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Research Article

The Spread of a Noise Field in a Dispersive Medium

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We discuss the production of induced noise by a pulse and the propagation of the noise in a dispersive medium. We present a simple model where the noise is the sum of pulses and where the mean of each pulse is random. We obtain explicit expressions for the standard deviation of the spatial noise as a function of time. We also formulate the problem in terms of a time-frequency phase space approach and in particular we use the Wigner distribution to define the spatial/spatial-frequency distribution.

1. Introduction

In many situations noise is induced by a pulse due to the scattering of the pulse from many sources. To take a specific example, consider a pulse that hits a school of fish; each fish scatters the wave and the acoustic field seen at an arbitrary point is the sum of the waves received from each fish; the sum of which is noise like. Another example is the creation of a set of bubbles by a propeller in a finite region in space. When each bubble explodes it produces a wave and the acoustic pressure seen is the sum of the waves produced by each bubble. Of course this is a simplified view since there could be many other effects such as multiple scattering. Now as time evolves the noise field is propagating and can be changing in a significant way if the medium has dispersion. Our aim here is to investigate how a noise field which is composed of a group of pulses behave as it is propagating and in particular we want to investigate the spreading of the field as a function of time. Suppose we consider a space-time signal composed of the sum of elementary signal, u_n,

$$\psi(x,t) = A\sum_{n=1}^{N} a_n u_n(x,t;\varepsilon_n), \tag{1}$$

where u_n is a deterministic function and a_n and ε_n are random parameters. We have put in an overall A for normalization convenience. The production of noise by expressions like (1) are sometimes called FOM models of noise production [1–7]. Here, we consider the case where the only random variables are the means of each of the elementary signals and where all u_n are the same. Hence we write

$$\psi(x,t) = A\sum_{n=1}^{N} a_n u(x-x_n,t), \tag{2}$$

where x_n are the means of the elementary signals, assuming that the mean of $u(x,t)$ is zero. The approach we take is the following. At time $t=0$ we form an ensemble of signals

$$\psi(x,0) = A\sum_{n=1}^{N} a_n u(x-x_n,0). \tag{3}$$

We then evolve $u(x-x_n,0)$ into $u(x-x_n,t)$ in a medium with dispersion giving (2) and calculate the appropriate ensemble averaged moments of $\psi(x,t)$.

For the elementary signal we define the moments in the standard way

$$\langle x^n\rangle_t = \int x^n |u(x,t)|^2 dx. \tag{4}$$

Now if there are random parameters as for example the means then use $E[\cdot]$ to signify ensemble averaging. In particular

$$E[\langle x^n\rangle_t] = \int x^n E\Big[|u(x,t)|^2\Big] dx. \tag{5}$$

Note that in general it is not the case that the ensemble averaging and ordinary averaging can be interchanged.

Which is to be used depends on the quantity we are considering. In this paper we will assume that

$$E[a_n^* a_m] = |a|^2 \delta_{nm}. \tag{6}$$

As to the means x_n we assume they are taken from a distribution $P(x_n)$ and we define the ensemble mean and standard deviation by

$$E[x_n] = \int x_n P(x_n) dx_n, \tag{7}$$

$$E[x_n^2] = \int x_n^2 P(x_n) dx_n, \tag{8}$$

$$\mu^2 = E[x_n^2] - E^2[x_n]. \tag{9}$$

2. Pulse Propagation

We briefly review some of our previous results regarding pulse propagation in a dispersive medium. We consider one mode and also assume that the dispersion relation $\omega = \omega(k)$ is real, which means there is no attenuation. The solution for each mode is [8–13],

