Research Article
Budget Allocation in a Competitive Communication Spectrum Economy

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This study discusses how to adjust "monetary budget" to meet each user's physical power demand, or balance all individual utilities in a competitive "spectrum market" of a communication system. In the market, multiple users share a common frequency or tone band and each of them uses the budget to purchase its own transmit power spectra (taking others as given) in maximizing its Shannon utility or pay-off function that includes the effect of interferences. A market equilibrium is a budget allocation, price spectrum, and tone power distribution that independently and simultaneously maximizes each user's utility. The equilibrium conditions of the market are formulated and analyzed, and the existence of an equilibrium is proved. Computational results and comparisons between the competitive equilibrium and Nash equilibrium solutions are also presented, which show that the competitive market equilibrium solution often provides more efficient power distribution.

1. Introduction

The competitive economy equilibrium problem of a communication system consists of finding a set of prices and distributions of frequency or tone power spectra to users such that each user maximizes his/her utility, subject to his/her budget constraints, and the limited power bandwidth resource is efficiently utilized. Although the study of the competitive equilibrium can date back to Walras [1] work in 1874, the concepts applied to a communication system just emerged few years ago because of the great advances in communication technology recently. In a modern communication system such as cognitive radio or digital subscriber lines (DSL), users share the same frequency band and how to mitigate interference is a major design and management concern. The Frequency Division Multiple Access (FDMA) mechanism is a standard approach to eliminate interference by dividing the spectrum into multiple tones and preassigning them to the users on a nonoverlapping basis.

However, this approach may lead to high system overhead and low bandwidth utilization. Therefore, how to optimize users' utilities without sacrificing the bandwidth utilization through spectrum management becomes an important issue. That is why the spectrum management problem has recently become a topic of intensive research in the signal processing and digital communication community.

From the optimization perspective, the problem can be formulated either as a noncooperative Nash game [2–5]; or as a cooperative utility maximization problem [6, 7]. Several algorithms were proposed to compute a Nash equilibrium solution (Iterative Waterfilling Algorithm (IWFA) [2, 4]; Linear Complementarity Problem (LCP) [3]) or globally optimal power allocations (Dual decomposition method, [8–10]) for the cooperative game. Due to the problem's nonconvex nature, these algorithms either lack global convergence or may converge to a poor spectrum sharing strategy. Moreover, the Nash equilibrium solution may not achieve social communication economic efficiency; and, on

the other hand, an aggregate social utility (i.e., the sum of all users' utilities) maximization model may not simultaneously optimize each user's individual utility.

Recently, Ye [11] proposed a competitive economy equilibrium solution that may achieve both social economic efficiency and individual optimality in dynamic spectrum management. He proved that a competitive equilibrium always exists for the communication spectrum market with Shannon utility for spectrum users, and under a weak-interference condition the equilibrium can be computed in a polynomial time. In [11], Ye assumes that the budget is fixed, but this paper deals how adjusting the budget can further improve the social utility and/or meet each individual physical demand. This adds another level of resource control to improve spectrum utilization.

This study investigates how to allocate budget between users to meet each user's physical power demand or balance all individual utilities in the competitive communication spectrum economy. We prove what follows.

(1) A competitive equilibrium that satisfies each user's physical power demand always exists for the communication spectrum market with Shannon utilities if the total power demand is less than or equal to the available total power supply.

(2) A competitive equilibrium where all users have identical utility value always exists for the communication spectrum market with Shannon utilities.

Computational results and comparisons between the competitive equilibrium and Nash equilibrium solutions are also presented. The simulation results indicate that the competitive economy equilibrium solution provides more efficient power distribution to achieve a higher social utility in most cases. Besides, the competitive economy equilibrium solution can make more users to obtain higher individual utilities than the Nash equilibrium solution does in most cases. Moreover, the competitive economy equilibrium takes the power supply capacity of each channel into account, while the Nash equilibrium model assumes the supply unlimited where each user just needs to satisfy its power demand.

The remainder of this paper is organized as follows. The mathematical notations are illustrated in Section 2. Section 3 describes the competitive communication spectrum market considered in this study. Section 4 formulates two competitive equilibrium models that address budget allocation on satisfying power demands and budget allocation on balancing individual utilities. Section 5 demonstrates a toy example of two users and two channels. Section 6 describes how to solve the market equilibrium and presents computational results. Finally, conclusions are made in the last section.

2. Mathematical Notations

First, a few mathematical notations. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space; $\mathbf{R}^n_+$ denote the subset of $\mathbf{R}^n$ where each coordinate is nonnegative. $\mathbf{R}$ and $\mathbf{R}_+$ denote the set of real numbers and the set of nonnegative real numbers, respectively.

Let $X \in \mathbf{R}^{mn}_+$ denote the set of ordered m-tuples $X = (\mathbf{x}_1, \ldots, \mathbf{x}_m)$ and let $\overline{X}_i \in \mathbf{R}^{(m-1)n}_+$ denote the set of ordered $(m-1)$-tuples $\overline{X} = (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_m)$, where $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in X_i \subset \mathbf{R}^n_+$ for $i = 1, \ldots, m$. For each i, suppose there is a real utility function u_i, defined over X. Let $A_i(\overline{\mathbf{x}}_i)$ be a subset of $\overline{X}_i$ defining for each point $\overline{\mathbf{x}}_i \in \overline{X}_i$, then the sequence $[X_1, \ldots, X_m, u_1, \ldots, u_m, A_1(\overline{\mathbf{x}}_1), \ldots, A_m(\overline{\mathbf{x}}_m)]$ will be termed an abstract economy. Here $A_i(\overline{\mathbf{x}}_i)$ represents the feasible action set of agent i that is possibly restricted by the actions of others, such as the budget restraint that the cost of the goods chosen at current prices dose not exceed his income, and the prices and possibly some or all of the components of his income are determined by choices made by other agents. Similarly, utility function $u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i)$ for agent i depends on his or her actions $\mathbf{x}_i$, as well as actions $\overline{\mathbf{x}}_i$ made by all other agents. Also, denote $\mathbf{x}_j = (x_{1j}, \ldots, x_{mj}) \in \mathbf{R}^m$ for a given $\mathbf{x} \in X$.

A function $u : \mathbf{R}^n_+ \to \mathbf{R}_+$ is said to be *concave* if for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$ and any $0 \leq \alpha \leq 1$, we have $u(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) \geq \alpha u(\mathbf{x}) + (1 - \alpha)u(\mathbf{y})$; and it is strictly concave if $u(\alpha\mathbf{x} + (1 - \alpha)\mathbf{y}) > \alpha u(\mathbf{x}) + (1 - \alpha)u(\mathbf{y})$ for $0 < \alpha < 1$. It is *monotone increasing* if for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n_+$, $\mathbf{x} \geq \mathbf{y}$ implies that $u(\mathbf{x}) \geq u(\mathbf{y})$.

3. Competitive Communication Spectrum Market

Let the multiuser communication system consist of m transmitter-receiver pairs sharing a common frequency band discretized by n tones. For simplicity, we will call each of such transmitter-receiver pair a "User." Each user i will be endowed a "monetary" budget $w_i > 0$ and use it to "purchase" powers, x_{ij}, across tones $j = 1, \ldots, n$, from an open market so as to maximize its own utility $u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i)$:

$$\begin{aligned} \text{maximize}_{\mathbf{x}_i} \quad & u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i) \\ \text{subject to} \quad & \mathbf{p}^T \mathbf{x}_i = \sum_j p_j x_{ij} \leq w_i, \\ & \mathbf{x}_i \geq \mathbf{0}, \end{aligned} \quad (1)$$

where $\mathbf{x}_i = (x_{i1}, \ldots, x_{in}) \in \mathbf{R}^n_+$ and $\overline{\mathbf{x}}_i \in \mathbf{R}^{(m-1)n}_+$ are power units purchased by all other users, and p_j is the unit price for tone j in the market.

