Active Optical Lattice Filters

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Optical lattice filter structures including gains are introduced and analyzed. The photonic realization of the active, adaptive lattice filter is described. The algorithms which map between gains space and filter coefficients space are presented and studied. The sensitivities of filter parameters with respect to gains are derived and calculated. An example which is relevant to adaptive signal processing is also provided.

Keywords and phrases: active, adaptive, filter, optical, lattice.

1. INTRODUCTION

Dowling and MacFarlane [1, 2, 3] viewed a lightwave lattice filter design using multistage etalons and resonators as a discrete-time linear system. As such they discussed the behavior of the system in terms of the transmission and reflection transfer functions. Given a desired transfer function denominator for an all-pole system in transmission they developed a layer-peeling algorithm to determine the appropriate reflection coefficients. The reflection transfer function has this same denominator polynomial. They proved that any denominator polynomial, and hence any autoregressive (AR) transfer function in transmission, can be generated by proper choice of the reflection coefficients. Transmission coefficients as well as reflection coefficients can be chosen, but these transmission coefficients do not affect the denominator polynomial. If an autoregressive moving average (ARMA) transfer function is desired in transmission, then it must first be approximated by an AR transfer function of higher order, and a higher-order lattice filter be designed.

The problem with the above optical lattice filter is that it is a passive device, and once it is built with a given set of reflection (also transmission) coefficients, its filtering characteristics cannot be changed. Hence this precludes programming in the field to change the filter as needed. Moreover,

from the point of view of signal processing, the entire set of applications derived from adaptive signal processing is not accessible. Adaptive signal processing is used in communications, control, radar, sonar, seismology, biomedical, target tracking, and so forth [4]. Adaptation is needed if the statistics of a signal are unknown or if a system is time varying (nonstationary). Finite impulse response (FIR) filters have dominated the field, but infinite impulse response (IIR) filters can often provide a system of much lower order [5].

To add adjustment to these optical lattice filters we propose to add gains to each stage. These gains can be changed as needed and also serve to make our filters active instead of passive. Suppose we are given a target all-pole transfer function for transmission whose order is the same as the order (number of stages) of our optical filter. Once the reflection coefficients (by layer peeling) and transmission coefficients are determined and the device is built, then we tune the device by changing the gains. The question of interest is what all-pole transfer functions (or denominator polynomials) can we achieve by changing the gains? Certainly when the gains are all unity we have the denominator polynomial of the all-pole transfer function that we started. From the point of tunable signal processing it would be desirable to generate all "nearby" polynomials of the same degree.

It is interesting to note that after fixing these reflection and transmission coefficients, it is not possible to use the gains to generate an arbitrary all-pole transfer function in transmission. We present an example to illustrate this point. However, if we consider all gains as one when the reflection and transmission coefficients are fixed, then there is an open

neighborhood U of these ones in gains space and an open neighborhood V of the corresponding denominator polynomial coefficients in polynomial coefficients space, so that for each polynomial with coefficients in V there is a unique set of gains in U that produce that polynomial. All that needs to be checked is that the Jacobian matrix of the map from the gains (actually gains squared) to the polynomial coefficients is nonsingular when the gains are all one. If this matrix is nonsingular it is natural to solve for the gains as functions of the coefficients using the standard Newton-Raphson method [6, 7]. The equations expressing the relationship of the polynomial coefficients and the gains are easily generated and have a high degree of symmetry. Hence we have a procedure that will allow us to calculate all nearby (in terms of polynomial coefficients) denominator polynomials for our transfer functions. Thus we can use gains to adapt our filter. Of course we are only interested in stable filters when applications are considered.

For the map from the gains to the denominator polynomial coefficients we stress the case where the gains are all one. However, for a given denominator polynomial there may be other sets of gains that yield the same polynomial coefficients. If for such a point in gain space, the Jacobian matrix of the gains to polynomial coefficients map is nonsingular, then there is an open neighborhood U^* of the gain point and an open neighborhood V^* of the corresponding polynomial coefficient point in polynomial coefficient space on which the map is one-to-one and onto. Of course U and U^* cannot intersect, so we can adapt in the set U (or U^*) without disturbing the other set.

The main purpose of this paper is to provide design tools for active lattice filters that contain active gain elements. This discussion is particularly relevant to an optical architecture that is currently under early development. This photonic realization of the active lattice filter is described in Section 2 of this paper. In particular we advocate a semiconductor laser amplifier structure in which coupling between gain and delay stages is accomplished by surface gratings [8]. While there is a rich tradition of active electronic filters, optical filters with gain are not generally used. Gain elements allow filters with high-quality factors, and filters with programmable transfer functions. Further, gain elements are essential to providing an architecture that may be scaled to reasonable sophistication.

In Section 3 of this paper we discuss the Dowling-MacFarlane optical lattice filter and the layer-peeling procedure for computing the reflection coefficients. We indicate how gains are added to their structure. We then compute the transfer function denominator polynomials as functions of the gains assuming that all reflection and transmission coefficients have been fixed. In Section 4 we provide an example showing that an arbitrary transmission all-pole denominator cannot be generated using gains once the reflection and transmission coefficients have been chosen. Here we work with a two-stage filter and a second-order transfer function denominator polynomial. We generate the map from two-dimensional gains space to two-dimensional polynomial coefficients space. For given polynomial coefficients this map can be represented by two curves in gains space and there are three possible outcomes: transversal intersection where the Jacobian matrix is nonsingular, no intersection and no solution, and tangential intersection where the Jacobian matrix is singular. It is the first case that is of interest to us. It is important to understand that we start with a given polynomial and fix the reflection coefficients so that we have this case if the Jacobian matrix is nonsingular when all gains are unity. Then all "nearby polynomials" can be implemented by a proper choice of gains.