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int S(k,0) e^{-i\omega(k)t} e^{ikx} dk, \tag{10}$$

where $S(k,0)$ is the initial spatial spectrum

$$S(k,0) = \frac{1}{\sqrt{2\pi}} \int u(x,0) e^{-ikx} dx. \tag{11}$$

The pulse is normalized so that

$$\int |u(x,t)|^2 dx = 1. \tag{12}$$

If we define

$$S(k,t) = S(k,0) e^{-i\omega(k)t} \tag{13}$$

then $S(k,t)$ and $u(x,t)$ form Fourier transform pairs for all time and hence the spatial moments can be obtained by way of

$$\langle x^n \rangle_t = \int S^*(k,t) \mathcal{X}^n S(k,t) dk, \tag{14}$$

where $\mathcal{X}$ is the position operator in the k representation

$$\mathcal{X} = i\frac{\partial}{\partial k}. \tag{15}$$

The first two moments and standard deviation are

$$\begin{aligned} \langle x \rangle_t &= \int x |u(x,t)|^2 dx = \int S^*(k,t) \mathcal{X} S(k,t) dk, \\ \langle x^2 \rangle_t &= \int x^2 |u(x,t)|^2 dx = \int S^*(k,t) \mathcal{X}^2 S(k,t) dk, \\ \sigma_{x|t}^2 &= \langle x^2 \rangle_t - \langle x \rangle_t^2 = \int S^*(k,t) (\mathcal{X} - \langle x \rangle_t)^2 S(k,t) dk. \end{aligned} \tag{16}$$

These expressions have been explicitly obtained [14, 15] to give

$$\langle x \rangle_t = \langle x \rangle_0 + Vt, \tag{17}$$

$$\langle x^2 \rangle_t = \langle x^2 \rangle_0 + t \langle \nu\mathcal{X} + \mathcal{X}\nu \rangle_0 + t^2 \left\langle \nu_g^2 \right\rangle, \tag{18}$$

$$\sigma_{x|t}^2 = \sigma_{x|0}^2 + 2t\mathrm{Cov}_{x\nu|0} + t^2 \sigma_\nu^2, \tag{19}$$

where

$$\nu(k) = \frac{d\omega(k)}{dk} \tag{20}$$

is the group velocity and where

$$\begin{aligned} V &= \langle \nu \rangle_0 = \int \nu(k) |S(k,0)|^2 dk = \langle \nu \rangle_0, \\ \sigma_\nu^2 &= \int (\nu(k) - V)^2 |S(k,0)|^2 dk, \\ \mathrm{Cov}_{x\nu|0} &= \frac{1}{2} \langle \nu\mathcal{X} + \mathcal{X}\nu \rangle_0 - \langle \nu \rangle_0 \langle x \rangle_0. \end{aligned} \tag{21}$$

2.1. An Exactly Solvable Case. An interesting and exactly solvable example that we will use is where the dispersion relation is

$$\omega(k) = ck + \frac{\gamma k^2}{2} \tag{22}$$

and the initial pulse is taken to be

$$u(x,0) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2 + i\beta x^2/2 + ik_0 x}. \tag{23}$$

At $t = 0$ the means and standard deviations of x and k are

$$\begin{aligned} \langle x \rangle_0 &= 0; \qquad \langle k \rangle_0 = k_0, \\ \sigma_{x|0}^2 &= \frac{1}{2\alpha}; \qquad \sigma_{k|0}^2 = \frac{\alpha^2 + \beta^2}{2\alpha}. \end{aligned} \tag{24}$$

Using (17)–(21), one obtains that

$$\begin{aligned} \mathrm{Cov}_{x\nu_g} &= \frac{\gamma\beta}{2\alpha}, \\ \left\langle \nu_g \right\rangle &= c + \gamma k_0, \\ \left\langle \nu_g^2 \right\rangle &= \gamma^2 \frac{\alpha^2 + \beta^2}{2\alpha} - (c + \gamma k_0)^2, \\ \sigma_{\nu_g}^2 &= \gamma^2 \frac{\alpha^2 + \beta^2}{2\alpha}, \end{aligned} \tag{25}$$

and these yield

$$\begin{aligned} \langle x \rangle_t &= (c + \gamma k_0) t, \\ \sigma_{x|t}^2 &= \sigma_{x|0}^2 [1 + 2\beta\gamma t + \gamma^2 (\alpha^2 + \beta^2) t^2]. \end{aligned} \tag{26}$$