A commonly recognized utility for user i, $i = 1, \ldots, m$, in communication is the Shannon utility [12]:

$$u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i) = \sum_{j=1}^n \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + \sum_{k \neq i} a^i_{kj} x_{kj}}\right), \quad (2)$$

where parameter σ_{ij} denotes the normalized background noise power for user i at tone j, and parameter a^i_{kj} is the normalized crosstalk ratio from user k to user i at tone j. Due to normalization we have $a^i_{ij} = 1$ for all i, j. Clearly, $u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i)$ is a continuous concave and monotone increasing function in $\mathbf{x}_i \in \mathbf{R}^n_+$ for every $\overline{\mathbf{x}}_i \in \mathbf{R}^{(m-1)n}_+$.
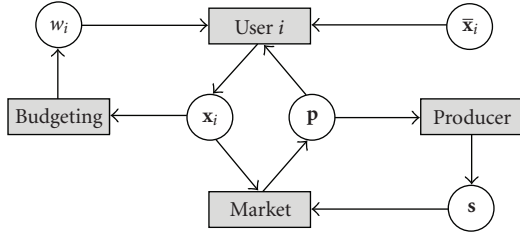
FIGURE 1: Interaction among four types of agents in the proposed competitive spectrum market.

There are four types of agents in this market. The first-type agents are users. Each user aims to maximize its own utility under its budget constraint and the decisions by all other users. The second-type agent, "Producer or Provider," who installs power capacity supply $s_j \geq 0$ to the market from a convex and compact set S to maximize his or her utility. We assume that they are fixed as $\bar{\mathbf{s}}$ in this paper, and $\sum_i d_i \leq \sum_j \bar{s}_j$, that is, the total power demand is less than or equal to the available total power supply.

The third agent, "Market," sets tone power unit "price" $p_j \geq 0$, which can be interpreted as a "preference or ranking" of tones $j = 1, \ldots, n$. For example, $p_1 = 1$ and $p_2 = 2$ simply mean that users may use one unit of $\bar{s}_2$ to trade for two units of $\bar{s}_1$.

The fourth agent, "Budgeting," allocates "monetary" budget $w_i > 0$ to user i from a bounded total budget, say $\sum_i w_i = m$.

Figure 1 illustrates the interaction among four types of agents in the proposed competitive spectrum market. Each user i determines its power allocation $\mathbf{x}_i$ under its budget w_i, power spectra unit price density $\mathbf{p}$ and the decisions by all other users $\bar{\mathbf{x}}_i$. The producer installs power capacity spectra density $\mathbf{s}$ based on power spectra unit price density $\mathbf{p}$ to maximize his or her utility. The market sets power spectra unit price density $\mathbf{p}$ based on tone power distribution $\mathbf{x}$ and power capacity spectra density $\mathbf{s}$ to make market clear. The budgeting agent allocates budget w_i to user i from a bounded total budget according to tone power distribution $\mathbf{x}$ for satisfying power demands or balancing individual utilities. In this study, we assume power capacity spectra density $\mathbf{s}$ is fixed. Therefore, $\mathbf{s}$ is determined first in the system.

4. Budget Allocation in Competitive Communication Spectrum Market

In this section, we discuss how to adjust "monetary" budget to satisfy each user's prespecified physical power demand or to balance all individual utilities in a competitive spectrum market.

4.1. Budget Allocation on Satisfying Individual Power Demands. The first question is whether or not the "Budgeting" agent can adjust "monetary budget" for each user to meet each user's desired total physical power demand d_i that may be composed from any tone combination. We give an affirmative answer in this section.

A competitive market equilibrium is a power distribution $[\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*, \mathbf{p}^*, \mathbf{w}^*]$ such that

(i) (user optimality) $\mathbf{x}_i^*$ is a maximizer of (1) given $\bar{\mathbf{x}}_i^*$, $\mathbf{p}^*$ and w_i^* for every i;

(ii) (market efficiency) $\mathbf{p}^* \geq \mathbf{0}$, $\sum_{i=1}^m x_{ij}^* \leq \bar{s}_j$, $p_j^*(\bar{s}_j - \sum_{i=1}^m x_{ij}^*) = 0$ for all j. This condition says that if tone power capacity $\bar{s}_j$ is greater than or equal to the total power consumption for tone j, $\sum_{i=1}^m x_{ij}^*$, then its equilibrium price $p_j^* = 0$;

(iii) (budgeting according to demands) given $\mathbf{x}^*$, $\mathbf{w}^*$ is a maximizer of

$$\max_{\mathbf{w}} \sum_i \left(\max \left\{ 0, d_i - \sum_j x_{ij}^* \right\} \right) w_i,$$
$$\text{s.t.} \sum_i w_i = m, \quad \mathbf{w} \geq \mathbf{0}. \tag{3}$$

This condition says that if user i's power demand is not met, that is, $d_i - \sum_j x_{ij} > 0$, then one should allocate more or all "money budget" to user i. Any budget allocation is optimal if $d_i - \sum_j x_{ij} \leq 0$ for all i, that is, if every user's physical power demand is met.

Since the "Budgeting" agent's problem is a bounded linear maximization, and all other agents' problems are identical to those in Ye [11], we have the following corollary.

Corollary 4.1. *The communication spectrum market with Shannon utilities has a competitive equilibrium that satisfies each user's tone power demand, if the total power demand is less than or equal to the available total power supply.*

Now consider the KKT conditions of (1):

$$\lambda \mathbf{p}^* - \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \bar{\mathbf{x}}_i^*) \geq \mathbf{0},$$
$$\lambda \geq 0,$$
$$\lambda \cdot \left((\mathbf{p}^*)^T \mathbf{x}_i^* - w_i^* \right) = 0,$$
$$(\mathbf{x}_i^*)^T \cdot \left(\lambda \mathbf{p}^* - \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \bar{\mathbf{x}}_i^*) \right) = 0, \tag{4}$$
$$(\mathbf{p}^*)^T \mathbf{x}_i^* - w_i^* \leq 0,$$
$$\mathbf{x}_i^* \geq \mathbf{0},$$

where $\nabla_{\mathbf{x}_i} u(\mathbf{x}_i, \bar{\mathbf{x}}_i) \in \mathbf{R}^n$ denotes any subgradient vector of $u(\mathbf{x}_i, \bar{\mathbf{x}}_i)$ with respect to $\mathbf{x}_i$.