Section 5 contains a development of the map from gains (squared) space to denominator polynomial coefficient space in the general case. Then we state and prove our main result using the inverse function theorem [9]. The assumption on the nonsingularity of the Jacobian matrix is the exact condition needed to employ the Newton-Raphson algorithm to solve our equations for the gains. We return to the example in Section 4 for the nonsingular case, vary the coefficients of the polynomial, and show the computations of the gains. We also provide an example that is relevant to adaptive signal processing, assuming that a feedback loop may be implemented that is fast enough to respond to a changing environment. We start with an output response of an AR system. Next we identify the all-pole transfer function and determine the appropriate reflection coefficients (through layer peeling), which we fix in our lattice filter. Then we allow the time response to change, identify the corresponding denominator polynomials for the new transfer functions, and compute the gains to deliver those polynomials. Here the time response could be replaced by a frequency response or by autocorrelations. In Section 6 we derive and calculate the sensitivities of filter parameters with respect to gains. Section 7 contains our conclusions and a discussion of future research.

2. A REALIZATION ARCHITECTURE

The optical gains that are modeled in this paper may be realized in a number of ways. For example, it is possible to imagine using erbium-doped fibers or waveguides separated by fiber Bragg gratings or an equivalent coupling interface. The passive version of this fiber based-filter architecture is very nicely described in the classic paper by Moslehi et al. [10]. The active fiber filter with gain would have particular interest for fiber sensors and instrumentation [11]. In the interests of faster updating speeds and adaptive operation we are envisioning a semiconductor laser amplifier structure in which coupling between gain and delay stages is accomplished by surface gratings. This architecture also offers faster clock frequencies, higher efficiencies and integrated manufacture advantages.

In Figure 1 is shown a schematic of the active optical lattice filter that uses semiconductor laser amplifier stages to provide gain and delays. By way of example, the figure shows a two-stage active lattice filter with two gain/delay regions and three interstage couplers. The figure is drawn in side view and shows a substrate with an epitaxial-grown quantum well active region. On the surface of this substrate are two electrodes through which injection current may enter the
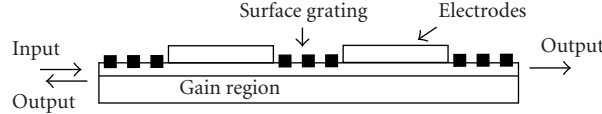
FIGURE 1: Block diagram of an active lattice filter based on surface grating interstage couplers and semiconductor optical amplifiers.

quantum well. The filter stage gain is controlled by the injection current to the individually addressable gains in the amplifying regions. These regions also provide the phase delays that lead to filter responses.

Also on the surface of the substrate are three grating couplers that provide transmission and reflection of optical signals between the two stages under the two electrodes. These gratings may be written by holographic lithography [8, 12] or by focused ion beam micromachining.

In operation, an optical signal is injected into the heterojunction of the structure where it undergoes gain and delay. At the grating couplers part of the signal is reflected and part is transmitted. Each of these components combines with signals from the adjacent stage, and this delay and mix action is the same as in coupled etalons, or in thin film filters. The additional presence of gain provides advantages in filter performance including a tunable response.

In practice, the semiconductor optical amplifier may suffer from two impairments: saturation and phase distortion. Since the theory presented herein applies to linear, time-invariant (LTI) systems, either impairment will render the results presented in this paper an approximation valid over a finite region of operation. Since the filter will be operated in a stable region, well below lasing threshold, the overall gains needed for useful filters will be small, and will vary over a limited range. Hence the linearity condition will often be justified. Similarly, it may also be argued that in this operating region, the injected current will not substantially impact the refractive index of the inversion region, particularly if the effective gain length is kept short and the epitaxy is properly designed. An additional possibility for adjusting and maintaining the correct phase is to add an additional phase control section to the filter as is done in many tunable lasers so as to allow an additional adjustment for phase control.

3. DOWLING-MACFARLANE FILTER

In 1944 MacFarlane and Dowling studied the analysis and design of purely passive coupled Fabry-Perot etalons and thin film filters using digital signal processing techniques [1, 2]. This approach allowed certain design optimizations, and brought an ease of use that proved helpful to electrical engineers working in the photonic telecommunications industry over the last decade. In Figure 2 is shown a z-transform-based block diagram for an exemplary three-stage photonic lattice filter. The interfaces are characterized by reflection and transmission coefficients, and the signal transit time between interfaces is described by a delay block. Considering the realization architecture discussed above, we follow the notation of [2]. For a lossless interface, the field reflection and

transmission coefficients obey an energy conservation condition:

$$r_i^2 + t_i^2 = 1. \tag{1}$$

The transmission transfer function denominator polynomial for one stage is

$$1 + r_0 r_1 z^{-1}, \tag{2}$$

for two stages is

$$1 + (r_0 r_1 + r_1 r_2)z^{-1} + r_0 r_2 z^{-2}, \tag{3}$$

and for three stages is

$$1 + (r_0 r_1 + r_1 r_2 + r_2 r_3)z^{-1} + (r_0 r_2 + r_0 r_1 r_2 r_3 + r_1 r_3)z^{-2} + r_0 r_3 z^{-3}. \tag{4}$$

Simple induction can derive the polynomial for n stages.

