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Wigner Distribution Moments Measured as Intensity Moments in Separable First-Order Optical Systems

Martin J. Bastiaans

Faculteit Elektrotechniek, Technische Universiteit Eindhoven, Postbus 513, 5600 MB Eindhoven, The Netherlands
Email: m.j.bastiaans@tue.nl

Tatiana Alieva

Facultad de Ciencias Físicas, Universidad Complutense de Madrid, Ciudad Universitaria s/n, Madrid 28040, Spain
Email: talieva@fis.ucm.es

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It is shown how all global Wigner distribution moments of arbitrary order can be measured as intensity moments in the output plane of an appropriate number of separable first-order optical systems (generally anamorphic ones). The minimum number of such systems that are needed for the determination of these moments is derived.

Keywords and phrases: Wigner distribution moments, beam characterization, first-order optical systems.

1. INTRODUCTION

After the introduction of the Wigner distribution (WD) [1] for the description of coherent and partially coherent optical fields [2], it became an important tool for optical signal/image analysis and beam characterization [3, 4, 5]. The WD completely describes the complex amplitude of a coherent optical field (up to a constant phase factor) or the mutual coherence function of a partially coherent field. As the WD of a two-dimensional optical field is a function of four variables, it is difficult to analyze. Therefore, the optical field is often represented not by the WD itself, but by its global moments. Beam characterization based on the second-order moments of the WD thus became the basis of an International Organization for Standardization standard [6].

Some of the WD moments can directly be determined from measurements of the intensity distributions in the image plane or the Fourier plane, but most of the moments cannot be determined in such an easy way. In order to calculate such moments, additional information is required. Since first-order optical systems [7]—also called *ABCD* systems—produce affine transformations of the WD in phase space, the intensity distributions measured at the output of such systems can provide such additional information. The application of *ABCD* systems for the measurements of the second-order WD moments has been reported in several publications [8, 9, 10, 11, 12, 13].

It is the aim of this paper to show how all global WD moments can be measured as intensity moments only. We show that not only the second-order moments, but also all other moments of the four-dimensional WD can be obtained from measurements of only intensity distributions in an appropriate number of (generally anamorphic) separable first-order optical systems.

2. WIGNER DISTRIBUTION

Let partially coherent light be described by a temporally stationary stochastic process $f(x, y; t)$; as far as the time dependence is concerned, the ensemble average of the product $f(x_1, y_1; t_1) f^*(x_2, y_2; t_2)$, where the asterisk denotes complex conjugation, is then only a function of the time difference $t_1 - t_2$:

$$\mathrm{E}\{f(x_1, y_1; t_1) f^*(x_2, y_2; t_2)\} = \gamma(x_1, x_2; y_1, y_2; t_1 - t_2). \quad (1)$$

The function $\gamma(x_1, x_2; y_1, y_2; \tau)$ is known as the mutual coherence function [14, 15, 16, 17] of the stochastic process $f(x, y; t)$. The mutual power spectrum [16, 17] or cross-spectral density function [18] $\Gamma(x_1, x_2; y_1, y_2; \omega)$ is defined as the temporal Fourier transform of the mutual coherence function:

$$\begin{aligned}\Gamma&(x_1, x_2; y_1, y_2; \omega)\\ &= \int_{-\infty}^{\infty} \gamma(x_1, x_2; y_1, y_2; \tau) \exp(j\omega\tau) d\tau.\end{aligned} \quad (2)$$

For $x_1 = x_2 = x$, $y_1 = y_2 = y$, the cross-spectral density function reduces to the (auto) power spectrum $\Gamma(x, x; y, y; \omega)$, which represents the intensity distribution of the light for the temporal frequency ω. Since in the present discussion the explicit temporal-frequency dependence is of no importance, we will, for the sake of convenience, omit the temporal-frequency variable ω from the formulas in the remainder of the paper.

The Wigner distribution of partially coherent light is defined in terms of the cross-spectral density function by

$$\begin{aligned} W(x,u;y,v) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma\Big(x+\frac{1}{2}x', x-\frac{1}{2}x'; y+\frac{1}{2}y', y-\frac{1}{2}y'\Big) \\ &\quad\times \exp\big[-j2\pi(ux'+vy')\big]dx'\,dy'. \end{aligned} \tag{3}$$

A distribution function according to definition (3) was first introduced in optics by Walther [19, 20], who called it the generalized radiance. The WD $W(x, u; y, v)$ represents partially coherent light in a combined space/spatial-frequency domain, the so-called phase space, where u is the spatial-frequency variable associated to the space variable x, and v the spatial-frequency variable associated to the space variable y.