Since $\lambda \mathbf{p}^* \geq \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \bar{\mathbf{x}}_i^*)$ and $(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i, \bar{\mathbf{x}}_i))_j = 1/\sigma_{ij} + \sum_{k \neq i} a_{kj}^i x_{kj} + x_{ij} > 0$, for all j, we have $\mathbf{p}^* > \mathbf{0}$ and $\lambda > 0$. Then, $(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i^*$. The optimality conditions in (4) can be simplified to

$$w_i^* \cdot \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \bar{\mathbf{x}}_i^*) \leq \left(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \bar{\mathbf{x}}_i^*)^T \mathbf{x}_i^* \right) \cdot \mathbf{p}^*,$$
$$(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i^*, \tag{5}$$
$$\mathbf{x}_i^* \geq \mathbf{0}.$$

The complete necessary and sufficient conditions for a competitive equilibrium with satisfied power demands can be summarized as

$$w_i^* \cdot \nabla_{\mathbf{x}_i} u_i\left(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*\right) \leq \left(\nabla_{\mathbf{x}_i} u_i\left(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*\right)^T \mathbf{x}_i^*\right) \cdot \mathbf{p}^*, \quad \forall i,$$

$$\sum_i x_{ij}^* \leq \bar{s}_j, \ \forall j,$$

$$\bar{\mathbf{s}}^T \mathbf{p}^* \leq \sum_i w_i^*,$$

$$\max\left\{0, \ d_i - \sum_j x_{ij}^*\right\} - \lambda \leq 0, \quad \forall i,$$

$$w_i^*\left(\max\left\{0, \ d_i - \sum_j x_{ij}^*\right\} - \lambda\right) = 0, \quad \forall i,$$

$$\sum_i w_i^* = m,$$

$$\mathbf{x}_i^*, \mathbf{p}^*, \mathbf{w}^* \geq \mathbf{0}, \quad \forall i. \tag{6}$$

Note that the conditions $(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i$ for all i are implied by the conditions in (6): multiplying $\mathbf{x}_i^* \geq \mathbf{0}$ to both sides of the first inequality, we have $(\mathbf{p}^*)^T \mathbf{x}_i^* \geq w_i$ for all i, which, together with other inequality conditions in (6), imply

$$\sum_i w_i \geq \bar{\mathbf{s}}^T \mathbf{p}^* \geq (\mathbf{p}^*)^T \left(\sum_i \mathbf{x}_i^*\right) = \sum_i (\mathbf{p}^*)^T \mathbf{x}_i^* \geq \sum_i w_i, \tag{7}$$

that is, every inequality in the sequence must be tight, which implies $(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i$ for all i.

On the other hand, the 4–6th conditions in (6) are optimality conditions of budget allocator's linear program, where λ is the dual variable. Then, we have a characterization theorem of a competitive equilibrium that satisfies power demands.

Theorem 4.2. *Every equilibrium of the discretized communication spectrum market with the Shannon utility that satisfies power demands has the following properties:*

(1) $\mathbf{p}^* > \mathbf{0}$ *(every tone power has a price);*

(2) $\sum_i \mathbf{x}_i^* = \bar{\mathbf{s}}$ *(all powers are allocated);*

(3) $(\mathbf{p}^*)^T \bar{\mathbf{s}} = \sum_i w_i^*$ *(all user budgets are spent);*

(4) $\sum_j x_{ij}^* \geq d_j$ *for all i (all user demands are met);*

(5) *If* $x_{ij}^* > 0$ *then* $(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)^T \mathbf{x}_i^*) \cdot p_j^* - w_i \cdot (\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*))_j = 0$ *for all i, j (every user only purchases most valuable tone power).*

Proof. Note that

$$\left(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i, \overline{\mathbf{x}}_i)\right)_j = \frac{1}{\sigma_{ij} + \sum_{k \neq i} a_{kj}^i x_{kj} + x_{ij}} > 0, \quad \forall \mathbf{x} \geq \mathbf{0}. \tag{8}$$

Since w_i cannot be zero for all i, there is at least one i such that

$$w_i \cdot \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) > \mathbf{0}, \tag{9}$$

so that the first inequality of (6) implies that $\mathbf{p}^* > \mathbf{0}$.

The second property is from $(\mathbf{p}^*)^T(\sum_i \mathbf{x}_i^*) = (\mathbf{p}^*)^T \bar{\mathbf{s}}$, $\sum_i \mathbf{x}_i^* \leq \bar{\mathbf{s}}$ and $\mathbf{p}^* > \mathbf{0}$.

The third is from $(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i$ for all i and $\sum_i \mathbf{x}_i^* = \bar{\mathbf{s}}$.

We prove the fourth property by contradiction. Suppose, $d_i - \sum_j x_{ij}^* > 0$ for $i \in \bar{I}$ for a nonempty index set $\bar{I}$. Then, $w_i = 0$ for all $i \notin \bar{I}$ so that $\mathbf{x}_i^* = \mathbf{0}$ for all $i \notin \bar{I}$. Then,

$$\sum_{i \in \bar{I}} d_i > \sum_{i \in \bar{I}} \sum_j x_{ij}^* = \sum_i \sum_j x_{ij}^* = \sum_j \bar{s}_j, \tag{10}$$

which is a contradiction to the assumption $\sum_i d_i \leq \sum_j \bar{s}_j$.

The last one is from the complementarity condition of user optimality. □

The fourth property of Theorem 4.2 implies that equilibrium conditions (6) can be simplified to

$$w_i^* \cdot \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) \leq \left(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)^T \mathbf{x}_i^*\right) \cdot \mathbf{p}^*, \quad \forall i,$$

$$\sum_j x_{ij}^* \geq d_i, \quad \forall i,$$

$$\sum_i x_{ij}^* \leq \bar{s}_j, \quad \forall j,$$

$$\bar{\mathbf{s}}^T \mathbf{p}^* \leq \sum_i w_i^*,$$

$$\sum_i w_i^* = m,$$

$$\mathbf{x}_i^*, \mathbf{p}^*, \mathbf{w}^* \geq \mathbf{0}, \quad \forall i. \tag{11}$$

Note that the constraint $\sum_i w_i^* = m$ is merely a normalizing constraint and it can be replaced by another type of normalizing constraint such as $\prod_i w_i^* \geq 1$. Moreover, multiple competitive equilibria may exist due to the nonconvexity of the optimality conditions of the spectrum management problem with minimal user power demands.

4.2. Budget Allocation on Balancing Individual Utilities. The second question is whether or not the "Budgeting" agent can adjust "monetary budget" for each user such that a certain fairness is achieved in the spectrum market; for example, every user obtains the same utility value, which is also a critical issue in spectrum management. We again give an affirmative answer in this section.

Here, a competitive market equilibrium is a density point $[\mathbf{x}_1^*, \dots, \mathbf{x}_m^*, \mathbf{p}^*, \mathbf{w}^*]$ such that

(i) (user optimality) $\mathbf{x}_i^*$ is a maximizer of (1) given $\overline{\mathbf{x}}_i^*$, $\mathbf{p}^*$ and w_i^* for every i;

(ii) (market efficiency) $\mathbf{p}^* \geq \mathbf{0}$, $\sum_{i=1}^m x_{ij}^* \leq \bar{s}_j$, $p_j^*(\bar{s}_j - \sum_{i=1}^m x_{ij}^*) = 0$ for all j;

(iii) (budgeting according to individual utilities) Given $\mathbf{x}^*$, $\mathbf{w}^*$ is a minimizer of

$$\min_{\mathbf{w}} \sum_i u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) w_i^*, \quad \text{s.t.} \quad \sum_i w_i = m, \ \mathbf{w} \geq \mathbf{0}. \qquad (12)$$

This condition says that if user i's utility is higher than any others', that is, $u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) > u_j(\mathbf{x}_j^*, \overline{\mathbf{x}}_j^*)$, then one should shift "money budget" from user i to user j. Any budget allocation is optimal if $u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)$ are identical for all i, that is, if every user has the same utility value.