In this paper we include gain as well as delay between the interfaces, and this block diagram for an active lattice filter is shown in Figure 3. Equations (2), (3), and (4) may be readily generalized to include this gain. For example, the transmission transfer function denominator polynomial for one stage with gains is

$$1 + r_0 r_1 G_1^2 z^{-1}, \tag{5}$$

for two stages with gains is

$$1 + (r_0 r_1 G_1^2 + r_1 r_2 G_2^2)z^{-1} + r_0 r_2 G_1^2 G_2^2 z^{-2}, \tag{6}$$

and for three stages with gains is

$$1 + (r_0 r_1 G_1^2 + r_1 r_2 G_2^2 + r_2 r_3 G_3^2)z^{-1} \\ + (r_0 r_2 G_1^2 G_2^2 + r_0 r_1 r_2 r_3 G_1^2 G_3^2 + r_1 r_3 G_2^2 G_3^2)z^{-2} \\ + r_0 r_3 G_1^2 G_2^2 G_3^2 z^{-3}. \tag{7}$$

Again, induction can be used to derive the polynomial for n stages. The gain for every stage appears to a power 2, regardless of the number of stages used.

Given a desired polynomial of degree n we can use the layer-peeling process [1, 2, 3] to compute $r_0, r_1, \ldots, r_n$ (actually r_0 can be chosen and the other reflection coefficients are computed). Fixing these reflection coefficients we can simply write the one-stage polynomial with gains as
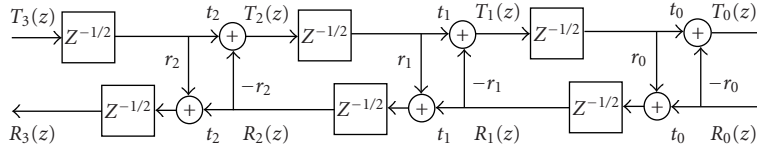
$$1 + c_1 G_1^2 z^{-1}, \tag{8}$$

FIGURE 2: Signal flow diagram for three stages of an optical lattice filter.
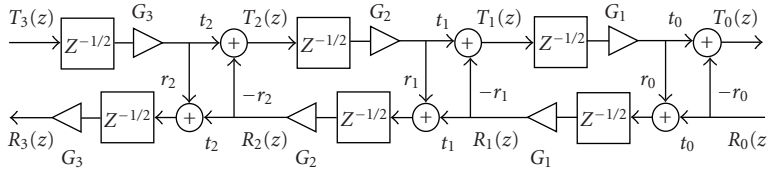

FIGURE 3: Signal flow diagram for three stages of an active optical lattice filter.

the two-stage polynomial with gains as

$$1 + (c_1 G_1^2 + c_2 G_2^2)z^{-1} + c_{12} G_1^2 G_2^2 z^{-2}, \qquad (9)$$

and the three-stage polynomial with gains as

$$
\begin{aligned}
1 &+ (c_1 G_1^2 + c_2 G_2^2 + c_3 G_3^2)z^{-1} \\
&+ (c_{12} G_1^2 G_2^2 + c_{13} G_1^2 G_3^2 + c_{23} G_2^2 G_3^2)z^{-2} + c_{123} G_1^2 G_2^2 G_3^2 z^{-3}.
\end{aligned}
\qquad (10)
$$

Here the c's are computed from the r's in the obvious way. The nth-degree polynomial with gains is

$$
\begin{aligned}
1 &+ \left(\sum_{i=1}^{n} c_i G_i^2 \right)z^{-1} + \left(\sum_{i,j=1,i<j}^{n} c_{ij} G_i^2 G_j^2 \right)z^{-2} \\
&+ \cdots + c_{12\cdots n} G_1^2 G_2^2 \cdots G_n^2 z^{-n}.
\end{aligned}
\qquad (11)
$$

Again the c coefficients are known numbers computed from the r's, and the gains G_i are allowed to vary. Since we are interested in only nonnegative real gains G_i we set $x_i = G_i^2$ and replace (11) by

$$
\begin{aligned}
1 &+ \left(\sum_{i=1}^{n} c_i x_i \right)z^{-1} + \left(\sum_{i,j=1,i<j}^{n} c_{ij} x_i x_j \right)z^{-2} \\
&+ \cdots + c_{12\cdots n} x_1 x_2 \cdots x_n z^{-n}.
\end{aligned}
\qquad (12)
$$

If we determine $x_1, x_2, \ldots, x_n$ then we know $G_1, G_2, \ldots, G_n$. We first consider a two-stage filter and return to the general case later.