In this paper we consider the normalized moments of the WD, where the normalization is with respect to the total energy E of the signal:

$$\begin{aligned} E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,u;y,v)\,dx\,du\,dy\,dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Gamma(x,x;y,y)\,dx\,dy. \end{aligned} \tag{4}$$

These normalized moments μ_{pqrs} of the WD are thus defined by

$$\begin{aligned} \mu_{pqrs}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,u;y,v)\,x^p u^q y^r v^s\,dx\,du\,dy\,dv \quad (p,q,r,s\geq 0) \\ &= \frac{1}{(4\pi j)^{q+s}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r \Big(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\Big)^q \Big(\frac{\partial}{\partial y_1}-\frac{\partial}{\partial y_2}\Big)^s \Gamma(x_1,x_2;y_1,y_2)\Big|_{x_1=x_2=x,\,y_1=y_2=y} dx\,dy. \end{aligned} \tag{5}$$

Note that for $q = s = 0$ we have intensity moments, which can easily be measured:

$$\begin{aligned} \mu_{p0r0}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(x,u;y,v) \\ &\qquad\times x^p y^r\,dx\,du\,dy\,dv \quad (p,r\geq 0) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^p y^r \Gamma(x,x;y,y)\,dx\,dy. \end{aligned} \tag{6}$$

The WD moments μ_{pqrs} provide valuable tools for the characterization of optical beams, see, for instance [21]. First-order moments yield the position of the beam (μ_{1000} and μ_{0010}) and its direction (μ_{0100} and μ_{0001}). Second-order moments give information about the spatial width of the beam (the shape μ_{2000} and μ_{0020} of the spatial ellipse and its orientation μ_{1010}) and the angular width in which the beam is radiating (the shape μ_{0200} and μ_{0002} of the spatial-frequency ellipse and its orientation μ_{0101}); moreover, they provide information about its curvature (μ_{1100} and μ_{0011}) and its twist (μ_{1001} and μ_{0110}). Many important beam characterizers, like the overall beam quality [12]

$$\begin{aligned} &(\mu_{2000}\mu_{0200}-\mu_{1100}^2)+(\mu_{0020}\mu_{0002}-\mu_{0011}^2) \\ &\quad +2(\mu_{1010}\mu_{0101}-\mu_{1001}\mu_{0110}), \end{aligned} \tag{7}$$

are based on second-order moments. Higher-order moments are used, for instance, to characterize the beam's symmetry and its sharpness [21].

3. SEPARABLE FIRST-ORDER OPTICAL SYSTEMS

It is well known that the input-output relationship between the WD $W_{\text{in}}(x, u; y, v)$ at the input plane and the WD $W_{\text{out}}(x, u; y, v)$ at the output plane of a separable first-order optical system reads [3, 4, 5]

$$\begin{aligned} &W_{\text{out}}(x,u;y,v) \\ &\quad = W_{\text{in}}(d_x x-b_x u, -c_x x+a_x u; d_y y-b_y v, -c_y y+a_y v). \end{aligned} \tag{8}$$

The coefficients a_x, b_x, c_x, d_x and a_y, b_y, c_y, d_y are the matrix entries of the symplectic ray transformation matrix [7] that relates the position x, y and direction u, v of an optical ray in the input and the output plane of the first-order optical system:

$$\begin{bmatrix} x_{\text{out}} \\ y_{\text{out}} \\ u_{\text{out}} \\ v_{\text{out}} \end{bmatrix} = \begin{bmatrix} a_x & 0 & b_x & 0 \\ 0 & a_y & 0 & b_y \\ c_x & 0 & d_x & 0 \\ 0 & c_y & 0 & d_y \end{bmatrix} \begin{bmatrix} x_{\text{in}} \\ y_{\text{in}} \\ u_{\text{in}} \\ v_{\text{in}} \end{bmatrix}. \tag{9}$$

For separable systems, symplecticity simply reads $a_x d_x - b_x c_x = 1$ and $a_y d_y - b_y c_y = 1$. Note that in a first-order optical system, with such a symplectic ray transformation matrix, the total energy E, see (4), is invariant.

