x)$ coincides with the normal cone to K at point x (cf. Property 1). Therefore, (F) can be written as

$$\dot{x}(t) \in -x(t) + Ax(t) + I - N_K(x(t)), \tag{16}$$

which represents a class of differential inclusions termed differential variational inequalities (DVIs) [12, Chapter 5].

Let $x_0 \in K$. A solution of (F) in $[0, \tilde{t}]$, with initial condition x_0, is a function x satisfying [12]: (a) $x(t) \in K$ for $t \in [0, \tilde{t}]$ and $x(0) = x_0$; (b) x is absolutely continuous on $[0, \tilde{t}]$, and for almost all (a.a.) $t \in [0, \tilde{t}]$ we have $\dot{x}(t) \in -x(t) + Ax(t) + I - N_K(x(t))$. By an equilibrium point (EP) we mean a constant solution $x(t) = \xi \in K$, $t \geq 0$, of (F). Note that $\xi \in K$ is an EP of (F) if and only if there exists $\gamma_\xi \in N_K(\xi)$ such that $0 = -\xi + A\xi + I - \gamma_\xi$, or equivalently, we have $(A - E_n)\xi + I \in N_K(\xi)$.

By exploiting the theory of DVIs, the next result has been proved in [11].

Property 2. For any $x_0 \in K$, there exists a unique solution x of (F) with initial condition $x(0) = x_0$, which is defined for all $t \geq 0$. Moreover, there exists at least an EP $\xi \in K$ of (F).

We will denote by $E \neq \emptyset$ the set of EPs of (F). It can be shown that E is a compact subset of K.

It is both of theoretic and practical interest to compare the solutions of the ideal model (F) with those of model (I). The next result shows that the solutions of (F) are the uniform limit, as the slope $m \to +\infty$, of the solutions of model (I).

Property 3. Let $x(t)$, $t \geq 0$, be the solution of (F) with initial condition $x(0) = x_0 \in K$. Moreover, for any $m = k = 1, 2, 3, \ldots$, let $x_k(t)$, $t \geq 0$, be the solution of model (I) such that $x_k(t) = x_0$. Then, $x_k(\cdot)$ converges uniformly to $x(\cdot)$ on any compact interval $[0, T] \subset [0, +\infty)$, as $k \to +\infty$.

Proof. See Appendix A. □

4. LaSalle's Invariance Principle for FR-CNNs

Consider the system of ordinary differential equations:

$$\dot{x} = f(x), \tag{17}$$

where $x \in \mathbb{R}^n$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable (candidate) Lyapunov function, and consider the vector field:

$$\delta(x) = \langle f(x), \nabla\phi(x) \rangle, \tag{18}$$

for all $x \in \mathbb{R}^n$. From the standard Lyapunov method for ordinary differential equations [17], it is known that for all times t the derivative of ϕ along a solution x of (17) can be evaluated from δ as follows:

$$\frac{d}{dt}\phi(x(t)) = \delta(x(t)). \tag{19}$$

Such a treatment cannot be directly applied to the differential inclusion (16) modeling the dynamics of a FR-CNN, since the vector field at the right-hand side of (16) assumes multiple values when some component x_i of x assumes the values ± 1. In what follows our goal is to introduce a suitable concept of derivative, which generalizes the definition of δ, for evaluating the time evolution of a candidate Lyapunov function along the solutions of the differential inclusion (16). Then, we prove a version of LaSalle's invariance principle generalizing to the differential inclusions (16) the classic version for ordinary differential equations [17]. In doing so, we need to take into account that the limiting sets for the solutions of (16) enjoy a weaker invariance property with respect to the solutions of the standard differential equations defined by a continuously differentiable vector field.

We begin by introducing the following definition of derivative.

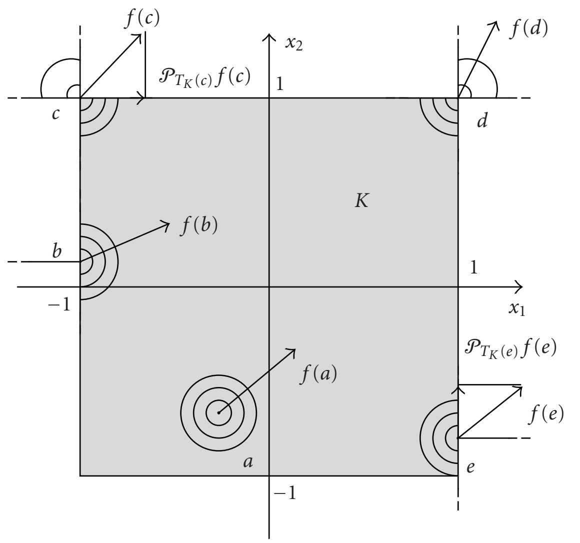

Figure 3: Vector fields involved in the definition of the derivative $D\phi$ for a second-order FR-CNN. Let $f(x) = \mathcal{A}x + I$. We have $\mathcal{P}_{T_K(x)}f(x) \in N_K^{\perp}(x)$, hence $D\phi(x)$ is a singleton, when x is one of the points $a, d, e \in K$. On the other hand, $\mathcal{P}_{T_K(x)}f(x) \notin N_K^{\perp}(x)$ and then $D\phi(x) = \varnothing$, when x is one of the points $b, c \in K$.