4. TWO-STAGE EXAMPLE

In the case of a two-stage active lattice filter, (12) becomes

$$1 + (c_1 x_1 + c_2 x_1)z^{-1} + c_{12} x_1 x_2 z^{-2}, \qquad (13)$$

where the c's are fixed and only the x_i can vary. If a desired denominator polynomial is

$$1 + a_1 z^{-1} + a_2 z^{-2}, \qquad (14)$$

then the equations to determine the gains as function of the polynomial coefficients are

$$
\begin{aligned}
c_1 x_1 + c_2 x_1 &= a_1, \\
c_{12} x_1 x_2 &= a_2.
\end{aligned}
\qquad (15)
$$

Geometrically, these equations represent a straight line and a hyperbola in the closed first quadrant in (x_1, x_2) space. With c_1, c_2, c_{12} determined and a_1, a_2 given, a solution is a point (x_1, x_2) where the line and the hyperbola intersect. These equations can also be thought of as a map from the gains (actually gains squared) space (x_1, x_2) to the desired polynomial coefficient space (a_1, a_2).

For example, if we set $r_0 = 1$, $r_1 = 1/4$, and $r_2 = 1/4$, (3) for the transfer function denominator becomes

$$1 + \left(\frac{5}{16} \right)z^{-1} + \left(\frac{1}{4} \right)z^{-2}. \qquad (16)$$

This discussion could have begun from the point of view of choosing coefficients $a_1 = 5/16$, $a_2 = 1/4$, and $r_0 = 1$, and using the Dowling-MacFarlane layer-peeling algorithm to find $r_1 = 1/4$ and $r_2 = 1/4$. We have stable systems in transmission and reflection since both poles are inside the unit circle. Equations (15) now becomes

$$
\begin{aligned}
\left(\frac{1}{4} \right)x_1 + \left(\frac{1}{16} \right)x_2 &= a_1, \\
\left(\frac{1}{4} \right)x_1 x_2 &= a_2.
\end{aligned}
\qquad (17)
$$

With $a_1 = 5/16$ and $a_2 = 1/4$ we have two solutions $(1, 1)$ and $(1/4, 4)$ for (x_1, x_2). The solution of interest to us
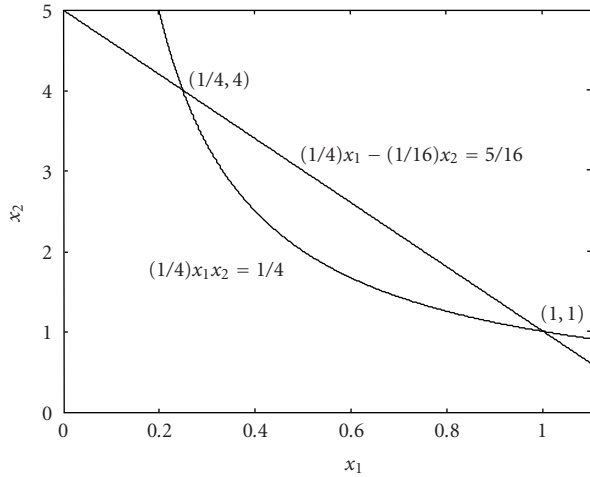
FIGURE 4: Intersection in (x_1, x_2) space for $a_1 = 5/16$ and $a_2 = 1/4$.

is $(1, 1)$. Considering (17) as a map from (x_1, x_2) space to (a_1, a_2) space, the Jacobian matrix is $\left[\begin{smallmatrix} 1/4 & 1/16 \\ (1/4)x_2 & (1/4)x_1 \end{smallmatrix}\right]$. We note that this matrix, evaluated at $(x_1, x_2) = (1, 1)$, is nonsingular. Thus by the inverse function theorem [9] there is an open set U containing $(1, 1)$ in (x_1, x_2) space and open set V containing $(5/16, 1/4)$ in (a_1, a_2) space. Hence for each coefficient pair (a_1, a_2) in V there is exactly one set of gains (squared) (x_1, x_2) in U that solves (17). Geometrically, for $a_1 = 5/16$ and $a_2 = 1/4$ the line and the hyperbola intersect transversally (nontangentially) at $(1, 1)$ in (x_1, x_2) space. If we change a_1 and a_2 slightly we still have a transverse intersection near $(1, 1)$. This intersection is shown graphically in Figure 4.

We now show that it is not possible to find the gains in (17) to do arbitrary pole placement for the same values of r_0, r_1, and r_2. If we let $a_1 = 5/16$ and $a_2 = 1$ in (17), then

$$
\begin{aligned}
\left(\frac{1}{4}\right)x_1 + \left(\frac{1}{16}\right)x_2 &= \frac{5}{16}, \\
\left(\frac{1}{4}\right)x_1 x_2 &= 1
\end{aligned}
\tag{18}
$$

have only complex solutions for x_1 and x_2. Of course the straight line and hyperbola in (17) do not intersect in (x_1, x_2) space. Hence arbitrary denominator polynomials cannot be realized for the transmission and reflection transfer functions if the reflection coefficients are held fixed and the gains varied, for given order (number of lattice stages).

There is one more case of interest. If $a_1 = 1/2$ and $a_2 = 1/4$ in (17), then

$$
\begin{aligned}
\left(\frac{1}{4}\right)x_1 + \left(\frac{1}{16}\right)x_2 &= \frac{1}{2}, \\
\left(\frac{1}{4}\right)x_1 x_2 &= 1
\end{aligned}
\tag{19}
$$

have only the solution $x_1 = 1$ and $x_2 = 4$. Hence the straight line and the hyperbola from (17) intersect at only one point, a point of tangency. Slight changes in a_1 from $1/2$ can result in either no points of intersection or two points of intersection. It is very important to note that the Jacobian matrix $\left[\begin{smallmatrix} 1/4 & 1/16 \\ (1/4)x_2 & (1/4)x_1 \end{smallmatrix}\right]$ of (17) is singular at the point $(x_1, x_2) = (1, 4)$.