Since the "Budgeting" agent's problem is again a bounded linear maximization, and all other agents' problems are identical to those in Ye [11], we have the following corollary.

Corollary 4.3. *The communication spectrum market with Shannon utilities has a competitive equilibrium that balances each user's utility value.*

The complete necessary and sufficient conditions for a competitive equilibrium with balanced utilities can be summarized as

$$w_i^* \cdot \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) \leq (\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)^T \mathbf{x}_i^*) \cdot \mathbf{p}^*, \quad \forall i,$$

$$\sum_i x_{ij}^* \leq \overline{s}_j, \quad \forall j,$$

$$\overline{\mathbf{s}}^T \mathbf{p}^* \leq \sum_i w_i^*,$$

$$u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) - \lambda \geq 0, \quad \forall i,$$

$$w_i^* \left(u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) - \lambda \right) = 0, \quad \forall i,$$

$$\sum_i w_i^* = m,$$

$$\mathbf{x}_i^*, \mathbf{p}^*, \mathbf{w}^* \geq \mathbf{0}, \quad \forall i. \qquad (13)$$

Note that the conditions $(\mathbf{p}^*)^T \mathbf{x}_i^* = w_i$ for all i are implied by the conditions in (13). On the other hand, the 4–6th conditions in (13) are optimality conditions of budget allocator's linear program for balancing utilities, where λ is the dual variable.

Again, we have a characterization theorem of a competitive equilibrium that balances individual utilities.

Theorem 4.4. *Every equilibrium of the discretized communication spectrum market with the Shannon utility that balances individual utilities has the following properties:*

(1) $\mathbf{p}^* > \mathbf{0}$ *(every tone power has a price);*

(2) $\sum_i \mathbf{x}_i^* = \overline{\mathbf{s}}$ *(all powers are allocated);*

(3) $(\mathbf{p}^*)^T \overline{\mathbf{s}} = \sum_i w_i^*$ *(all user budgets are spent);*

(4) $u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)$ *are identical for all i (all user utilities are the same);*

(5) *If $x_{ij}^* > 0$ then $(\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)^T \mathbf{x}_i^*) \cdot p_j^* - w_i \cdot (\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*))_j = 0$ for all i, j (every user only purchases most valuable tone power).*

Proof. The proof of properties 1, 2, 3, and 5 are the same as Theorem 4.2. The fourth property is from the 5th condition of (13). If $w_i = 0$, then the user cannot participate the game. Therefore, $w_i > 0$ and $u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) = \lambda$, $\forall i$ by the 5th condition of (13), which implies all user utilities are identical. $\square$

The fourth property of Theorem 4.4 implies that equilibrium conditions (13) can be simplified to

$$w_i^* \cdot \nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) \leq (\nabla_{\mathbf{x}_i} u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*)^T \mathbf{x}_i^*) \cdot \mathbf{p}^*, \quad \forall i,$$

$$u_i(\mathbf{x}_i^*, \overline{\mathbf{x}}_i^*) = \lambda, \quad \forall i,$$

$$\sum_i x_{ij}^* \leq \overline{s}_j, \quad \forall j,$$

$$\overline{\mathbf{s}}^T \mathbf{p}^* \leq \sum_i w_i^*,$$

$$\sum_i w_i^* = m,$$

$$\mathbf{x}_i^*, \mathbf{p}^*, \mathbf{w}^* \geq \mathbf{0}, \quad \forall i. \qquad (14)$$

5. An Illustration Example

Consider two channels f_1 and f_2, and two users $\mathbf{x}$ and $\mathbf{y}$. Let the Shannon utility function for user $\mathbf{x}$ be

$$\log\left(1 + \frac{x_1}{1 + y_1}\right) + \log\left(1 + \frac{x_2}{4 + y_2}\right), \qquad (15)$$

and one for user $\mathbf{y}$ be

$$\log\left(1 + \frac{y_1}{2 + x_1}\right) + \log\left(1 + \frac{y_2}{4 + x_2}\right), \qquad (16)$$

and let the aggregate social utility be the sum of the two individual user utilities.

Assume a competitive spectrum market with power supply for two channels is $s_1 = s_2 = 2$ and the initial endowments for two users is $w_x = w_y = 1$. Then the competitive solution is

$$p_1 = \frac{3}{5}, \quad p_2 = \frac{2}{5},$$

$$x_1 = \frac{5}{3}, \quad x_2 = 0, \qquad (17)$$

$$y_1 = \frac{1}{3}, \quad y_2 = 2,$$

where the utility of user $\mathbf{x}$ is 0.3522, the utility of user $\mathbf{y}$ is 0.2139, and the social utility has value 0.5661.

Now consider each of them has a physical power demand $d_x = d_y = 2$. From above example we find $x_1 + x_2 = 5/3$ can not satisfy user $\mathbf{x}$'s power demand $d_x = 2$ if $w_x = w_y = 1$. By the proposed method, we can adjust the initial budget endowments to $w_x = 6/5$ and $w_y = 4/5$, then the equilibrium price will remain the same and the equilibrium allocation will be

$$x_1 = 2, \quad x_2 = 0,$$

$$y_1 = 0, \quad y_2 = 2, \qquad (18)$$

Iterative algorithm for budget allocation on satisfying power demands

Step 1: Set power supply of each channel $\bar{s}_j = m$, $j = 1, \ldots, n$.
Step 2: Initialize budget assigned to each user $w_i = 1$, $i = 1, \ldots, m$.
Step 3: Loop:
    (i) Compute competitive economy equilibrium $[\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*, \mathbf{p}^*]$ under $\bar{s}_j$, w_i
       according to the model in [11].
    (ii) Obtain total allocated power for each user i, $\sum_j x_{ij}^*$.
    (iii) Calculate average power shortage, avg_short $= \sum_i (d_i - \sum_j x_{ij}^*)/m$,
       and minimal user budget, min_w $= \min_i w_i$.
    (iv) Update $w_i = w_i + (((d_i - \sum_j x_{ij}^*) - \text{avg\_short})/m \cdot n) \cdot \text{min\_w}$, $i = 1, \ldots, m$.
   Until $(d_i - \sum_j x_{ij}^*)/d_i \leq$ error tolerance, $i = 1, \ldots, m$.

ALGORITHM 1

Iterative algorithm for budget allocation on balancing individual utilities

Step 1: Set power supply of each channel $\bar{s}_j = m$, $j = 1, \ldots, n$.
Step 2: Initialize budget assigned to each user $w_i = 1$, $i = 1, \ldots, m$.
Step 3: Loop:
    (i) Compute competitive economy equilibrium $[\mathbf{x}_1^*, \ldots, \mathbf{x}_m^*, \mathbf{p}^*]$ under $\bar{s}_j$, w_i
       according to the model in [11].
    (ii) Obtain individual utility of each user $\mathbf{u}_i$.
    (iii) Calculate average reciprocal of individual utility, avg_rec_u $= \sum_i (1/u_i)/m$,
       and minimal user budget, min_w $= \min_i w_i$.
    (iv) Update $w_i = w_i + ((1/u_i - \text{avg\_rec\_u})/\text{avg\_rec\_u}) \cdot \text{min\_w}$, $i = 1, \ldots, m$.
   Until $(\max_i u_i - \min_i u_i)/\min_i u_i \leq$ difference tolerance.

ALGORITHM 2

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where the utility of user **x** is 0.4771, the utility of user **y** is 0.1761, and the social utility has value 0.6532.