Definition 1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function in $\mathbb{R}^n$. The derivative $D\phi(x)$ of function ϕ at a point $x \in K$ is given by

$$D\phi(x) = \langle \mathcal{P}_{T_K(x)}(\mathcal{A}x + I), \nabla\phi(x)\rangle, \tag{20}$$

if $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \in N_Q^{\perp}(x)$, while

$$D\phi(x) = \varnothing, \tag{21}$$

if $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \notin N_K^{\perp}(x)$.

We stress that, for any $x \in K$, $D\phi(x)$ is either the empty set or a singleton. These two different cases are illustrated in Figure 3 for a second-order FR-CNN. Moreover, if $\xi \in E$, then we have $D\phi(\xi) = 0$. Indeed, we have $\mathcal{A}\xi + I \in N_K(\xi)$, and then $\mathcal{P}_{T_Q(\xi)}(\mathcal{A}\xi + I) = 0 \in N_Q^{\perp}(\xi)$. Moreover, $\langle \mathcal{P}_{T_Q(\xi)}(\mathcal{A}\xi + I), \nabla\phi(\xi)\rangle = \langle 0, \nabla\phi(\xi)\rangle = 0$ and so $D\phi(\xi) = 0$.

Definition 2. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function in $\mathbb{R}^n$. We say that ϕ is a Lyapunov function for (F), if we have $D\phi(x) = \varnothing$ or $D\phi(x) \le 0$, for any $x \in K$. If, in addition, we have $D\phi(x) = 0$ if and only if x is an EP of (F), then ϕ is said to be a strict Lyapunov function for (F).

The next fundamental property can be proved.

Property 4. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function in $\mathbb{R}^n$, and let $x(t)$, $t \ge 0$, be a solution of (F). Then, for a.a. $t \ge 0$ we have

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t)). \tag{22}$$

If ϕ is a Lyapunov function for (F), then for a.a. $t \ge 0$ we have

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t)) \le 0, \tag{23}$$

hence $\phi(x(t))$ is a nonincreasing function for $t \ge 0$, and there exists the $\lim_{t\to+\infty}\phi(x(t)) = \phi_\infty > -\infty$.

Proof. The function $\phi(x(t))$, $t \ge 0$, is absolutely continuous on any compact interval in $[0, +\infty)$, since it is the composition of a continuously differentiable function ϕ and an absolutely continuous function x. Then, for a.a. $t \ge 0$ we have that $x(\cdot)$ and $\phi(x(\cdot))$ are differentiable at t. By [12, page 266, Proposition 2] we have that for a.a. $t \ge 0$

$$\dot{x}(t) \in \mathcal{P}_{T_K(x(t))}(\mathcal{A}x(t) + I). \tag{24}$$

Let $t > 0$ be such that x is differentiable at t. Let us show that $\dot{x}(t) \in N_K^{\perp}(x(t))$. Let $h > 0$, and note that since $x(t)$ and $x(t+h)$ belong to K, we have

$$\operatorname{dist}(x(t) + h\dot{x}(t), K) \le \|x(t) + h\dot{x}(t) - x(t+h)\|. \tag{25}$$

Dividing by h, and accounting for the differentiability of x at time t, we obtain

$$\lim_{h\to 0^+}\frac{\operatorname{dist}(x(t) + h\dot{x}(t), K)}{h} = 0, \tag{26}$$

and hence we have $\dot{x}(t) \in T_K(x(t))$.

Now, suppose that $h \in (-t, 0)$. Since once more $x(t)$ and $x(t+h)$ belong to K, we have

$$\begin{aligned} 0 &\le \frac{\operatorname{dist}(x(t) + (-h)(-\dot{x}(t)), K)}{-h} \\ &\le \frac{\|(x(t) + h\dot{x}(t) - x(t+h)\|}{-h}. \end{aligned} \tag{27}$$

Let $\rho = -h$. Then,

$$\lim_{\rho\to 0^+}\frac{\operatorname{dist}(x(t) + \rho(-\dot{x}(t)), K)}{\rho} = 0, \tag{28}$$

and hence, by definition, $-\dot{x}(t) \in T_K(x(t))$. Now, it suffices to observe that $T_K(x) \cap (-T_K(x)) = N_K^{\perp}(x)$ for any $x \in K$. In fact, if $v \in T_K(x) \cap (-T_K(x))$ and $p \in N_K(x)$, then $\langle v, p\rangle \le 0$ and $\langle -v, p\rangle \le 0$. This means that $\langle v, p\rangle = 0$, that is, $v \in N_K^{\perp}(x)$. Conversely, if $v \in N_K^{\perp}(x)$ and $p \in N_K(x)$, then we have $\langle v, p\rangle = 0$ and $\langle -v, p\rangle = 0$. Hence $v \in T_K(x) \cap (-T_K(x))$.