3. Spread of the Wave Group

For convenience we repeat the basic equations producing the noise field

$$\psi(x,t) = A\sum_{n=1}^{N} a_n u(x-x_n,t). \tag{27}$$

We normalize ψ to one by considering

$$\begin{aligned}&\int |\psi(x,t)|^2 dx\\&= A^2 \sum_{n,m=1}^{N} a_n^* a_m \int u^*(x-x_n,t)u(x-x_m,t)dx.\end{aligned} \tag{28}$$

Now taking the ensemble average we have

$$\begin{aligned}&E\left[\int |\psi(x,t)|^2 dx\right]\\&= A^2 E\left[\sum_{n,m=1}^{N} a_n^* a_m \int u^*(x-x_n,t)u(x-x_m,t)dx\right]\\&= A^2 \sum_{n,m=1}^{N} E[a_n^* a_m] E\left[\int u^*(x-x_n,t)u(x-x_m,t)dx\right]\\&= A^2|a|^2 \sum_{n,m=1}^{N} \delta_{nm} E\left[\int u^*(x-x_n,t)u(x-x_m,t)dx\right]\\&= A^2|a|^2 \sum_{n=1}^{N} E\left[\int |u(x-x_n,t)|^2 dx\right]\\&= A^2|a|^2 N\end{aligned} \tag{29}$$

and therefore we take A so that

$$A^2|a|^2N = 1. \tag{30}$$

In the above we have assumed that a_n and x_n are independent.

Now consider the mean

$$\begin{aligned}\langle x\rangle_t &= A^2 \int x|\psi(x,t)|^2 dx\\&= A^2 \sum_{n,m=1}^{N} a_n^* a_m \int x u^*(x-x_n,t)u(x-x_m,t)dx.\end{aligned} \tag{31}$$

The ensemble average is

$$\begin{aligned}E[\langle x\rangle_t] &= A^2|a|^2 \sum_{n,m=1}^{N} \delta_{nm} E\left[\int x u^*(x-x_n,t)u(x-x_m,t)dx\right]\\&= A^2|a|^2 \sum_{n=1}^{N} E\left[\int x|u(x-x_n,t)|^2 dx\right]\\&= A^2|a|^2 \sum_{n=1}^{N} E\left[\int (x+x_n)|u(x,t)|^2 dx\right]\end{aligned} \tag{32}$$

giving

$$E[\langle x\rangle_t] = \langle x\rangle_t + E[x_n], \tag{33}$$

where $E[x_n]$ is given by (7).

For the calculation of $\langle x^2\rangle_t$ the first few steps as above lead to

$$\begin{aligned}E[\langle x^2\rangle_t] &= A^2|a|^2 \sum_{n=1}^{N} E\left[\int x^2 |u(x-x_n,t)|^2 dx\right]\\&= A^2|a|^2 \sum_{n=1}^{N} E\left[\int (x+x_n)^2 |u(x,t)|^2 dx\right]\end{aligned} \tag{34}$$

which gives

$$E[\langle x^2\rangle_t] = \langle x^2\rangle_t + E[x_n^2] + 2\langle x\rangle_t E[x_n]. \tag{35}$$

The standard deviation is therefore

$$\Gamma_{x|t}^2 = E[\langle x^2\rangle_t] - E^2[\langle x\rangle_t] \tag{36}$$

which, combining (33) and (35) leads to

$$\Gamma_{x|t}^2 = \sigma_{x|t}^2 + \mu^2, \tag{37}$$

where μ is given by (9). Also, noting that

$$\Gamma_{x|0}^2 = \sigma_{x|0}^2 + \mu^2, \tag{38}$$

one has

$$\Gamma_{x|t}^2 = \Gamma_{x|0}^2 + \sigma_{x|t}^2 - \sigma_{x|0}^2. \tag{39}$$

Further, if we use (19) we then have

$$\Gamma_{x|t}^2 = \Gamma_{x|0}^2 + 2t\,\mathrm{Cov}_{xv|0} + t^2\sigma_v^2. \tag{40}$$