As examples of first-order optical systems we mention the following in particular:

(i) a section of free space in the paraxial approximation, or "parabolic" system [22] (with $a = d = 1$, $c = 0$, and b proportional to the propagation distance z),
(ii) a fractional Fourier transform system [23], or "elliptic" system [22] (with $a = d = \cos\alpha$ and $b = -c = \sin\alpha$),
(iii) a "hyperbolic" system [22] (with $a = d = \cosh\alpha$ and $b = c = \sinh\alpha$).

These three systems are characterized by one parameter. Other one-parameter first-order optical systems are

(i) a thin lens (with $a = d = 1$, $b = 0$, and c inverse proportional to the focal distance),
(ii) an ideal magnifier (with $a = m$, $d = 1/m$, $b = c = 0$).

The latter systems however—like all systems for which the input and output planes are conjugate planes—cannot be used to determine the moments, as we will see later, because they have the property $b \equiv 0$.

The normalized moments μ^{out}_{pqrs} of the output WD $W_{\text{out}}(x,u;y,v)$ are related to the normalized moments $\mu^{\text{in}}_{pqrs} = \mu_{pqrs}$ of the input WD $W_{\text{in}}(x,u;y,v)$ as

$$\begin{aligned}
\mu^{\text{out}}_{pqrs}E &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{out}}(x,u;y,v) \\
&\qquad\times x^p u^q y^r v^s \,dx\,du\,dy\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{in}}\big(d_x x - b_x u, -c_x x + a_x u; \\
&\qquad d_y y - b_y v, -c_y y + a_y v\big) \\
&\qquad\times x^p u^q y^r v^s \,dx\,du\,dy\,dv \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W_{\text{in}}(x,u;y,v)\left(a_x x + b_x u\right)^p \\
&\qquad\times \left(c_x x + d_x u\right)^q \left(a_y y + b_y v\right)^r \\
&\qquad\times \left(c_y y + d_y v\right)^s \,dx\,du\,dy\,dv \\
&= E\sum_{k=0}^{p}\sum_{l=0}^{q}\sum_{m=0}^{r}\sum_{n=0}^{s}\binom{p}{k}\binom{q}{l}\binom{r}{m}\binom{s}{n} a_x^{p-k} b_x^k c_x^l \\
&\qquad\times d_x^{q-l} a_y^{r-m} b_y^m c_y^n d_y^{s-n} \\
&\qquad\times \mu_{p-k+l,q-l+k,r-m+n,s-n+m},
\end{aligned} \tag{10}$$

and for the intensity moments in particular (i.e., $q = s = 0$) we have

$$\mu^{\text{out}}_{p0r0} = \sum_{k=0}^{p}\sum_{m=0}^{r}\binom{p}{k}\binom{r}{m} a_x^{p-k} b_x^k a_y^{r-m} b_y^m \,\mu_{p-k,k,r-m,m}. \tag{11}$$

The remainder of this paper is based on (11), in which the output intensity moments μ^{out}_{p0r0} are expressed in terms of the input moments μ_{pqrs} and the system parameters a_x, a_y, b_x, and b_y. Note that only the parameters a and b enter this equation; the parameters c and d can be chosen freely, as long as the symplecticity condition $a_x d_x - b_x c_x = a_y d_y - b_y c_y = 1$ is satisfied.

4. RELATIONS BETWEEN INPUT AND OUTPUT MOMENTS

4.1. First-order moments

For the first-order moments, the following two equations are relevant:

$$\mu^{\text{out}}_{1000} = a_x\mu_{1000} + b_x\mu_{0100}, \tag{12}$$

$$\mu^{\text{out}}_{0010} = a_y\mu_{0010} + b_y\mu_{0001}, \tag{13}$$

which correspond to (11) with $pqrs = 1000$ and $pqrs = 0010$, respectively, and the four input moments μ_{1000}, μ_{0100}, μ_{0010}, and μ_{0001} can be determined by measuring the intensity moments μ^{out}_{1000} and μ^{out}_{0010} in the output planes of two systems with different values of a and b, see (12) and (13), respectively.