For a.a. $t \ge 0$ we have

$$\begin{aligned} \frac{d}{dt}\phi(x(t)) &= \langle \dot{x}(t), \nabla\phi(x(t))\rangle \\ &= \langle \mathcal{P}_{T_K(x(t))}(\mathcal{A}x(t) + I), \nabla\phi(x(t))\rangle, \end{aligned} \tag{29}$$

and hence, by Definition 1,

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t)). \tag{30}$$

Now, suppose that ϕ is a Lyapunov function for (F). Then, for a.a. $t \geq 0$ we have

$$\frac{d}{dt}\phi(x(t)) = D\phi(x(t)) \leq 0, \tag{31}$$

and hence $\phi(x(t))$, $t \geq 0$, is a monotone nonincreasing function. Moreover, ϕbeing a continuous function, it attains a minimum over the compact set K. Since we have $x(t) \in K$ for all $t \geq 0$, the function $\phi(x(t))$, $t \geq 0$, is bounded from below, and there exists the $\lim_{t\to+\infty}\phi(x(t)) = \phi_\infty > -\infty$. □

It is important to stress that, as in the standard Lyapunov approach for differential equations, $D\phi$ permits to evaluate $d\phi(x(t))/dt$ for a.a. $t \geq 0$ directly from the vector field $\mathcal{A}x+I$, without involving integrations of (F) (see Property 4).

We are now in a position to prove the next extended version of LaSalle's invariance principle for FR-CNNs.

Theorem 1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function in $\mathbb{R}^n$, which is a Lyapunov function for* (F). *Let $Z = \{x \in K : D\phi(x) = 0\}$, and let M be the largest positively invariant subset of* (F) *in* cl(Z). *Then, any solution $x(t)$, $t \geq 0$, of* (F) *converges to M as $t \to +\infty$, that is,* $\lim_{t\to+\infty} dist(x(t), M) = 0$.

Proof. Consider the differential inclusion

$$\dot{x} \in F_r(x) = \mathcal{A}x + I - [N_K(x) \cap \mathrm{cl}(B(0,r))], \tag{32}$$

where $+\infty > r > \sup_K \|\mathcal{A}x + I\|$ and F_r from K into $\mathbb{R}^n$ is an upper-semicontinuous set-valued map with nonempty compact convex values. By [11, Proposition 5] we have that if $x(t)$, $t \geq 0$, is a solution of (F), then x is also a solution of (32) for $t \geq 0$.

Denote by ω_x the ω-limit set of the solution $x(t)$, $t \geq 0$, that is, the set of points $y \in \mathbb{R}^n$ such that there exists a sequence $\{t_k\}$, with $t_k \to +\infty$ as $k \to +\infty$, such that $\lim_{k\to+\infty} x(t_k) = y$. It is known that ω_x is a nonempty compact connected subset of K, and $x(t) \to \omega_x$ as $t \to +\infty$ [18, pages 129, 130]. Furthermore, due to the uniqueness of the solution with respect to the initial conditions (Property 2), ω_x is positively invariant for the solutions of (F) [18, pages 129, 130].

Now, it suffices to show that $\omega_x \subseteq M$. It is known from Property 4 that $\phi(x(t))$, $t \geq 0$, is a nonincreasing function on $[0, +\infty)$ and $\phi(x(t)) \to \phi(\infty) > -\infty$ as $t \to +\infty$. For any $y \in \omega_x$, there exists a sequence $\{t_k\}$, with $t_k \to +\infty$ as $k \to +\infty$, such that $x(t_k) \to y$ as $k \to +\infty$. From the continuity of ϕ, we have $\phi(y) = \lim_{t_k\to+\infty}\phi(x(t_k)) = \phi(\infty)$, hence ϕ is constant on ω_x.

Let $y_0 \in \omega_x$ and let $y(t)$, $t \geq 0$, be the solution of (F) such that $y(0) = y_0$. Since ω_x is positively invariant, we have $y(t) \subseteq \omega_x$ for $t \geq 0$. It follows that $\phi(y(t)) = \phi(\infty)$ for $t \geq 0$ and hence, by Property 4, for a.a. $t \geq 0$ we have $0 = d\phi(y(t))/dt = D\phi(y(t))$. This means that $y(t) \in Z$ for a.a. $t \geq 0$. Hence, $y(t) \in \mathrm{cl}(Z)$ for all $t \geq 0$. In fact, if we had $y(t^*) \notin \mathrm{cl}(Z)$ for some $t^* \geq 0$, then we could find $\delta > 0$ such that $y([t^*, t^* + \delta)) \cap Z = \varnothing$, which is a contradiction. Now, note that in particular we have $y_0 = y(0) \in \mathrm{cl}(Z)$. y_0 being an arbitrary point of ω_x, we conclude that $\omega_x \subset \mathrm{cl}(Z)$. Finally, since ω_x is positively invariant, it follows that $\omega_x \subseteq M$. □

5. Convergence of Symmetric FR-CNNs

In this section, we exploit the extended LaSalle's invariance principle in Theorem 1 in order to prove convergence of FR-CNNs with a symmetric neuron interconnection matrix.

Definition 3. The FR-CNN (F) is said to be quasiconvergent if we have $\lim_{t\to+\infty} \mathrm{dist}(x(t), E) = 0$ for any solution $x(t)$, $t \geq 0$, of (F). Moreover, (F) is said to be convergent if for any solution $x(t)$, $t \geq 0$, of (F) there exists an EP ξ such that $\lim_{t\to+\infty} x(t) = \xi$.

Suppose that $A = A'$ is a symmetric matrix, and consider for (F) the (candidate) quadratic Lyapunov function

$$\phi(x) = -\frac{1}{2}x'\mathcal{A}x - x'I, \tag{33}$$

where $x \in \mathbb{R}^n$.