5.   MAIN RESULTS

We return to the general case for n stages with gains in Section 3. Equation (12) of interest is

$$
1 + \left(\sum_{i=1}^{n} c_i x_i\right)z^{-1} + \left(\sum_{i,j=1, i<j}^{n} c_{ij} x_i x_j\right)z^{-2} + \cdots + c_{12\ldots n} x_1 x_2 \cdots x_n z^{-n}.
\tag{20}
$$

Here the c's (through the r's) have been chosen by using layer peeling to deliver a specified transmission transfer function denominator without using the gains (or equivalently, when all the $x_i = 1$). If another desired denominator polynomial to be achieved by computing only the gains is

$$
1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n},
\tag{21}
$$

then the equations to determine the gains as functions of the polynomial coefficients are

$$
\begin{aligned}
\left(\sum_{i=1}^{n} c_i x_i\right) &= a_1, \\
\left(\sum_{i,j=1, i<j}^{n} c_{ij} x_i x_j\right) &= a_2, \\
&\;\;\vdots \\
c_{12\ldots n} x_1 x_2 \cdots x_n &= a_n.
\end{aligned}
\tag{22}
$$

The n by n Jacobian matrix of the gains (or gains squared) to polynomial coefficient map is

$$
\mathbf{J} = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ \sum_{j=2}^{n} c_{1j} x_j & c_{12} x_1 + \sum_{j=3}^{n} c_{2j} x_j & \cdots & \sum_{i=1}^{n-1} c_{in} x_i \\ \vdots & \vdots & \ddots & \vdots \\ c_{12\ldots n} x_2 x_3 \cdots x_n & c_{12\ldots n} x_1 x_3 \cdots x_n & \cdots & c_{12\ldots n} x_1 x_2 \cdots x_{n-1} \end{bmatrix}
\tag{23}
$$

which is easily computed by taking partial derivatives in (22). When all gains (squared) x_i in (22) are unity, we denote the corresponding polynomial coefficients by $\hat{a}_1, \hat{a}_2, \ldots, \hat{a}_n$. By the inverse function theorem [9], these are the coefficients of the transfer function denominator polynomial for which the reflection coefficients are fixed.

If we move the a_i from the right-hand side to the left-hand side in (22), we have a system of equations of the form

$$f_1(x_1, x_2, \ldots, x_n) = 0,$$
$$f_2(x_1, x_2, \ldots, x_n) = 0,$$
$$\vdots \qquad\qquad (24)$$
$$f_n(x_1, x_2, \ldots, x_n) = 0.$$

Equations (24) may be solved by the Newton-Raphson algorithm to find the $(x_1, x_2, \ldots, x_n)$ as a function of the $(a_1, a_2, \ldots, a_n)$, so long as the Jacobian matrix of this system is nonsingular.

As an example, we now return to the two-stage example in Section 4 and apply the Newton-Raphson algorithm to compute the correct gains as we vary the desired polynomial coefficients. We use Matlab to implement Newton-Raphson [13]. Equations (17) of interest are

$$\left(\frac{1}{4}\right)x_1 + \left(\frac{1}{16}\right)x_2 = a_1,$$
$$\left(\frac{1}{4}\right)x_1 x_2 = a_2. \qquad (25)$$

The starting values are $a_1 = 5/16$ and $a_2 = 1/4$ for which the solutions are $x_1 = 1$ and $x_2 = 1$. Of course this implies that $G_1 = 1$ and $G_2 = 1$. To make the changes in a_1 and a_2 somewhat random we choose

$$a_1 = \frac{5}{16} \pm 0.05 \times (\text{rand} - 0.5),$$
$$a_2 = \frac{1}{4} \pm 0.05 \times (\text{rand} - 0.5), \qquad (26)$$

where rand is a Matlab command for a uniformly distributed random variable with values between 0 and 1. Then we solve for x_1, x_2, G_1, and G_2. Our computations yield the results in Table 1.

The Newton-Raphson algorithm actually provides two solutions (x_1, x_2) for each a_1, a_2, but we take only that solution which is on the same side of the straight line $x_2 = 4x_1$ as the point $(1, 1)$ (recall that we work in an open neighborhood of $(1, 1)$). The line $x_2 = 4x_1$ represents the set of points where the Jacobin matrix $\left[\begin{smallmatrix} 1/4 & 1/16 \\ (1/4)x_2 & (1/4)x_1 \end{smallmatrix}\right]$ is singular.