Since the Nash equilibrium model only considers each user's power demand, we set the power constraints of user **x** and user **y** as 2 and get a Nash equilibrium $x_1 = 2$, $x_2 = 0$, $y_1 = 1$, $y_2 = 1$, where the utility of user **x** is 0.3010, the utility of user **y** is 0.1938, and the social utility has value 0.4948. Since the power resource supply of each channel is assumed to be unconstrained in the Nash model, we see that Channel 1 supplies 3 units power and Channel 2 supplies 1. Even though, comparing the competitive equilibrium and Nash equilibrium solutions, one can see that the competitive equilibrium provides a power distribution that not only meets physical power demand and supply constraints but also achieves a much higher social utility than the Nash equilibrium does.

Now consider user **x** and user **y** need to have more balanced individual utilities. By the proposed method, we can adjust the initial endowments to $w_x = 4/5$ and $w_y = 6/5$, then the equilibrium price will remain the same and the equilibrium power distribution will be

$$x_1 = \frac{4}{3}, \quad x_2 = 0,$$
$$y_1 = \frac{2}{3}, \quad y_2 = 2, \quad (19)$$

where the utilities of user **x** and user **y** are both 0.25527, and the social utility is 0.51054.

If the power constraints of user **x** and user **y** are set as 4/3 and 8/3, respectively, then the Nash equilibrium will be $x_1 = 4/3$, $x_2 = 0$, $y_1 = 5/3$, $y_2 = 1$, where the utility of user **x** is 0.1761 , the utility of user **y** is 0.2730, and the social utility has value 0.4491. Comparing the competitive equilibrium and Nash equilibrium solutions again, one can see that the competitive equilibrium provides a power distribution that not only makes both users with an identical utility value but also achieves a higher social utility than the Nash equilibrium does.

6. Numerical Simulations

This section presents some computer simulation results on using two different approaches to achieve budget allocation for satisfying each user's power demand or balancing individual utilities. We compare the competitive equilibrium solution with Nash equilibrium solution in social utility and individual utilities under various number of channels and number of users in a weak-interference communication environment. In a weak-interference communication channel, the Shannon utility function is approximated by

$$u_i(\mathbf{x}_i, \bar{\mathbf{x}}_i) = \sum_{j=1}^n \log\left(1 + \frac{x_{ij}}{\sigma_{ij} + a_j^i(\sum_{k \neq i} x_{kj})}\right), \quad (20)$$

where a_j^i represent the average of normalized crosstalk ratios for $k \neq i$. Furthermore, we assume $0 \leq a_j^i \leq 1$, that is,
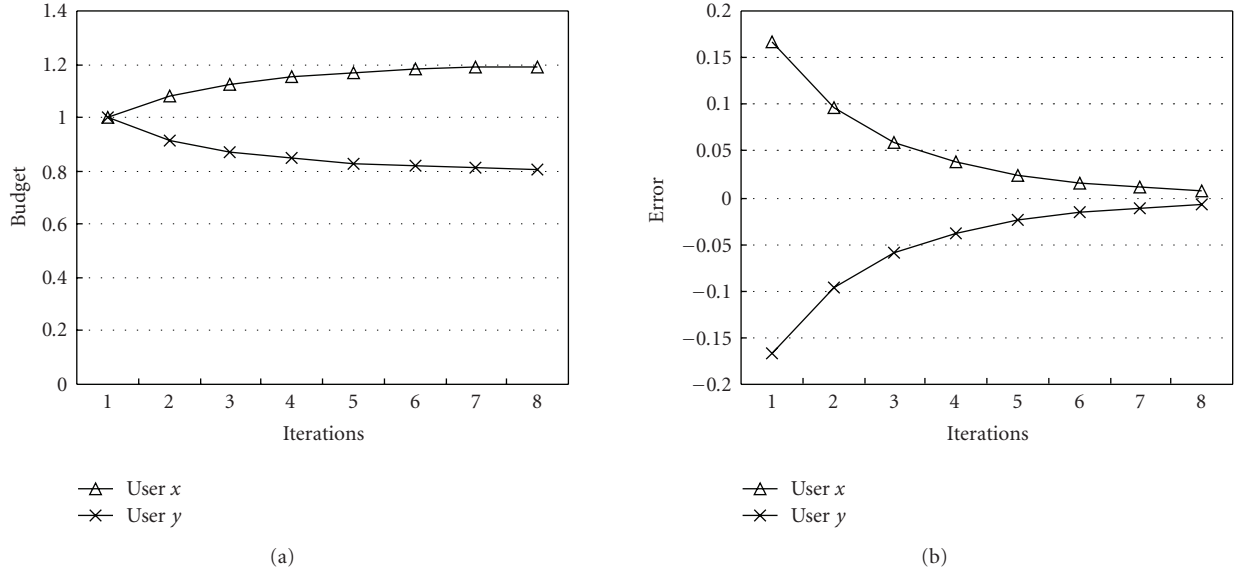
FIGURE 2: Convergence of iterative algorithm for satisfying power demands.

TABLE 1: Number of iterations required to achieve the budget allocation where the competitive equilibrium satisfies power demands $d_i = 0.5(\sum_j \bar{s}_j/m)$ and $d_i = \sum_j \bar{s}_j/m$ by the iterative algorithm, error tolerance = 0.01, average of 10 simulation runs.

No. of channels	$d_i = 0.5(\sum_j \bar{s}_j/m)$ No. of users					$d_i = \sum_j \bar{s}_j/m$ No. of users				
	2	4	6	8	10	2	4	6	8	10
2	1	1	1	1	1	5	14	20	22	38
4	1	1	1	1	1	5	18	44	49	85
6	1	1	1	1	1	5	20	37	47	65
8	1	1	1	1	1	7	21	35	52	66
10	1	1	1	1	1	6	18	33	53	66
12	1	1	1	1	1	6	18	35	56	71
14	1	1	1	1	1	6	20	35	50	70
16	1	1	1	1	1	6	18	35	49	74
18	1	1	1	1	1	6	18	35	45	70
20	1	1	1	1	1	6	18	31	50	68

TABLE 2: Comparisons of CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium satisfies power demands $d_i = \sum_j \bar{s}_j/m$ between two approaches, error tolerance = 0.01 and average of 10 simulation runs.

No. of channels	No. of users									
	2		4		6		8		10	
	M1*	M2†	M1	M2	M1	M2	M1	M2	M1	M2
2	0.033	1.085	0.049	1.228	0.069	1.358	0.088	1.624	0.136	1.882
4	0.022	1.164	0.080	1.479	0.267	2.255	0.463	3.465	1.011	6.450
6	0.028	1.270	0.106	2.207	0.312	5.129	0.639	10.545	1.947	19.406
8	0.025	1.516	0.103	3.788	0.510	10.305	0.875	25.697	2.592	51.210
10	0.035	1.889	0.130	7.222	0.525	27.027	0.938	44.909	2.455	111.270
12	0.028	2.482	0.158	12.558	0.603	41.747	1.816	93.028	3.164	190.489
14	0.028	3.231	0.161	20.454	0.528	66.719	2.464	150.099	2.708	322.979
16	0.039	4.793	0.184	33.251	0.684	102.846	1.260	263.820	6.006	519.137
18	0.041	6.529	0.250	46.043	0.627	150.047	2.181	385.401	5.781	773.646
20	0.042	9.322	0.247	66.839	0.703	215.038	2.645	553.401	4.689	1179.129

*M1: iterative algorithm
†M2: solving the entire optimal conditions.