Property 5. If $A = A'$, then for function ϕ as in (33) we have

$$D\phi(x) = -\left\|\mathcal{P}_{T_K(x)}(\mathcal{A}x + I)\right\|^2 \leq 0, \tag{34}$$

if $P_{T_K(x)}(\mathcal{A}x + I) \in N_K^\perp(x)$, while

$$D\phi(x) = \varnothing, \tag{35}$$

if $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \notin N_K^\perp(x)$. Furthermore, $\mathcal{D}\phi(x) = 0$ if and only if x is an EP of (F), that is, ϕ is a strict Lyapunov function for (F).

Proof. Let $x \in K$ and suppose that $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \in N_K^\perp(x)$. Observe that $\nabla\phi(x) = -(\mathcal{A}x + I)$. Moreover, since $N_K(x)$ is the negative polar cone of $T_K(x)$ [12, page 220, Proposition 2], we have [12, page 26, Proposition 3]

$$\mathcal{A}x + I = \mathcal{P}_{T_K(x)}(\mathcal{A}x + I) + \mathcal{P}_{N_K(x)}(\mathcal{A}x + I), \tag{36}$$

with $\langle \mathcal{P}_{T_K(x)}(\mathcal{A}x + I), \mathcal{P}_{N_K(x)}(\mathcal{A}x + I)\rangle = 0$.

Accounting for Definition 1, we have

$$\begin{aligned} D\phi(x) &= \langle \mathcal{P}_{T_K(x)}(\mathcal{A}x + I), \nabla\phi(x)\rangle \\ &= \langle \mathcal{P}_{T_K(x)}(\mathcal{A}x + I), -\mathcal{P}_{T_K(x)}(\mathcal{A}x + I)\rangle \\ &\quad + \langle \mathcal{P}_{T_K(x)}(\mathcal{A}x + I), -\mathcal{P}_{N_K(x)}(\mathcal{A}x + I)\rangle \\ &= -\left\|\mathcal{P}_{T_K(x)}(\mathcal{A}x + I)\right\|^2 \leq 0. \end{aligned} \tag{37}$$

Hence, ϕ is a Lyapunov function for (F). It remains to show that it is strict. If x is an EP of (F), then we have $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) = 0$ and hence $D\phi(x) = 0$. Conversely, if $D\phi(x) = 0$, then we have $\|\mathcal{P}_{T_K(x)}(\mathcal{A}x + I)\| = 0$. Thus, x is an EP for (F). □

Property 5 and Theorem 1 yield the following.

Theorem 2. *Suppose that $A = A'$. Then,* (F) *is quasiconvergent, and it is convergent if the EPs of* (F) *are isolated.*

Proof. Since ϕ is a strict Lyapunov function for (F), we have $Z = E$. Let M be the largest positively invariant set of (F) contained in Z. Due to the uniqueness of the solutions for (F) (Property 2), it follows that $E \subseteq M$. On the other hand,

E is a closed set and hence $E = \mathrm{cl}(E) = \mathrm{cl}(Z) \supseteq M$. In conclusion, $M = E$. Then, Theorem 1 implies that any solution $x(t)$, $t \geq 0$, of (F) converges to E as $t \to +\infty$. Hence (F) is quasiconvergent. Suppose in addition that the equilibrium points of (F) are isolated. Observe that ω_x is a connected subset of $M = E$. This implies that there exists $\xi \in E$ such that $\omega_x = \xi$. Since $x(t) \to \omega_x$, we have $x(t) \to \xi$ as $t \to +\infty$. □

6. Remarks and Discussion

Here, we discuss the significance of the result in Theorem 2 by comparing it with existing results in the literature on convergence of FR-CNNs and S-CNNs. Furthermore, we briefly discuss the possible extensions of the proposed Lyapunov approach to neural network models described by more general classes of differential inclusions.

(1) Theorem 2 coincides with the result on convergence obtained in [11, Theorem 1]. In what follows we point out some advantages with respect to that paper. It is stressed that the proof of Theorem 2 is a direct consequence of the extended version of LaSalle's invariance principle in this paper. The proof of [11, Theorem 1], which is not based on an invariance principle, is comparatively more complex, and in particular it requires an elaborate analysis of the behavior of the solutions of (F) close to the set of equilibrium points of (F). Also the mathematical machinery employed in [11] is more complex than that in the present paper. In fact, in [11] use is made of extended Lyapunov functions assuming the value $+\infty$ outside K and a generalized version of the chain-rule for computing the derivative of the extended-valued functions along the solutions of (F). Here, instead, we have analyzed convergence of (F) by means of a simple quadratic Lyapunov function as in (33).

(2) Consider the S-CNN model (S) and suppose that the neuron interconnection matrix $A = A'$ is symmetric. It has been shown in [5] that (S) admits the Lyapunov function:

$$\psi(x) = -\frac{1}{2}G'(x)(A - E_n)G(x) - G'(x)I, \tag{38}$$

where $x \in \mathbb{R}^n$. One key problem is that ψ *is not a strict Lyapunov function* for the symmetric S-CNN (S), since in partial and total saturation regions of (S) the time derivative of ψ along solutions of (S) may vanish in sets of points that are larger than the sets of equilibrium points of (S). Then, in order to prove quasiconvergence or convergence of (S), it is needed to investigate the geometry of the largest invariant sets of (S) where the time derivative of ψ along solutions of (S) vanishes [7]. Such an analysis is quite elaborate and complex (see [19] for the details). It is worth to remark once more that, according to Theorem 2, ϕ as in (33) *is a strict Lyapunov function* for a symmetric FR-CNN, hence the proof of quasiconvergence or convergence of (F) is a direct consequence of the generalized version of LaSalle's invariance principle in this paper.