In the case of a fractional Fourier transform system we can choose, for instance, [24, 25], the fractional angles $\alpha_x = \alpha_y = 0$ (leading to $a_x = a_y = 1$ and $b_x = b_y = 0$) and $\alpha_x = \alpha_y = \pi/2$ (leading to $a_x = a_y = 0$ and $b_x = b_y = 1$), but any other choice could be made as well, as long as it leads to four independent equations. In the case of free space propagation, we simply choose two different values of the propagation distance z, corresponding to two different values of b_x and b_y (with $a_x = a_y = 1$, of course).

Note that the two first-order optical systems can always be chosen such that they are isotropic, $a_x = a_y = a_i$, $b_x = b_y = b_i$, and so forth ($i = 1, 2$), with identical behavior in the x and the y direction.

4.2. Second-order moments

For the $3 + 4 + 3 = 10$ second-order moments, the following equations are relevant:

$$\mu^{\text{out}}_{2000} = a_x^2\mu_{2000} + 2a_x b_x\mu_{1100} + b_x^2\mu_{0200}, \tag{14}$$

$$\mu^{\text{out}}_{1010} = a_x a_y\mu_{1010} + a_x b_y\mu_{1001} + b_x a_y\mu_{0110} + b_x b_y\mu_{0101}, \tag{15}$$

$$\mu^{\text{out}}_{0020} = a_y^2\mu_{0020} + 2a_y b_y\mu_{0011} + b_y^2\mu_{0002}, \tag{16}$$

which equations correspond to (11) with $pqrs = 2000$, $pqrs = 1010$, and $pqrs = 0020$, respectively.

The three input moments μ_{2000}, μ_{1100}, and μ_{0200} can be determined by measuring the intensity moment μ^{out}_{2000} in the output planes of three systems with different values of a_x and b_x, see (14). Likewise, with the transversal coordinate x replaced by y, the three input moments μ_{0020}, μ_{0011}, and μ_{0002} can be determined by measuring the intensity moment μ^{out}_{0020}

in the output planes of three systems with different values of a_y and b_y, see (15). Note that we can choose $a_x = a_y = a_i$ and $b_x = b_y = b_i$ $(i = 1, 2, 3)$ for these three systems, in which case we are obviously using isotropic systems.

The other four input moments μ_{1010}, μ_{1001}, μ_{0110}, and μ_{0101} follow from measuring the intensity moment μ_{1010}^{out} in the output planes of four different systems, see (15). However, if we would use only isotropic systems, like we could do for (14) and (16), (15) would reduce to

$$\mu_{1010}^{\text{out}} = a^2\mu_{1010} + ab(\mu_{1001} + \mu_{0110}) + b^2\mu_{0101} \tag{17}$$

and we can only determine the combination $\mu_{1001} + \mu_{0110}$. Hence, while three systems may be isotropic again—and, for instance, be identical to the ones that we used when we were dealing with (14) and (16)—at least one system should be anamorphic.

We conclude that all ten second-order moments can be determined from the knowledge of the output intensities of four first-order optical systems, where one of them has to be anamorphic. In the case of fractional Fourier transform systems we could choose, for instance [24, 25], the fractional angles $\alpha_x = \alpha_y = 0$ (leading to $a_x = a_y = 1$ and $b_x = b_y = 0$), $\alpha_x = \alpha_y = \pi/4$ (leading to $a_x = a_y = b_x = b_y = \sqrt{2}/2$), $\alpha_x = \alpha_y = \pi/2$ (leading to $a_x = a_y = 0$ and $b_x = b_y = 1$), and the anamorphic combination $\alpha_x = \pi/2\pi$ and $\alpha_y = 0$ (leading to $a_x = b_y = 0$ and $a_y = b_x = 1$). If we decide to determine the moments using free space propagation, we should be aware of the fact that an anamorphic free space system cannot be realized by mere free space, but can only be simulated by using a proper arrangement of cylindrical lenses.

Of course, optical schemes to determine all ten second-order moments have been described before, see, for instance [8, 9, 11, 12, 13], but the way to determine these moments as presented in this paper is based on a general scheme that can also be used for the determination of arbitrary higher-order moments.