4. The Spread in Wave Number

We now consider the spatial spectrum. For the total wave we define

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int F(k,t)e^{ikx} dk, \tag{41}$$

$$F(k,t) = \frac{1}{\sqrt{2\pi}} \int \psi(x,t)e^{-ikx} dx \tag{42}$$

and for the elementary wave we define

$$\begin{aligned}u(x,t) &= \frac{1}{\sqrt{2\pi}} \int S(k,t)e^{ikx} dk,\\S(k,t) &= \frac{1}{\sqrt{2\pi}} \int u(x,t)e^{-ikx} dx.\end{aligned} \tag{43}$$

The standard wave number moments are defined by

$$\langle k^n\rangle_t = \int k^n |S(k,t)|^2 dk. \tag{44}$$

Substituting $\psi(x,t)$ into (42), we have

$$F(k,t) = \frac{1}{\sqrt{2\pi}}\int \psi(x,t)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}A\sum_{n=1}^{N}a_n\int u(x-x_n,t)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}A\sum_{n=1}^{N}a_n\int u(x,t)e^{-ik(x+x_n)}dx. \quad (45)$$

Or

$$F(k,t) = AS(k,t)\sum_{n=1}^{N}a_n e^{-ikx_n}. \quad (46)$$

Now consider the ensemble average of the wave number

$$\begin{aligned} E[\langle k\rangle_t] &= E\left[\int k|F(k,t)|^2dk\right] \\ &= A^2\int |S(k,t)|^2\sum_{n,m=1}^{N}ka_n^*a_mE\left[e^{-ik(x_n-x_m)}\right]dk \\ &= A^2|a|^2\int |S(k,t)|^2\sum_{n,m=1}^{N}k\delta_{nm}E\left[e^{-ik(x_n-x_m)}\right]dk \\ &= A^2|a|^2N\int k|S(k,t)|^2 = A^2|a|^2N\langle k\rangle_t = \langle k\rangle_t \end{aligned} \quad (47)$$

and hence

$$E[\langle k\rangle_t] = \langle k\rangle_t. \quad (48)$$

Similarly

$$E[\langle k^2\rangle_t] = \langle k^2\rangle_t \quad (49)$$

and further

$$E\left[\sigma_{x|t}^2\right] = \sigma_{x|t}^2. \quad (50)$$

Thus we see that the ensemble of the wave number moments are the moments of the individual pulse. This is the case because our model deals with random spatial translations only.

4.1. Example. For the example we described in Section 2.1 we substitute (26) into (33) and (39) to obtain

$$\begin{aligned} E[\langle x\rangle_t] &= (c+\gamma k_0)t + E[x_n], \\ \Gamma_{x|t}^2 &= \sigma_{x|0}^2[1+2\beta\gamma t+\gamma^2(\alpha^2+\beta^2)t^2]+\mu^2. \end{aligned} \quad (51)$$

Also using (40) we have

$$\Gamma_{x|t}^2 = \Gamma_{x|0}^2 + \frac{\beta\gamma}{\alpha}t + \frac{\gamma^2(\alpha^2+\beta^2)}{2\alpha}t^2. \quad (52)$$

To be concrete we now take a particular distribution for x_n,

$$P(x_n) = \frac{1}{\sqrt{2\pi q^2}}e^{-(x_n-z)^2/2q^2} \quad (53)$$

in which case

$$\begin{aligned} E[x] &= z, \\ \mu^2 &= q^2 \end{aligned} \quad (54)$$

and therefore we have

$$\begin{aligned} E[\langle x\rangle_t] &= (c+\gamma k_0)t + z, \\ \Gamma_{x|t}^2 &= \frac{1}{2\alpha}[1+2\beta\ \gamma t+\gamma^2(\alpha^2+\beta^2)t^2]+q^2. \end{aligned} \quad (55)$$