(3) The derivative $D\phi$ in Definition 1 and the extended version of LaSalle's invariance principle in Theorem 1 have been inspired by analogous concepts previously developed by Shevitz and Paden [20] and later improved by Bacciotti and Ceragioli [21].

Next, we briefly compare the derivative $D\phi$ with the derivative $\widetilde{D}\phi$ proposed in [21]. If we consider that ϕ is continuously differentiable in $\mathbb{R}^n$, then we have

$$\widetilde{D}\phi(x) = \{\langle v, \nabla\phi(x)\rangle,\ v \in \mathcal{A}x + I - N_K(x)\}, \tag{39}$$

for any $x \in K$. Note that $\widetilde{D}\phi$ is in general set valued, that is, it may assume an entire interval of values. Since $\mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \cap N_K^{\perp}(x) \subseteq \mathcal{P}_{T_K(x)}(\mathcal{A}x + I) \subseteq \mathcal{A}x + I - N_K(x)$, we have

$$D\phi(x) \subseteq \widetilde{D}\phi(x), \tag{40}$$

for any $x \in K$. An analogous inclusion holds when comparing $D\phi$ with the derivative in [20].

Consider now the following second-order symmetric FR-CNN:

$$\dot{x} = -x + Ax + I - N_K(x) = f(x) - N_K(x), \tag{41}$$

where $x = (x_1, x_2)' \in \mathbb{R}^2$,

$$A = \begin{pmatrix} 2 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}, \qquad I = \begin{pmatrix} 0 \\ \frac{2}{3} \end{pmatrix}, \tag{42}$$

whose solutions evolve in the square $K = [-1, 1]^2$. Also consider the candidate Lyapunov function ϕ given in (33), namely,

$$\phi(x) = -\frac{1}{2}x'\mathcal{A}x - I'x = -\frac{1}{2}x_1(x_1 - x_2) - \frac{2}{3}x_2. \tag{43}$$

Simple computations show that, for any $x = (x_1, x_2)' \in K$ such that $x_2 = 1$, it holds $\mathcal{P}_{T_K(x)}f(x) \in N_K^{\perp}(x)$. As a consequence, if a solution of the FR-CNN (41) passes through a point belonging to the upper edge of K, then the solution will slide along that edge during some time interval.

Now, consider the point $x^* = (0, 1)'$, lying on the upper edge of K. We have $f(x^*) = (-1/2, 2/3)'$, $\nabla\phi(x^*) = -f(x^*) = (1/2, -2/3)'$ and, from Definition 1,

$$\begin{aligned} D\phi(x^*) &= \langle P_{T_K(x^*)}(f(x^*)), \nabla V(x^*)\rangle \\ &= \left\langle \left(-\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{2}{2}\right)\right\rangle = -\frac{1}{4} < 0. \end{aligned} \tag{44}$$

On the other hand, we obtain

$$\begin{aligned} \widetilde{D}\phi(x^*) &= \{\langle v, \nabla\phi(x^*)\rangle,\ v \in f(x^*) - N_K(x^*)\} \\ &= \left[-\frac{25}{36}, +\infty\right). \end{aligned} \tag{45}$$

It is seen that $\widetilde{D}\phi(x^*)$ assume both positive and negative values; see Figure 4 for a geometric interpretation.

Therefore, by means of the derivative $D\phi$ we can conclude that ϕ as in (33) is a Lyapunov function for the FR-CNN, while it cannot be concluded that ϕ is a Lyapunov function for the FR-CNN using the derivative $\widetilde{D}\phi$.