4.3. *Higher-order moments*

For higher-order moments we can proceed analogously. For the $4 + 6 + 6 + 4 = 20$ third-order moments, the following equations are relevant:

$$\mu_{3000}^{\text{out}} = a_x^3\mu_{3000} + 3a_x^2b_x\mu_{2100} + 3a_xb_x^2\mu_{1200} + b_x^3\mu_{0300}, \tag{18}$$

$$\begin{aligned}\mu_{2010}^{\text{out}} &= a_x^2a_y\mu_{2010} + a_x^2b_y\mu_{2001} + 2a_xb_xa_y\mu_{1110} \\ &\quad + 2a_xb_xb_y\mu_{1101} + b_x^2a_y\mu_{0210} + b_x^2b_y\mu_{0201},\end{aligned} \tag{19}$$

$$\begin{aligned}\mu_{1020}^{\text{out}} &= a_xa_y^2\mu_{1020} + 2a_xa_yb_y\mu_{1011} + a_xb_y^2\mu_{1002} \\ &\quad + b_xa_y^2\mu_{0120} + 2b_xa_yb_y\mu_{0111} + b_xb_y^2\mu_{0102},\end{aligned} \tag{20}$$

$$\mu_{0030}^{\text{out}} = a_y^3\mu_{0030} + 3a_y^2b_y\mu_{0021} + 3a_yb_y^2\mu_{0012} + b_y^3\mu_{0003}. \tag{21}$$

Note again that these equations correspond to (11) with $pqrs = 3000$, $pqrs = 2010$, $pqrs = 1020$, and $pqrs = 0030$, respectively. The 20 third-order moments can be determined from the knowledge of the output intensities of six first-order optical systems, where two of them have to be anamorphic.

We consider in more detail how the third-order moments could be determined.

(i) The four input moments μ_{3000}, μ_{2100}, μ_{1200}, and μ_{0300} can be determined by measuring the intensity moment $\mu_{3000,i}^{\text{out}}$ $(i = 1, 2, 3, 4)$ in the output planes of four systems with different values of a_x and b_x, see (18). Likewise, with the transversal coordinate x replaced by y, the four input moments μ_{0030}, μ_{0021}, μ_{0012}, and μ_{0003} can be determined by measuring the intensity moment $\mu_{0030,i}^{\text{out}}$ $(i = 1, 2, 3, 4)$ in the output planes of four systems with different values of a_y and b_y, see (21). Note that we can choose $a_x = a_y = a_i$ and $b_x = b_y = b_i$ $(i = 1, 2, 3, 4)$ for these four different systems, in which case we are obviously using isotropic systems. This then leads to the set of four equations

$$\begin{aligned}a_i^3\mu_{3000} + 3a_i^2b_i\mu_{2100} + 3a_ib_i^2\mu_{1200} + b_i^3\mu_{0300} \\ = \mu_{3000,i}^{\text{out}} \quad (i = 1, 2, 3, 4)\end{aligned} \tag{22}$$

based on (18) and a similar set of four equations

$$\begin{aligned}a_i^3\mu_{0030} + 3a_i^2b_i\mu_{0021} + 3a_ib_i^2\mu_{0012} + b_i^3\mu_{0003} \\ = \mu_{0030,i}^{\text{out}} \quad (i = 1, 2, 3, 4),\end{aligned} \tag{23}$$

based on (21). Possible system choices are, for instance, four sections of free space, with $a_i = 1$ and b_i proportional to the four different propagation distances z_i $(i = 1, 2, 3, 4)$; or four isotropic fractional Fourier transform systems with $a_i = \cos\alpha_i$ and $b_i = \sin\alpha_i$, and α_i $(i = 1, 2, 3, 4)$ four different fractional angles.

(ii) Using the same four isotropic systems as above, the two input moments μ_{2010} and μ_{0201}, together with the two moment combinations $\mu_{2001} + 2\mu_{1110}$ and $2\mu_{1101} + \mu_{0210}$, follow from measuring the intensity moment $\mu_{2010,i}^{\text{out}}$ $(i = 1, 2, 3, 4)$ in the output planes of these four systems, see (19), while the two input moments μ_{1020} and μ_{0102}, together with the two moment combinations $2\mu_{1011} + \mu_{0120}$ and $\mu_{1002} + 2\mu_{0111}$, follow from measuring the intensity moment $\mu_{1020,i}^{\text{out}}$ $(i = 1, 2, 3, 4)$, see (20). This leads to the set of four equations