TABLE 3: Number of iterations and CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium satisfies power demands $d_i = 0.95(\sum_j \bar{s}_j/m)$ in large-scale problems by the iterative method, error tolerance = 0.05 and average of 10 simulation runs.

| No. of channels | $d_i = 0.95(\sum_j \bar{s}_j/m)$ | | | | | | | |
| | 2 users | | 10 users | | 50 users | | 100 users | |
	Iterations	Time	Iterations	Time	Iterations	Time	Iterations	Time
256	1	0.072	1	2.162	5	400.092	17	4751.578
512	1	0.255	1	5.656	1	148.097	3	2277.144
1024	1	0.388	1	15.978	1	365.290	1	1720.400

TABLE 4: Comparisons of social utility and individual utility between competitive equilibrium(CE) with power demands $d_i = \sum_j \bar{s}_j/m$ and Nash equilibrium(NE), error tolerance = 0.01 and average of 100 simulation runs.

| No. of channels | No. of users | | | | | | | | | |
| | 2 | | 4 | | 6 | | 8 | | 10 | |
	Social*	Indiv†	Social	Indiv	Social	Indiv	Social	Indiv	Social	Indiv
2	9.20%	83%	8.51%	58%	7.85%	53%	8.42%	51%	9.38%	51%
4	6.78%	87%	6.21%	70%	6.21%	62%	6.40%	58%	6.11%	57%
6	5.91%	88%	6.20%	81%	5.68%	71%	5.59%	64%	5.57%	62%
8	6.83%	92%	5.26%	78%	5.65%	71%	5.46%	69%	5.19%	67%
10	6.14%	94%	5.82%	80%	5.31%	74%	5.26%	70%	5.05%	67%
12	6.18%	94%	5.76%	84%	5.50%	77%	5.50%	74%	5.24%	71%
14	5.73%	95%	5.49%	84%	5.55%	79%	5.26%	74%	5.04%	70%
16	6.24%	97%	5.35%	83%	5.27%	81%	5.03%	75%	5.02%	74%
18	5.62%	96%	5.64%	86%	5.33%	82%	5.22%	77%	5.28%	76%
20	5.83%	97%	5.26%	88%	5.34%	85%	5.25%	81%	5.02%	74%

*Social: (average social utility in CE − average social utility in NE)/average social utility in NE

†Indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.

the average cross-interference ratio is not above 1 or it is less than the self-interference ratio (always normalized to 1). In all simulated cases, the channel background noise levels σ_{ij} are chosen randomly from the interval $(0, m]$, and the normalized crosstalk ratios a_j^i are chosen randomly from the interval [0, 1]. The power supply of each channel j is $\bar{s}_j = m$, $j = 1, \dots n$. The total budget is $\sum_i w_i = m$. All simulations are run on a Genuine Intel CPU 1.66 GHz Notebook.

6.1. Budget Allocation on Satisfying Individual Power Demands. In this section, we compute the budget allocation where the competitive equilibrium meets power demands $d_i = 0.5(\sum_j \bar{s}_j/m)$ or $d_i = \sum_j \bar{s}_j/m$ for all users under various number of channels and number of users. Two approaches are adopted to find out the budget allocation strategy: one is solving the entire optimality conditions in (11) by optimization solver LINGO; the other is iteratively adjusting total budget m among different users based on whether their power demands are satisfied or not. In the iterative algorithm, all user budgets w_i are set as 1 initially, then the competitive equilibrium can be derived from given channel capacity and user budget. If some user's power demand is not satisfied in the resulting competitive equilibrium, the budgeting agent reallocates budget to users and computes a new competitive equilibrium. The procedure reiterates until a desired competitive equilibrium is reached for satisfying

power demands. The iterative algorithm that allocates more budget to the users with more power shortage and keeps the total budget as m is summarized in Algorithm 1.

In each iteration, given channel capacity $\bar{s}_j$ and user budget w_i, the competitive equilibrium is derived by an iterative waterfilling method [13]. Since the competitive equilibrium in each iteration satisfies $\sum_i x_{ij}^* = \bar{s}_j = m$ and $\sum_i w_i^* = m$, and each user optimizes his own utility under his budget constraint and the equilibrium prices, relatively increasing one user's budget makes him obtain more powers and others obtain fewer powers. In Algorithm 1, the user budget is reassigned according to the power shortage of each user in the equilibrium solution. The idea of comparing the user's power shortage with average shortage makes more budget be allocated to the users with higher power shortage and the total budget remains m. The term min_w aims to keep new w_i not less than 0. The power demand value and the error tolerance have a significant impact on the number of iterations required to converge to the budget allocation where the competitive equilibrium meets the power demands. Figure 2 indicates the convergence behavior of the iterative algorithm for satisfying power demands for the case of 2 users and 2 channels illustrated in Section 5. Each user has a physical power demand $d_x = d_y = 2$. The error tolerance is set as 0.01. As the figure shows, at first, user **x** has power shortage and user **y** has power surplus, then the algorithm converges after eight iterations and the errors

TABLE 5: Number of iterations required to achieve the budget allocation where the competitive equilibrium has balanced individual utilities by the iterative algorithm, difference tolerance = 0.01 and average of 10 simulation runs.

No. of channels	No. of users									
	2		4		6		8		10	
	Iter*	Diff+	Iter	Diff	Iter	Diff	Iter	Diff	Iter	Diff
2	5	0.0076	10	0.0078	14	0.0079	18	0.0080	97	0.0080
4	4	0.0075	20	0.0079	27	0.0080	21	0.0080	74	0.0080
6	4	0.0077	9	0.0079	18	0.0080	22	0.0081	46	0.0081
8	4	0.0080	8	0.0078	40	0.0079	149	0.0081	33	0.0080
10	4	0.0080	13	0.0081	17	0.0079	57	0.0081	24	0.0080
12	4	0.0078	16	0.0080	35	0.0079	31	0.0080	29	0.0080
14	5	0.0078	8	0.0080	13	0.0079	21	0.0080	67	0.0080
16	5	0.0078	9	0.0080	12	0.0079	27	0.0080	48	0.0080
18	4	0.0076	6	0.0079	10	0.0078	18	0.0079	26	0.0080
20	4	0.0077	7	0.0078	8	0.0079	11	0.0079	20	0.0079

*Iter: number of iterations
+Diff: $(\max_i u_i - \min_i u_i)/\min_i u_i$.

TABLE 6: Comparisons of CPU time (seconds) required to achieve the budget allocation where competitive equilibrium has balanced individual utilities between two approaches, difference tolerance = 0.01 and average of 10 simulation runs.

No. of channels	No. of users									
	2		4		6		8		10	
	M1*	M2†	M1	M2	M1	M2	M1	M2	M1	M2
2	0.048	0.330	0.056	0.364	0.061	0.447	0.060	0.575	0.239	0.837
4	0.046	0.377	0.088	0.647	0.127	1.100	0.148	1.892	0.738	3.606
6	0.048	0.467	0.075	1.005	0.116	2.469	0.353	5.425	1.550	11.273
8	0.041	0.641	0.069	2.052	0.319	5.555	1.663	13.305	1.422	26.173
10	0.070	0.872	0.113	3.366	0.214	10.264	1.759	27.294	1.056	54.902
12	0.063	1.247	0.139	6.345	0.397	19.048	0.919	47.069	1.428	101.013
14	0.064	1.822	0.095	9.692	0.217	32.551	0.577	81.780	2.633	168.536
16	0.056	2.542	0.119	14.928	0.216	52.972	0.953	123.817	3.320	274.966
18	0.058	3.328	0.103	22.686	0.261	74.310	1.117	191.992	1.733	401.128
20	0.057	4.333	0.098	31.805	0.192	102.436	0.506	272.994	1.674	557.339

*M1: iterative algorithm
†M2: solving entire optimal conditions.