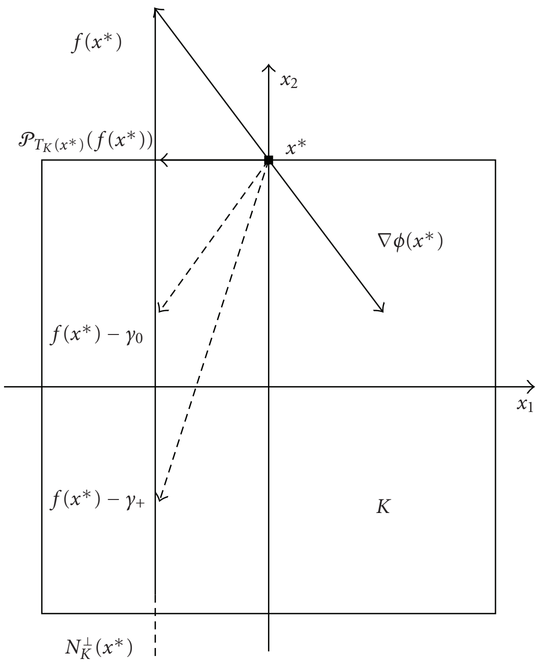

Figure 4: Comparison between the derivative $D\phi$ in Definition 1, and the derivative $\widetilde{D}\phi$ in [21], for the second-order FR-CNN (41). The point $x^* = (0,1)'$ lies on an edge of K such that $T_K(x^*) = \{(x_1,x_2) \in \mathbb{R}^2 : -\infty < x_1 < +\infty,\ x_2 \leq 0\}$, $N_K(x^*) = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 = 0,\ x_2 \geq 0\}$ and $N_K^\perp(x^*) = \{(x_1,x_2) \in \mathbb{R}^2 : -\infty < x_1 < +\infty,\ x_2 = 0\}$. We have $\mathcal{P}_{T_K(x^*)} f(x^*) \in N_K^\perp(x)$ and $D\phi(x^*) = \langle \mathcal{P}_{T_K(x^*)} f(x^*), \nabla\phi(x^*)\rangle = -1/4 < 0$. The derivative $\widetilde{D}\phi(x^*)$ is given by $\widetilde{D}\phi(x^*) = \{\langle v, \nabla\phi(x^*)\rangle,\ v \in f(x^*) - N_K(x^*)\} = [-25/36, +\infty)$, hence it assumes both positive and negative values. For example, the figure shows a vector $\gamma_0 \in N_K(x^*)$ such that we have $\widetilde{D}\phi(x^*) \ni 0 = \langle f(x^*) - \gamma_0, \nabla\phi(x^*)\rangle$, and a vector $\gamma_+ \in N_K(x^*)$ for which we have $\widetilde{D}\phi(x^*) \ni \langle f(x^*) - \gamma_+, \nabla\phi(x^*)\rangle > 0$.

(4) The Lyapunov approach in this paper has been developed in relation to the differential inclusion modeling the FR model of CNNs, that is, a class of DVIs (16) where the dynamics defined by an affine vector field $\mathcal{A}x + I$ are constrained to evolve within the hypercube $K = [-1,1]^n$. The approach can be generalized to a wider class of DVIs, by substituting K with an arbitrary compact convex set $Q \subset \mathbb{R}^n$, or by substituting the affine vector field with a more general (possibly nonsmooth) vector field. In the latter case, it is needed to use nondifferentiable Lyapunov functions and a generalized nonsmooth version of the derivative given in Definition 1. The details on these extensions can be found in the recent paper [13].

7. Conclusion

The paper has developed a generalized Lyapunov approach, which is based on an extended version of LaSalle's invariance principle, for studying stability and convergence of the FR model of CNNs. The approach has been applied to give a rigorous proof of convergence for symmetric FR-CNNs.

The results obtained have shown that, by means of the developed Lyapunov approach, the analysis of convergence of symmetric FR-CNNs is much more simple than that of the symmetric S-CNNs. In fact, one basic result proved here is that a symmetric FR-CNN admits a strict Lyapunov function, and thus it is convergent as a direct consequence of the extended version of LaSalle's invariance principle.

Future work will be devoted to investigate the possibility to apply the proposed methodology for addressing stability of other classes of FR-CNNs that are used in the solution of signal processing tasks in real time. Particular attention will be devoted to certain classes of FR-CNNs with nonsymmetric interconnection matrices. Another interesting issue is the possibility to extend the approach in order to consider the presence of delays in the FR-CNN neuron interconnections.

Appendices

A. Proof of Property 3

Let $M_i = \sum_{j=1}^{n}(|A_{ij}| + |I_i|)$, $i = 1,2,\ldots,n$, and $M = \max\{M_1, M_2, \ldots, M_n\} \geq 0$. We have $\|Ax + I\|_\infty \leq M + 1$ for all $x \in K$.

We need to define the following maps. For $k = 1,2,3,\ldots$, let $H_k(x) = (h_k(x_1), h_k(x_2), \ldots, h_k(x_n))'$, $x \in \mathbb{R}^n$, where

$$h_k(\rho) = \begin{cases} -M-1, & \text{if } \rho < -1 - \dfrac{(M+1)}{k}, \\ k\ell(\rho), & \text{if } |\rho| \leq 1 + \dfrac{(M+1)}{k}, \\ M+1, & \text{if } \rho > 1 + \dfrac{(M+1)}{k}, \end{cases} \tag{A.1}$$

and $\ell(\cdot)$ is defined in (13). Then, let $H_\infty(x) = (h_\infty(x_1), h_\infty(x_2), \ldots, h_\infty(x_n))'$, $x \in \mathbb{R}^n$, where

$$h_\infty(\rho) = \begin{cases} -(M+1), & \text{if } \rho < -1, \\ [-M-1, 0], & \text{if } \rho = -1, \\ 0, & \text{if } |\rho| < 1, \\ [0, M+1], & \text{if } \rho = 1, \\ M+1, & \text{if } \rho > 1. \end{cases} \tag{A.2}$$

Finally, let $B_M(x) = (b_m(x_1), b_m(x_2), \ldots, b_m(x_n))'$, $x \in \mathbb{R}^n$, where

$$b_m(\rho) = \begin{cases} -(M+1), & \text{if } \rho < -(1+M), \\ \rho, & \text{if } |\rho| \leq 1+M, \\ M+1, & \text{if } \rho > 1+M. \end{cases} \tag{A.3}$$

The three maps h_m, h_∞ and b_m are represented in Figure 5.

The proof of Property 3 consists of the three main steps detailed below.