5. Wigner Spectrum Approach

We now show that an effective method to study these types of problems is using phase-space methods. The advantage is that one can study nonstationary noise in a direct manner. Suppose we have a random function $\mathbf{z}(\mathbf{t})$, one can think of a particular realization and substitute into the Wigner distribution and then take the ensemble average of it [16–19]

$$\overline{W}(t,\omega) = \frac{1}{2\pi}\int E\left[\mathbf{z}^*\left(t-\frac{1}{2}\tau\right)\mathbf{z}\left(t+\frac{1}{2}\tau\right)\right]e^{-i\tau\omega}d\tau. \quad (56)$$

$\overline{W}_{\mathbf{z}}(t,\omega)$ is called the Wigner spectrum and satisfies the marginal conditions

$$\begin{aligned} \int \overline{W}(t,\omega)d\omega &= E\left[|\mathbf{z}(t)|^2\right], \\ \int \overline{W}(t,\omega)dt &= E\left[|\mathbf{Z}(\omega)|^2\right], \end{aligned} \quad (57)$$

where $\mathbf{Z}(\omega)$ is the Fourier transform of $\mathbf{z}(t)$. As standard we define the autocorrelation function by way of

$$R(t_1,t_2) = E[\mathbf{z}(t_1)\mathbf{z}^*(t_2)] \quad (58)$$

and hence the Wigner spectrum can be written as

$$\overline{W}_{\mathbf{z}}(t,\omega) = \frac{1}{2\pi}\int R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right)e^{-i\tau\omega}d\tau. \quad (59)$$

Taking the inverse

$$R\left(t+\frac{\tau}{2},t-\frac{\tau}{2}\right) = \int \overline{W}(t,\omega,x)e^{i\tau\omega}d\omega \quad (60)$$

and letting $t_2 = t-\tau/2$ and $t_1 = t+\tau/2$, we also have that

$$R(t_1,t_2) = \int \overline{W}\left(\frac{t_1+t_2}{2},\omega\right)e^{-i(t_2-t_1)\omega}d\omega. \quad (61)$$

However in this paper we are dealing with spatial noise and hence we have a spatial random function $\mathbf{z}(\mathbf{x})$. We define the spatial Wigner spectrum by

$$\overline{W}(x,k) = \frac{1}{2\pi}\int E\left[\mathbf{z}^*\left(x-\frac{1}{2}\tau\right)\mathbf{z}\left(x+\frac{1}{2}\tau\right)\right]e^{-i\tau k}d\tau. \quad (62)$$

The spatial autocorrelation is defined by

$$R(x_1, x_2) = E[\mathbf{x}(t_1)x^*(t_2)] \tag{63}$$

and therefore

$$\overline{W}(x,k) = \frac{1}{2\pi}\int R\left(x+\frac{\tau}{2}, x-\frac{\tau}{2}\right)e^{-i\tau k}d\tau. \tag{64}$$

To specialize to our case we have

$$\overline{W}(x,k,t) = \frac{1}{2\pi}\int E\left[\psi^*\left(x-\frac{1}{2}\tau,t\right)\psi\left(x+\frac{1}{2}\tau,t\right)\right]e^{-i\tau k}d\tau \tag{65}$$

and we can substitute (2) into (65). However, it is more effective if we work in the Fourier domain. The Wigner spectrum can be written as

$$\overline{W}(x,k,t) = \frac{1}{2\pi}\int E\left[F^*\left(k+\frac{\theta}{2},t\right)\ F\left(k-\frac{\theta}{2},t\right)\right]\ e^{-i\theta x}d\theta, \tag{66}$$

where $F(k,t)$ is given by (42). Now combining (13) and (46), we have

$$\begin{aligned} F(k,t) &= AS(k,t)\sum_{n=1}^{N} a_n e^{-ikx_n} \\ &= AS(k,0)e^{-i\omega(k)t}\sum_{n=1}^{N} a_n e^{-ikx_n} \end{aligned} \tag{67}$$

and therefore

$$F(k,t) = F(k,0)e^{-i\omega(k)t}. \tag{68}$$

Substituting this into (66), we obtain

$$\begin{aligned} \overline{W}(x,k,t) = \frac{1}{2\pi}\int E\left[F^*\left(k+\frac{\theta}{2},0\right)F\left(k-\frac{\theta}{2},0\right)\right] \\ \times e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t}e^{-i\theta x}d\theta. \end{aligned} \tag{69}$$