$(d_i - \sum_j x_{ij}^*)/d_i$ for user **x** and user **y** are both below error tolerance 0.01.

Table 1 lists the number of iterations required to find out the budget allocation with $d_i = 0.5(\sum_j \bar{s}_j/m)$ and $d_i = \sum_j \bar{s}_j/m$ by the above iterative algorithm. The cases of $d_i = \sum_j \bar{s}_j/m$ need more iterations since the total power demand $\sum_i d_i$ is equal to the total channel capacity $\sum_j \bar{s}_j$. This requirement is tight and the budget allocation makes each user get the same physical power in the competitive equilibrium, that is, $\sum_j x_{ij}^* = n$, for all i. Table 2 compares the CPU time used by two different approaches under power demands $d_i = \sum_j \bar{s}_j/m$. The iterative algorithm spends much less time than the method of solving entire optimal conditions on finding out the budget allocation and the competitive equilibrium. We can also use the iterative method to solve large scale problems. The number of iterations and the CPU time required to solve large-scale problems are listed in Table 3. We observe that more iterations and CPU time spending

for 100 users and 256 channels than those spending for 100 users and 1024 channels because the stop condition of the iterative algorithm is "$(d_i - \sum_j x_{ij}^*)/d_i \leq$ error tolerance." In our simulations in Table 3, $d_i = 0.95 * 256$ for 100 users and 256 channels and $d_i = 0.95 * 1024$ for 100 users and 1024 channels, therefore the case of 100 users and 1024 channels requires fewer iterations and less total CPU time to reach the error tolerance 0.05 than the case of 100 users and 256 channels does. However the CPU time spending for one iteration in the case of 100 users and 256 channels is less than that in the case of 100 users and 1024 channels.

In comparing competitive equilibrium with Nash equilibrium, the total power allocated to user i, $\sum_j x_{ij}^*$, in competitive equilibrium is used as the power constraint for user i in Nash equilibrium model to derive a Nash equilibrium. The simulation results averaged over 100 independent runs indicates that the average social utility of competitive equilibrium is higher than that of Nash equilibrium in

TABLE 7: Number of iterations and CPU time (seconds) required to achieve the budget allocation where the competitive equilibrium has balanced individual utilities in large-scale problems by the iterative method, difference tolerance = 0.05 and average of 10 simulation runs.

No. of channels	2 users		10 users		50 users		100 users	
	Iterations	Time	Iterations	Time	Iterations	Time	Iterations	Time
256	1	0.119	4	6.775	8	646.620	16	4005.344
512	1	0.211	3	14.309	6	964.164	8	4750.842
1024	1	0.452	3	35.631	5	1663.111	5	6326.120

TABLE 8: Comparisons of social utility and individual utility between competitive equilibrium (CE) with balanced individual utilities and Nash equilibrium (NE), difference tolerance = 0.01 and average of 100 simulation runs.

No. of channels	No. of users									
	2		4		6		8		10	
	Social*	Indiv†	Social	Indiv	Social	Indiv	Social	Indiv	Social	Indiv
2	0.96%	46%	−0.59%	45%	−0.41%	47%	−0.34%	48%	−0.44%	48%
4	1.05%	55%	0.83%	53%	0.42%	50%	0.01%	48%	−0.42%	48%
6	1.14%	56%	0.08%	50%	−0.15%	48%	−0.08%	48%	0.01%	49%
8	1.27%	58%	0.66%	57%	0.17%	51%	0.00%	49%	0.19%	52%
10	1.15%	61%	0.88%	58%	0.52%	54%	0.22%	52%	−0.08%	53%
12	1.35%	66%	0.78%	57%	0.50%	56%	0.39%	54%	0.16%	52%
14	1.43%	67%	0.85%	60%	0.28%	55%	0.10%	52%	0.16%	54%
16	1.60%	75%	0.88%	60%	0.42%	55%	0.30%	54%	0.29%	55%
18	1.63%	72%	0.71%	59%	0.34%	54%	0.14%	55%	0.16%	52%
20	1.50%	73%	0.80%	59%	0.30%	56%	0.11%	54%	0.17%	54%

*Social: (average social utility in CE − average social utility in NE)/average social utility in NE
†Indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.

all cases with $d_i = 0.5(\sum_j \bar{s}_j/m)$ and in most cases with $d_i = \sum_j \bar{s}_j/m$, even though the difference is not significant. However, in certain type of problems, for instance, the channels being divided into two categories: high quality and low quality, the competitive equilibrium solution performs much better than the Nash equilibrium solution does. Table 4 compares social utility and individual utility between the competitive equilibrium and the Nash equilibrium when one half of channels with $\sigma_{ij}, j = 1, \ldots, n/2$, chosen randomly from the interval $(0, 0.1]$ and the other half of channels with $\sigma_{ij}, j = n/2 + 1, \ldots, n$, chosen randomly from interval $[1, m]$. One can see that the competitive equilibrium significantly outperforms the Nash equilibrium in the social utility value and a much higher portion of users obtain higher individual utilities in the competitive equilibrium than those in the Nash equilibrium.

6.2. Budget Allocation on Balancing Individual Utilities. To consider fairness, we adjust each user's endowed monetary budget w_i to reach a competitive equilibrium where the individual utilities are balanced. Herein we also adopt two approaches to find out the budget allocation: one is solving the entire optimality conditions in (14) by optimization solver LINGO; the other is iteratively adjusting total budget m among different users based on their individual utilities. The iterative algorithm that shifts some budget from high-utility users to low-utility users and keeps the total budget as m is summarized in Algorithm 2.

Algorithm 2 is similar to Algorithm 1 for budget allocation on satisfying power demands. For balancing individual utilities, herein the user budget is adjusted based on the individual utility in the equilibrium solution. The idea of using the reciprocal of individual utility makes some budget be transferred from the high-utility users to low-utility users. Since relatively increasing one user's budget makes him obtain more powers and others obtain fewer powers, this will decrease the difference between highest individual utility and lowest individual utility. The term min_w aims to keep new w_i not less than 0. The difference tolerance significantly affects the number of iterations required to converge to the budget allocation. Figure 3 indicates the convergence behavior of the iterative algorithm for balancing individual utilities for the case of 2 users and 2 channels illustrated in Section 5. The difference tolerance is set as 0.01. As the figure shows, at first, the difference $(\max_i u_i - \min_i u_i)/\min_i u_i$ is higher than 0.6, then the algorithm converges after eighteen iterations and the difference is below difference tolerance 0.01.