Step 1. Let $x(t)$, $t \geq 0$, be the solution of (F) such that $x(0) = x_0 \in K$. We want to verify that x is also a solution of

$$\dot{x}(t) \in -B_M(x(t)) + AG(x(t)) + I - H_\infty(x(t)), \tag{A.4}$$

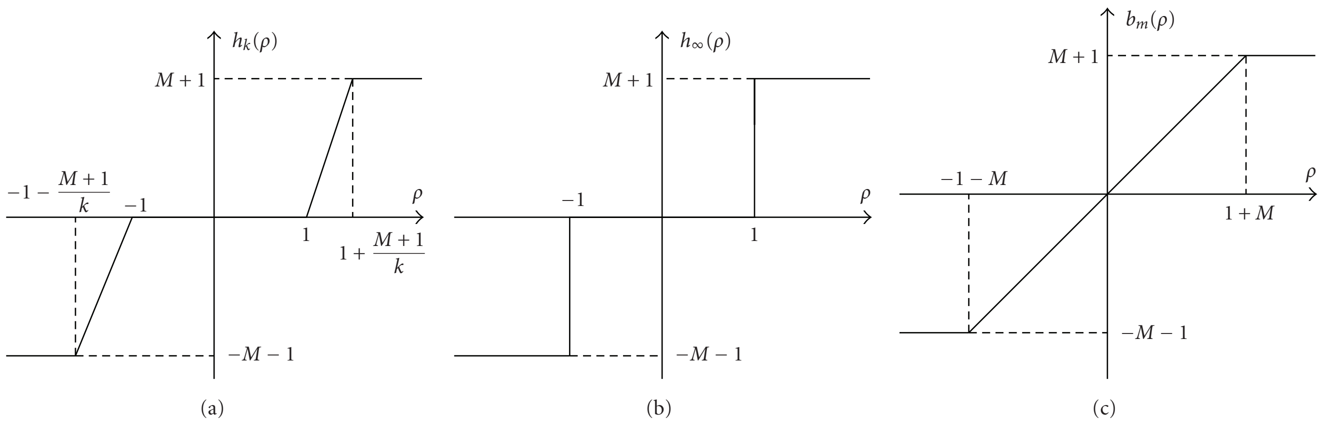

Figure 5: Auxiliary maps (a) h_m, (b) h_∞, and (c) b_m employed in the proof of Property 3.

for $t \geq 0$, where $G(x) = (g(x_1), g(x_2), \dots, g(x_n))'$, $x \in \mathbb{R}^n$, and $g(\cdot)$ is given in (11).

On the basis of [12, page 266, Proposition 2], for a.a. $t \geq 0$ we have

$$\dot{x}(t) = \mathcal{P}_{T_K(x(t))}(\mathcal{A}x(t) + I)) \\ = -x(t) + Ax(t) + I - \mathcal{P}_{N_K(x(t))}(\mathcal{A}x(t) + I), \tag{A.5}$$

where $\mathcal{P}_{N_K(x(t))}(\mathcal{A}x(t) + I) \in N_K(x(t))$ [12, page 24, Proposition 2; page 26, Proposition 3]. Since for any $t \geq 0$ we have $\|\mathcal{A}x(t) + I\|_\infty \leq M + 1$, by applying the result in Lemma 1 in Appendix B, we obtain $\mathcal{P}_{N_K(x(t))}(\mathcal{A}x(t) + I) \in H_\infty(x(t))$. Furthermore, considering that for any $t \geq 0$ we have $x(t) \in K$, it follows that $B_M(x(t)) = x(t) = G(x(t))$. In conclusion, for a.a. $t \geq 0$ we have

$$\dot{x}(t) \in -B_M(x(t)) + AG(x(t)) + I - H_\infty(x(t)). \tag{A.6}$$

Step 2. For any $k = 1, 2, 3, \dots$, let $x_k(t)$, $t \geq 0$, be the solution of (I) such that $x_k(0) = x_0 \in K$. We want to show that x_k is also a solution of

$$\dot{x}(t) \in -B_M(x(t)) + AG(x(t)) + I - H_k(x(t)), \tag{A.7}$$

for $t \geq 0$. For any $i \in \{1, 2, \dots, n\}$ and $t \geq 0$ we have from [2, equation 12]

$$\begin{aligned} |x_i^k(t)| &\leq \frac{M+k}{k+1} + \left(1 - \frac{M+k}{k+1}\right)\exp(-(k+1)t) \\ &= 1 + \frac{M-1}{k+1}(1 - \exp(-(k+1)t)) \\ &\leq 1 + \frac{|M-1|}{k+1} \leq 1 + \min\left\{M, \frac{M+1}{k}\right\}. \end{aligned} \tag{A.8}$$

Then, $B_M(x(t)) = x(t) = G(x(t))$ and $H_k(x(t)) = kL(x(t))$, for $t \geq 0$.

Step 3. Consider the map $\Phi_\infty(x) = -B_M(x) + AG(x) + I - H_\infty(x)$, $x \in \mathbb{R}^n$, and for $k = 1, 2, 3, \dots$, the maps $\Phi_k(x) = -B_M(x) + AG(x) + I - H_k(x)$, $x \in \mathbb{R}^n$, which are upper semicontinuous in $\mathbb{R}^n$ with nonempty compact convex values.