These results follow from the point of view that given the desired second-degree denominator polynomial, we program the gains to achieve that polynomial. We next merge

TABLE 1: Filter coefficients and gains.

a_1	a_2	x_1	x_2	G_1	G_2
5/16	1/4	1	1	1	1
0.3350	0.2366	1.13	0.83	1.063	0.910
0.31780	0.2493	1.028	0.969	1.013	0.984
0.3321	0.2631	1.086	0.968	1.042	0.983
0.3103	0.2259	1.019	0.886	1.009	0.9412
0.3286	0.2472	1.08	0.909	1.039	0.953
0.3067	0.2646	0.947	1.116	0.973	1.056
0.2914	0.2619	0.861	1.215	0.9279	1.1022
0.3172	0.2338	1.045	0.894	1.022	0.945
0.2907	0.2708	0.840	1.28	0.916	1.131
0.3170	0.2697	0.997	1.081	0.998	1.039
0.3346	0.2574	1.10	0.931	1.048	0.964
0.2968	0.2245	0.872	1.25	0.933	1.118
0.3306	0.2649	1.076	0.984	1.037	0.991
0.3276	0.2448	1.084	0.902	1.041	0.949
0.3239	0.2651	1.04	1.018	1.019	1.008
0.3152	0.2623	0.998	1.051	0.998	1.025
0.2909	0.2483	0.882	1.125	0.939	1.06
0.3166	0.2673	0.998	1.07	0.998	1.034
0.3112	0.2351	1.01	0.928	1.004	0.963

our gain computation procedure with an adaptive process so that we start with data, identify the AR transfer function, and compute the appropriate gains.

We generate data according to the linear time-varying difference equation

$$y(n) + \left(\frac{13}{24}\right)y(n-1) + \left(\frac{5}{8}\right)y(n-2)$$
$$+ \left[\frac{8}{27} + \frac{1}{27}\cos(n-3)\right]y(n-3) = u(n) \qquad (27)$$

with zero initial conditions $y(-1) = y(-2) = y(-3) = 0$ and input $u(n) = \delta(n)$, the discrete delta function. We assume that the data is associated with the time-invariant linear system

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 y(n-3) = u(n) \quad (28)$$

and we identify a_1, a_2, a_3 from the data as n moves. These a_1, a_2, a_3 are the coefficients of the desired AR denominator polynomial $1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}$. Then we compute the x_1, x_2, x_3 (and hence the gains G_1, G_2, G_3) from the equations

$$c_1 x_1 + c_2 x_2 + c_3 x_3 = a_1,$$
$$c_{12} x_1 x_2 + c_{13} x_1 x_3 + c_{23} x_2 x_3 = a_2, \qquad (29)$$
$$c_{123} x_1 x_2 x_3 = a_3.$$

For the starting point we take the difference equation

$$y(n) + \left(\frac{13}{24}\right)y(n-1) + \left(\frac{5}{8}\right)y(n-2) + \left(\frac{1}{3}\right)y(n-3) \\ = u(n) \tag{30}$$

and thus the polynomial $1 + (13/24)z^{-1} + (5/8)z^{-2} + (1/3)z^{-3}$ results. This difference equation can be found by substituting $u(n) = \delta(n)$ into (27) with zero initial conditions, evaluating $y(0), y(1), y(2), y(3)$, and computing a_1, a_2, a_3 from (28) using 3 linear equations (for $y(n) = y(1), y(2), y(3)$) in 3 unknowns. Choosing $r_0 = 1$ we compute the reflection coef-

ficients $r_1 = 1/4$, $r_2 = 1/2$, $r_3 = 1/3$ by layer peeling. Thus (29) becomes (see (7) and (10))

$$\left(\frac{1}{4}\right)x_1 + \left(\frac{1}{8}\right)x_2 + \left(\frac{1}{6}\right)x_3 = a_1,$$
$$\left(\frac{1}{2}\right)x_1x_2 + \left(\frac{1}{24}\right)x_1x_3 + \left(\frac{1}{12}\right)x_2x_3 = a_2, \tag{31}$$
$$\left(\frac{1}{3}\right)x_1x_2x_3 = a_3.$$

The Jacobian matrix is

$$\begin{bmatrix} \left(\frac{1}{4}\right) & \left(\frac{1}{8}\right) & \left(\frac{1}{6}\right) \\ \left(\frac{1}{2}\right)x_2 + \left(\frac{1}{24}\right)x_3 & \left(\frac{1}{2}\right)x_1 + \left(\frac{1}{12}\right)x_3 & \left(\frac{1}{24}\right)x_1 + \left(\frac{1}{12}\right)x_2 \\ \left(\frac{1}{3}\right)x_2x_3 & \left(\frac{1}{3}\right)x_1x_3 & \left(\frac{1}{3}\right)x_1x_2 \end{bmatrix}. \tag{32}$$

As discussed above, this matrix is nonsingular when x_1, x_2, x_3 are all unity. We then substitute in the varying values for a_1, a_2, a_3 and compute the corresponding x_1, x_2, x_3 (from (31)) and G_1, G_2, G_3. The first computation is for $a_1 = 13/24$, $a_2 = 5/8$, and $a_3 = 1/3$, and the solution is $x_1 = 1, x_2 = 1$, and $x_3 = 1$.

The second computation involves a_1, a_2, a_3 calculated by substituting $u(n) = \delta(n)$ into (27) with zero initial conditions, evaluating $y(0), y(1), y(2), y(3), y(4)$, and computing a_1, a_2, a_3 from (28) using 3 linear equations (for $y(n) = y(2), y(3), y(4)$) in 3 unknowns. The third and higher computations proceed in the obvious way. We record our results in Table 2

For this example we could have driven the system by noise, computed the autocorrelations, arrived at the polynomial coefficients through standard algorithms, and then computed the gains using our technique. Moreover, we also could have estimated the polynomial coefficients using a recursive least squares method based on the Kalman filter. However, we decided to keep things simple and work straight with the data from the difference equation, using sets of 3 equations in 3 unknowns. With noisy data we also could have used more linear equations in the 3 unknowns and found pseudoinverse solutions.