Table 5 lists the number of iterations required to converge to the budget allocation for balancing individual utilities by the iterative algorithm. Table 6 compares the CPU time used by two different approaches to achieve the budget allocation. The iterative algorithm spends less CPU time than the method of solving the entire optimal conditions. Treating the budget allocation problem by solving the entire optimal conditions can obtain a budget allocation where the competitive equilibrium has exactly identical individual utility value for each user. Table 7 lists the number of iterations

TABLE 9: Comparisons of social utility and individual utility between competitive equilibrium(CE) with balanced individual utilities and Nash equilibrium(NE) under two tiers of channels, difference tolerance = 0.01 and average of 100 simulation runs.

| No. of channels | No. of users | | | | | | | | | |
| | 2 | | 4 | | 6 | | 8 | | 10 | |
	Social*	Indiv†	Social	Indiv	Social	Indiv	Social	Indiv	Social	Indiv
2	9.02%	81%	9.86%	73%	10.01%	69%	8.85%	67%	9.62%	67%
4	7.68%	80%	7.67%	78%	8.30%	77%	8.54%	71%	8.23%	71%
6	6.06%	87%	6.55%	81%	6.87%	77%	7.43%	76%	7.16%	75%
8	5.56%	88%	6.24%	81%	6.41%	78%	6.80%	76%	6.75%	75%
10	5.46%	87%	6.12%	84%	6.18%	80%	6.47%	77%	6.38%	75%
12	5.66%	88%	5.75%	84%	5.88%	80%	5.94%	79%	6.38%	77%
14	5.65%	91%	6.01%	85%	5.65%	82%	5.80%	81%	5.86%	77%
16	5.63%	92%	5.84%	89%	5.75%	83%	5.86%	82%	5.70%	79%
18	5.78%	94%	5.84%	88%	5.80%	83%	5.81%	83%	5.54%	80%
20	5.66%	94%	5.58%	88%	5.77%	86%	5.72%	81%	5.63%	80%

*Social: (average social utility in CE − average social utility in NE)/average social utility in NE
†Indiv: average percentage in number of users obtaining higher individual utilities in CE than in NE.
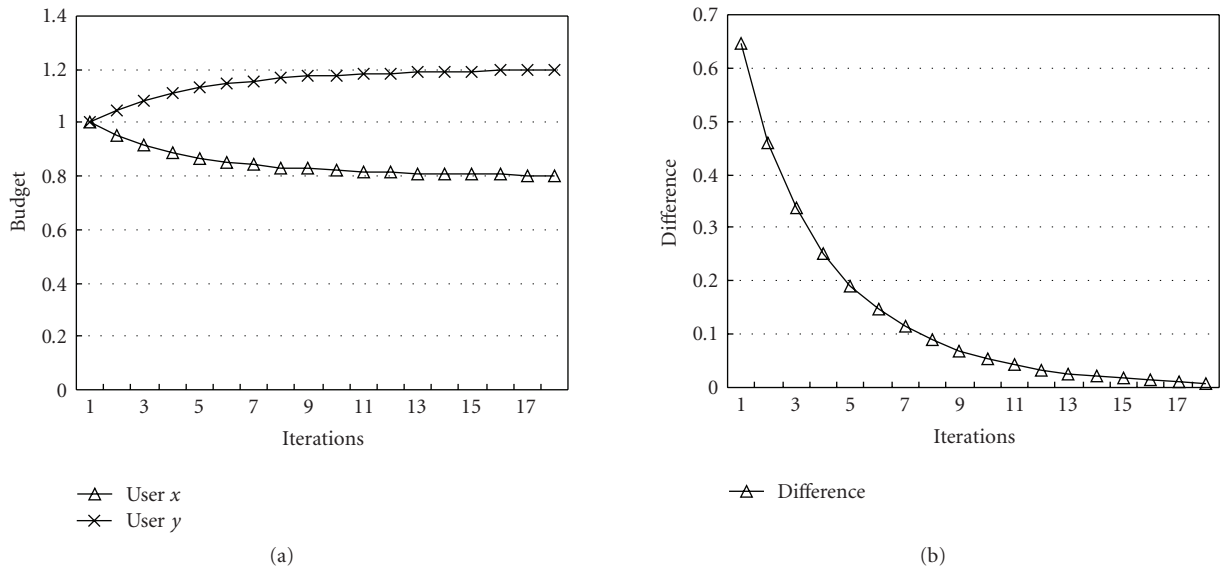

(a)

(b)

FIGURE 3: Convergence of iterative algorithm for balancing individual utilities.

and the CPU time required to solve large-scale problems for balancing utilities by the iterative method. We observe that more iterations are required for 100 users and 256 channels than those required for 100 users and 1024 channels because the stop condition of the proposed algorithm is "$(\max_i u_i - \min_i u_i)/\min_i u_i \leq$ difference tolerance." In our simulations in Table 7, the balanced individual utilities for 100 users and 1024 channels are higher than those for 100 users and 256 channels, therefore the case of 100 users and 1024 channels requires fewer iterations to reach the difference tolerance 0.05 than the case of 100 users and 256 channels does. However the CPU time spending for one iteration in the case of 100 users and 256 channels is less than that in the case of 100 users and 1024 channels.

In comparing competitive equilibrium with Nash equilibrium, the total power allocated to each user in competitive equilibrium is also used as the power constraint to derive a Nash equilibrium. The simulation results averaged over 100 independent runs are displayed in Table 8. We find that, in most cases, more users get higher individual utilities in competitive equilibrium than those in Nash equilibrium and the social utility of competitive equilibrium remains higher than that of Nash equilibrium. Table 9 lists the comparisons in the communication environment involving two tiers of channels, one half of channels with $\sigma_{ij}, j = 1, \ldots, n/2$, chosen randomly from the interval $(0, 0.1]$ and the other half of channels with $\sigma_{ij}, j = n/2 + 1, \ldots, n$, chosen randomly from the interval $[1, m]$. We can observe that the

competitive equilibrium not only makes more users obtain higher individual utilities but also significantly enhances the social utility. In other words, using budget allocation we can derive a competitive equilibrium that provides a power allocation strategy to balance individual utilities without sacrificing the social utility. Moreover, in the competitive equilibrium model with balanced individual utilities, all users have identical utility value. However, in the Nash equilibrium model the average difference between maximal individual utility and minimal individual utility is over 15%.

7. Conclusions

This study proposes two competitive equilibrium models: (1) to satisfy each user's physical power demand and (2) to balance all individual utilities in a competitive communication spectrum economy. Theoretically, we prove that a competitive equilibrium with physical power demand requirements always exists for the communication spectrum market with Shannon utility if the total power demand is less than or equal to the available total power supply. A competitive equilibrium with identical individual utilities also exists for the communication spectrum market with Shannon utility. Computationally, we use two approaches to find out the budget allocation where the competitive equilibrium satisfies power demand or balances individual utilities: one solves the characteristic equilibrium conditions and the other employs an iterative tatonament -type method by adjusting budget to each user. The iterative method performs significantly faster and can efficiently solve large-scale problems, which makes the competitive economy equilibrium model applicable in real-time spectrum management.

In comparing with the Nash equilibrium solution under the identical power usage of each user obtained from the competitive equilibrium model, our computational results show that the social utility of the competitive equilibrium solution is better than that of the Nash equilibrium solution in most cases. Under the equilibrium condition with balanced individual utilities, the competitive economy equilibrium solution makes more users obtain higher individual utilities than Nash equilibrium solution does without sacrificing the social utility.

In this study, we propose a centralized algorithm to reach a desired competitive equilibrium for satisfying power demands or balancing individual utilities. In the future, a distributed algorithm should be developed especially when a centralized controller is not available in the network. Besides, although the iterative method works well in our computational experiments, its convergence is unproven. We plan to do so in future work. We would also consider further study in how to adjust another exogenous factor **s** (power supply) to achieve a better social solution while maintaining individual satisfaction. That is, how to set the power supply capacity for each channel to make spectrum power allocation more efficient under the competitive equilibrium market model.

Acknowledgments

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