Setting $t = 0$, we have

$$\overline{W}(x,k,0) = \frac{1}{2\pi}\int E\left[F^*\left(k+\frac{\theta}{2},0\right)F\left(k-\frac{\theta}{2},0\right)\right]e^{-i\theta x}d\theta \tag{70}$$

and taking the Fourier inverse we have that

$$E\left[F^*\left(k+\frac{\theta}{2},0\right)F\left(k-\frac{\theta}{2},0\right)\right] = \overline{W}(x,k,0)^{i\theta x}dx. \tag{71}$$

Substituting this back into (69), we obtain

$$\begin{aligned} &\overline{W}(x,k,t) \\ &= \frac{1}{2\pi}\int \overline{W}(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t}e^{-i\theta(x-x')}d\theta dx'. \end{aligned} \tag{72}$$

This expresses the Wigner spectrum at an arbitrary time given the spectrum at time zero.

One can further simplify by defining the Wigner distribution for each pulse as

$$W_u(x,k,t) = \frac{1}{2\pi}\int u^*\left(x-\frac{1}{2}\tau,t\right)u\left(x+\frac{1}{2}\tau,t\right)e^{-i\tau k}d\tau. \tag{73}$$

Now consider

$$\begin{aligned} &W_\psi(x,k,t) \\ &= \frac{1}{2\pi}\int \psi^*\left(x-\frac{1}{2}\tau,t\right)\psi\left(x+\frac{1}{2}\tau,t\right)e^{-i\tau k}d\tau \\ &= \frac{A^2}{2\pi}\sum_{n,m=1}^{N} a_n^* a_m \int u^*\left(x-\frac{1}{2}\tau-x_n,t\right) \\ &\qquad \times u\left(x+\frac{1}{2}\tau-x_m,t\right)e^{-i\tau k}d\tau. \end{aligned} \tag{74}$$

Now taking the ensemble average we have

$$\begin{aligned} &\overline{W}(x,k,t) \\ &= \frac{A^2}{2\pi}E\left[\sum_{n,m=1}^{N} a_n^* a_m \int u^*\left(x-\frac{1}{2}\tau-x_n,t\right)\right. \\ &\qquad \left. \times u\left(x+\frac{1}{2}\tau-x_m,t\right)e^{-i\tau k}d\tau\right] \\ &= \frac{A^2}{2\pi}|a|^2\sum_{n,m=1}^{N}\delta_{nm}E\left[\int u^*\left(x-\frac{1}{2}\tau-x_n,t\right)\right. \\ &\qquad \left. \times u\left(x+\frac{1}{2}\tau-x_m,t\right)e^{-i\tau k}d\tau\right] \\ &= \frac{A^2}{2\pi}|a|^2\sum_{n=1}^{N}E\left[\int u^*\left(x-\frac{1}{2}\tau-x_n,t\right)\right. \\ &\qquad \left. \times u\left(x+\frac{1}{2}\tau-x_n,t\right)e^{-i\tau k}d\tau\right] \end{aligned} \tag{75}$$

and therefore we have that in general

$$\overline{W}(x,k,t) = A^2|a|^2E[W_u(x-x_n,k,t)]. \tag{76}$$

Setting $t = 0$, we also have

$$\overline{W}(x,k,0) = A^2|a|^2E[W_u(x-x_n,k,0)]. \tag{77}$$

We now aim at expressing $\overline{W}(x,k,t)$ in terms of $\overline{W}(x,k,0)$. Using (72), we have that

$$\begin{aligned} \overline{W}(x,k,t) &= \frac{1}{2\pi}\int E[W_u(x'-x_n,k,0)]e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \\ &\quad \times e^{-i\theta(x-x')}d\theta dx' \\ &= \frac{1}{2\pi}\int W_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \\ &\quad \times E\left[e^{-i\theta(x-x_n-x')}\right]d\theta dx'. \end{aligned} \tag{78}$$