6. PARAMETER SENSITIVITY ANALYSIS

The goal is to quantify the variation of denominator coefficients as the gains vary. Since the gains are physically realized their numerical value is prone to change. This might result in a significantly different transfer function. Hence the

perturbation of the coefficients a_i's due to error in the realization of the gains G_j's is worth analyzing. We quantify the perturbation relations by a measurable parameter called sensitivity. For ease of notation we analyze the sensitivity of the coefficients a_i with respect to the gains squared x_j.

The traditional definition of sensitivity of a parameter a with respect to a parameter x is given by

$$S = \frac{da/a}{dx/x}. \tag{33}$$

This can be viewed as the relative change of a with respect to change in x. Low values of S means the coefficients are somewhat insensitive to gain changes. The next step is to see how S fits into the nonlinear gains squared-coefficients relations.

The general gain squared-coefficients relation according to (22) can be expressed in the form

$$f_1(x_1, x_2, \ldots, x_n) = a_1,$$
$$f_2(x_1, x_2, \ldots, x_n) = a_2,$$
$$\vdots \tag{34}$$
$$f_n(x_1, x_2, \ldots, x_n) = a_n.$$

The above set of equations can be represented in a vector form as

$$f(\mathbf{x}) = \mathbf{a}, \tag{35}$$

where $\mathbf{x} = [x_1, \ldots, x_n]^T$ and $\mathbf{a} = [a_1, \ldots, a_n]^T$ and T is the matrix-transpose operator.

TABLE 2: Filter coefficients and gains.

a_1	a_2	a_3	x_1	x_2	x_3	G_1	G_2	G_3
13/24	5/8	1/3	1	1	1	1	1	1
13/24	5/8	0.3163	1.06948	0.95293	0.93108	1.0342	0.9762	0.9649
13/24	5/8	0.2809	1.19	0.883	0.8006	1.0909	0.9397	0.8948
13/24	5/8	0.2596	1.255	0.852	0.7275	1.1203	0.9230	0.8529
13/24	5/8	0.2721	1.2182	0.8702	0.7699	1.1037	0.9328	0.8774
13/24	5/8	0.3068	1.1045	0.93142	0.89466	1.0509	0.9651	0.9459
13/24	5/8	0.3319	1.00627	0.9954	0.9938	1.0031	0.9977	0.9969
13/24	5/8	0.3242	1.03851	0.9731	0.96238	1.0191	0.9865	0.981
13/24	5/8	0.2909	1.1589	0.90059	0.83612	1.0765	0.94899	0.91439
13/24	5/8	0.2626	1.24651	0.85683	0.73761	1.1130	0.9276	0.8639
13/24	5/8	0.2652	1.23887	0.86039	0.74641	1.0678	0.9544	0.9254
13/24	5/8	0.2965	1.0.1487	0.98940	0.98565	1.0074	0.9946	0.9928
13/24	5/8	0.3299	1.12353	0.92031	0.87447	1.0599	0.9607	0.93513
13/24	5/8	0.3014	1.22996	0.8646	0.75661	1.1090	0.9298	0.8698
13/24	5/8	0.2682	1.25177	0.85441	0.73155	1.1188	0.9243	0.8553
13/24	5/8	0.2608	1.17455	0.89230	0.81895	1.0838	0.9446	0.9049
13/24	5/8	0.2861	1.05207	0.96415	0.94878	1.0257	0.9819	0.9924
13/24	5/8	0.3329	1.08782	0.94150	0.91214	1.0429	0.9703	0.955
13/24	5/8	0.3114	1.20631	0.8761	0.78347	1.0983	0.9360	0.8851
13/24	5/8	0.2760	1.25612	0.85242	0.72651	1.1207	0.9233	0.8523

Writing (35) in differential form we get

$$\mathbf{J}\mathbf{d_x} = \mathbf{d_a},\qquad(36)$$

where $\mathbf{d_x} = [dx_1,\ldots,dx_n]^T$, $\mathbf{d_a} = [da_1,\ldots,da_n]^T$, and

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial f_1(\mathbf{x})}{\partial x_1} & \dfrac{\partial f_1(\mathbf{x})}{\partial x_2} & \cdots & \dfrac{\partial f_1(\mathbf{x})}{\partial x_n} \\[2mm] \dfrac{\partial f_2(\mathbf{x})}{\partial x_1} & \dfrac{\partial f_2(\mathbf{x})}{\partial x_2} & \cdots & \dfrac{\partial f_2(\mathbf{x})}{\partial x_n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \dfrac{\partial f_n(\mathbf{x})}{\partial x_1} & \dfrac{\partial f_n(\mathbf{x})}{\partial x_2} & \cdots & \dfrac{\partial f_n(\mathbf{x})}{\partial x_n} \end{bmatrix},\qquad(37)$$

where $\mathbf{J}$ is the Jacobian matrix. The entries of the matrix $\mathbf{J}$ are $\mathbf{J}_{i,j}$. It is easy to see from (36) that if only x_j changes, then

$$\mathbf{J}_{i,j}\mathbf{d_{x_j}} = \mathbf{d_{a_i}}.\qquad(38)$$

Plugging (38) into (33) and a little manipulation gives

$$\mathbf{S}_{i,j} = \frac{\mathbf{J}_{i,j}\mathbf{x}_j}{\mathbf{a}_i},\qquad(39)$$

where $\mathbf{S}_{i,j}$ relates the sensitivity of $\mathbf{a}_i$ to changes in $\mathbf{x}_j$. Thus we can find a sensitivity matrix $\mathbf{S}$ with $\mathbf{S}_{i,j}$ as its elements. $\mathbf{S}$ gives the one-to-one sensitivity relations between the square of the gains and the coefficients.