Let $\text{graph}(H_\infty) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y = H_\infty(x)\}$ and $\text{graph}(H_k) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y = H_k(x)\}$. Given any $\delta > 0$, for sufficiently large k, say $k > k_\delta$, we have

$$\text{graph}(H_k) \subseteq \text{graph}(H_\infty) + B(0, \delta). \tag{A.9}$$

By applying [12, page 105, Proposition 1] it follows that for any $\tilde{\epsilon} > 0$, $T > 0$, and for any $k > k_\delta$, there exists a solution $\hat{x}_k(t)$, $t \in [0, T]$, of (A.4), such that $\max_{[0,T]} \|\hat{x}_k(t) - x_k(t)\| < \tilde{\epsilon}$.

Choose $\tilde{\epsilon} = \epsilon \exp(-\|A\|_2 T/2)/2$, where $\epsilon > 0$, $\|A\|_2 = (\lambda_M(A'A))^{1/2}$ and $\lambda_M(A'A)$ denotes the maximum eigenvalue of the symmetric matrix $A'A$. Then, we obtain

$$\begin{aligned} \|\hat{x}_k(0) - x(0)\| &= \|\hat{x}_k(0) - x_k(0)\| \\ &\leq \max_{[0,T]} \|\hat{x}_k(t) - x_k(t)\| \\ &< \frac{\epsilon}{2} \exp(-\|A\|_2 T). \end{aligned} \tag{A.10}$$

By Property 6 in Appendix C we have $\max_{[0,T]} \|\hat{x}_k(t) - x(t)\| < \epsilon/2$. Then,

$$\begin{aligned} \max_{[0,T]} \|x_k(t) - x(t)\| &\leq \max_{[0,T]} \|\hat{x}_k(t) - x(t)\| \\ &\quad + \max_{[0,T]} \|\hat{x}_k(t) - x_k(t)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned} \tag{A.11}$$

for all $t \in [0, T]$.

B. Lemma 1 and Its Proof

Lemma 1. *For any $x \in K$, and any $v \in \mathbb{R}^n$ such that $\|v\|_\infty \leq M + 1$, we have $\mathcal{P}_{N_K(x)}(v) \in H_\infty(x)$.*

Proof. For any $i \in \{1, 2, \dots, n\}$ we have

$$[\mathcal{P}_{N_K(x)}(v)]_i = \begin{cases} v_i, & \text{if } |x_i| = 1,\ x_i v_i > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{B.1}$$

If $\mathcal{P}_{N_K(x)}(v)_i = 0$, we immediately obtain $[\mathcal{P}_{N_K(x)}(v)]_i \in h_\infty(x_i)$. If $x_i = 1$ and $x_i v_i > 0$, we may proceed as follows. We have $h_\infty(x_i) = h_\infty(1) = [0, M+1]$. On the other hand, $0 < v_i \leq M+1$ and so $[\mathcal{P}_{N_K(x)}(v)]_i = v_i \in [0, M+1] = h_\infty(x_i)$. We can proceed in a similar way in the case $x_i = -1$ and $x_i v_i > 0$. □

C. Property 6 and Its Proof

Property 6. Let $\epsilon > 0$. For any $y_0, z_0 \in \mathbb{R}^n$ such that

$$\|z_0 - y_0\| < \epsilon \exp\left(-\frac{\|A\|_2 T}{2}\right), \tag{C.1}$$

we have $\max_{[0,T]}\|z(t) - y(t)\| < \epsilon$, where y and z are the solutions of (A.4) such that $y(0) = y_0$ and $z(0) = z_0$, respectively.

Proof. Let $\varphi(t) = \|z(t) - y(t)\|^2/2$, $t \in [0, T]$. Due to (C.1), for a.a. $t \in [0, T]$ we have

$$\begin{aligned} \dot{\varphi}(t) &= \langle z(t) - y(t), \dot{z}(t) - \dot{y}(t)\rangle \\ &= -\langle z(t) - y(t), B_M(z(t)) - B_M(y(t))\rangle \\ &\quad + \langle z(t) - y(t), A(G(z(t)) - G(y(t)))\rangle \\ &\quad - \langle z(t) - y(t), \gamma_y(t) - \gamma_z(t)\rangle, \end{aligned} \tag{C.2}$$

where $\gamma_y(t) \in H_\infty(y(t))$ and $\gamma_z(t) \in H_\infty(z(t))$. It is seen that B_M is a monotone map in $\mathbb{R}^n$ [12, page 159, Proposition 1], that is, for any $x, y \in \mathbb{R}^n$ and any $\gamma_x \in B_M(x)$, $\gamma_y \in B_M(y)$, we have $\langle x - y, \gamma_x - \gamma_y\rangle \geq 0$. Also H_∞ is a monotone map in $\mathbb{R}^n$. Then, we obtain

$$\begin{aligned} \dot{\varphi}(t) &\leq \langle z(t) - y(t), A(G(z(t)) - G(y(t)))\rangle \\ &\leq \|A\|\,\|z(t) - y(t)\|^2 \\ &= 2\|A\|\varphi(t). \end{aligned} \tag{C.3}$$

Gronwall's lemma yields $\varphi(t) \leq \varphi(0)e^{\|A\|T}$, and so

$$\|z(t) - y(t)\| = \sqrt{2\varphi(t)} \leq \sqrt{2\varphi(0)e^{\|A\|T}} < \epsilon, \tag{C.4}$$

for $t \in [0, T]$. □

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