Or

$$\overline{W}(x,k,t) = \frac{1}{2\pi}\int W_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \times e^{-i\theta(x-x')}E\Big[e^{i\theta x_n}\Big]d\theta dx'. \tag{79}$$

This allows for the calculation of the Wigner spectrum in a simple and direct manner. We also point out that $E[e^{i\theta x_n}]$ may be considered as the characteristic function

$$M(\theta) = E\Big[e^{i\theta x_n}\Big] = \int e^{i\theta x_n}P(x_n)dx_n \tag{80}$$

and hence we may write (79) as

$$\overline{W}(x,k,t) = \frac{1}{2\pi}\int W_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \times e^{-i\theta(x-x')}M(\theta)d\theta dx'. \tag{81}$$

Our previous results can be obtained using this Wigner spectrum. For example, consider the first conditional moment

$$\begin{aligned}
&E[\langle x\rangle_t] \\
&= \int x\overline{W}(x',k,t)dx \\
&= \frac{1}{2\pi}\int xW_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \\
&\quad \times e^{-i\theta(x-x')}M(\theta)d\theta dx'\,dxdk \\
&= -\frac{1}{2\pi i}\int W_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \\
&\quad \times \left\{\frac{\partial}{\partial\theta}e^{-i\theta(x-x')}\right\}M(\theta)d\theta dx'\,dxdk \\
&= -\frac{1}{i}\int W_u(x',k,0)e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t} \\
&\quad \times e^{i\theta x'}\left\{\frac{\partial}{\partial\theta}\delta(\theta)\right\}M(\theta)d\theta dx'\,dk \\
&= \frac{1}{i}\int W_u(x,k,0)\,\frac{\partial}{\partial\theta}e^{i[\omega(k+\theta/2)-\omega(k-\theta/2)]t}e^{i\theta x'}M(\theta)\bigg|_{\theta=0}dx'\,dk
\end{aligned} \tag{82}$$

which evaluates to (33).

6. Conclusion

We emphasize that the above model assumed that the only random variable was the mean of the individual pulses. In a future paper we will consider more general cases and of particular interest is to allow the standard deviation of each pulse to be a random variable. For example instead of (2) we can write

$$\psi(x,t) = A\sum_{n=1}^{N}a_n\sqrt{\alpha_n}u(\alpha_n(x-x_n),t), \tag{83}$$

where $\sqrt{\alpha_n}$ is inserted for normalization purposes. The combination of x_n and α_n allows us to make both the means and standard deviation of each pulse random variable.

We now discuss our main result, namely, (37) and (40) which we repeat here for convenience

$$\begin{aligned}
E[\langle x\rangle_t] &= \langle x\rangle_t + E[x_n], \\
\Gamma^2_{x|t} &= \sigma^2_{x|t} + \mu^2, \\
\Gamma^2_{x|t} &= \Gamma^2_{x|0} + 2t\mathrm{Cov}_{xv|0} + t^2\sigma^2_v.
\end{aligned} \tag{84}$$

First we point out that one can consdier the case for the ensemble as one particle having an initial mean given $\langle x\rangle_0 + E[x_n]$ and an initial standard deviation $\Gamma^2_{x|0} = \sigma^2_{x|t} + \mu^2$.

For the case of no dispersion then $\sigma^2_{x|t} = \sigma^2_{x|0}$ and therefore

$$\Gamma^2_{x|t} = \Gamma^2_{x|0} \quad \text{no dispersion.} \tag{85}$$

For large times the dominant term is $t^2\sigma^2_v$ and since the coefficient of t^2 is manifestly positive we have that

$$\Gamma^2_{x|t} \longrightarrow t^2\sigma^2_v t \longrightarrow \infty. \tag{86}$$

Hence for large times the spread, $\Gamma_{x|t}$, is a linear function of time.

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