An upper bound on the sensitivity matrix, when one or more gains vary, is estimated using the 2-norm as follows. Equation (36) can be written as

$$\mathbf{J}\mathbf{d_x} = \mathbf{d_a}.\qquad(40)$$

Applying norm to both sides of the equation we get

$$||\mathbf{J}\mathbf{d_x}|| = ||\mathbf{d_a}|| \implies ||\mathbf{d_a}|| \leq ||\mathbf{J}||\,||\mathbf{d_x}||.\qquad(41)$$

Here $||\cdot||$ denotes $\sqrt{\max(\text{eigenvalues of } \mathbf{J}^T\mathbf{J})}$ for matrices and 2-norm for vectors. Thus we get

$$\frac{||\mathbf{d_a}||}{||\mathbf{d_x}||} \leq ||\mathbf{J}||\qquad(42)$$

or equivalently,

$$\frac{||\mathbf{d_a}||/||\mathbf{a}||}{||\mathbf{d_x}||/||\mathbf{x}||} = s \leq \frac{||\mathbf{J}||\,||\mathbf{x}||}{||\mathbf{a}||}.\qquad(43)$$

Here s is sensitivity with respect to gain squared. Thus (43) gives an upper bound on sensitivity. The sensitivity bound (s) is tabulated in Table 3 for different gains squared.

TABLE 3: Sensitivity of filter parameters respective to gains.

a_1	a_2	a_3	x_1	x_2	x_3	s
13/24	5/8	1/3	1	1	1	1.9702
13/24	5/8	0.3163	1.06948	0.95293	0.93108	1.9359
13/24	5/8	0.2809	1.19	0.883	0.8006	1.9109
13/24	5/8	0.2596	1.255	0.852	0.7275	1.9139
13/24	5/8	0.2721	1.2182	0.8702	0.7699	1.9109
13/24	5/8	0.3068	1.1045	0.93142	0.89466	1.9243
13/24	5/8	0.3319	1.00627	0.9954	0.9938	1.9664
13/24	5/8	0.3242	1.03851	0.9731	0.96238	1.9492
13/24	5/8	0.2909	1.1589	0.90059	0.83612	1.9135
13/24	5/8	0.2626	1.24651	0.85683	0.73761	1.9129
13/24	5/8	0.2652	1.23887	0.86039	0.74641	1.9121
13/24	5/8	0.2965	1.0.1487	0.9894	0.98565	1.9162
13/24	5/8	0.3299	1.12353	0.92031	0.87447	1.9615
13/24	5/8	0.3014	1.22996	0.8646	0.75661	1.9196
13/24	5/8	0.2682	1.25177	0.85441	0.73155	1.9115
13/24	5/8	0.2608	1.17455	0.8923	0.81895	1.9135
13/24	5/8	0.2861	1.05207	0.96415	0.94878	1.9119
13/24	5/8	0.3329	1.08782	0.9415	0.91214	1.969
13/24	5/8	0.3114	1.20631	0.8761	0.78347	1.9294
13/24	5/8	0.276	1.25612	0.85242	0.72651	1.9106

The results in Table 3 show that the active lattice is reasonably robust with respect to gain tolerances. That is to say that the filter response does not appreciably change with small injection current uncertainties. This advantageous result is somewhat expected given the traditional lattice filter's robustness with respect to round off error or the reflection coefficient variations [2, 4, 13].

7. CONCLUSIONS AND FUTURE RESEARCH

We have added gains to the optical lattice filters [1, 2, 3] in order to make the response functions programmable. To determine the denominator polynomials achievable by gains only we set up a map from the gains (squared) to the polynomial coefficients. Applying the inverse function theorem we provide a result giving sufficient conditions that all polynomials in an open set can be generated by appropriated choices of gains. This result naturally leads to a method for computing the gains using the Newton-Raphson algorithm. Two interesting examples are given, one of which stressed the adaptive signal processing point of view. We also performed a sensitivity analysis, measuring how the denominator coefficients vary as the gains vary.

We have shown that the presence of gains in the lattice filter can provide additional flexibility in the filter response. These gains may be implemented in a semiconductor laser amplifier, and hence these will be adjustable at GHz rates. In accompanying experimental work, we will be building and testing these devices, and exploring their range of operation both theoretically and empirically. This range will be bounded on one end by the noise figure and stability on the other. In addition, theoretical and experimental work is also underway in two-dimensional active lattice filters based on four-directional couplers [14].

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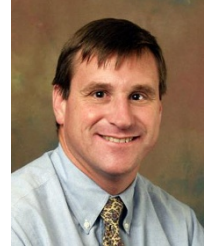
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