$$\begin{aligned}a_i^3\mu_{2010} + a_i^2b_i(\mu_{2001} + 2\mu_{1110}) + a_ib_i^2(2\mu_{1101} + \mu_{0210}) \\ + b_i^3\mu_{0201} = \mu_{2010,i}^{\text{out}} \quad (i = 1, 2, 3, 4)\end{aligned} \tag{24}$$

based on (19) and a similar set of four equations

$$\begin{aligned}a_i^3\mu_{1020} + a_i^2b_i(2\mu_{1011} + \mu_{0120}) + a_ib_i^2(\mu_{1002} + 2\mu_{0111}) \\ + b_i^3\mu_{0102} = \mu_{1020,i}^{\text{out}} \quad (i = 1, 2, 3, 4)\end{aligned} \tag{25}$$

based on (20).

(iii) Twelve of the 20 input moments (together with four moment combinations) can thus be determined by using four isotropic systems. To determine the remaining eight moments, we need four more equations based on (19) and (20), for which we have to use two more systems (labeled $i = 5$ and $i = 6$), which should now

TABLE 1: The number of nth-order moments N, and the required number of first-order optical systems to determine these N moments, given as a function of n.

n	Number of nth-order moments	N	N_t	N_a
0	1	1	1	0
1	2 + 2	4	2	0
2	3 + 4 + 3	10	4	1
3	4 + 6 + 6 + 4	20	6	2
4	5 + 8 + 9 + 8 + 5	35	9	4
5	6 + 10 + 12 + 12 + 10 + 6	56	12	6
6	7 + 12 + 15 + 16 + 15 + 12 + 7	84	16	9
⋮	⋮	⋮	⋮	⋮

be anamorphic. Among the many possibilities, an easy choice would be a system with $a_x = b_y = 0$, $b_x \neq 0$, $a_y \neq 0$, leading to

$$\begin{aligned} b_x^2 a_y \mu_{0210} &= \mu_{2010,5}^{\text{out}}, \\ b_x a_y^2 \mu_{0120} &= \mu_{1020,5}^{\text{out}}, \end{aligned} \tag{26}$$

and a system with $b_x = a_y = 0$, $a_x \neq 0$, $b_y \neq 0$, leading to

$$\begin{aligned} a_x^2 b_y \mu_{2001} &= \mu_{2010,6}^{\text{out}}, \\ a_x b_y^2 \mu_{1002} &= \mu_{1020,6}^{\text{out}}. \end{aligned} \tag{27}$$

The former system may be an anamorphic fractional Fourier transform system with fractional angles $\alpha_x = \pi/2$ and $\alpha_y = 0$ (and hence $a_x = b_y = 0$ and $b_x = a_y = 1$), while the latter may be an anamorphic fractional Fourier transform system with $\alpha_x = 0$ and $\alpha_y = \pi/2$ (and hence $b_x = a_y = 0$ and $b_y = a_x = 1$).

Altogether we have thus constructed 20 equations for the 20 third-order moments, using a total of six first-order systems: four isotropic systems where we measure the 16 output intensity moments $\mu_{3000,i}$, $\mu_{0030,i}$, $\mu_{2010,i}$, and $\mu_{1020,i}$ ($i = 1, 2, 3, 4$), and two anamorphic systems where we measure the four output intensity moments $\mu_{2010,i}^{\text{out}}$ and $\mu_{1020,i}^{\text{out}}$ ($i = 5, 6$).

For the $5 + 8 + 9 + 8 + 5 = 35$ fourth-order moments, the relevant equations follow from (11) with $pqrs = 4000$, $pqrs = 3010$, $pqrs = 2020$, $pqrs = 1030$, and $pqrs = 0040$, respectively. The 35 fourth-order moments can be determined from the knowledge of the output intensities of nine first-order optical systems spectra, where four of them have to be anamorphic. Constructing a measuring scheme along the lines described above for the second-order case and the third-order case, is rather straightforward.

To find the number of nth-order moments N, and the total number of first-order optical systems N_t (with N_a the number of anamorphic ones) that we need to determine these N moments, use can be made of the triangle presented in Table 1, which can easily be extended to higher order.

Note that N (the number of nth-order moments) is equal to the sum of the values in the nth row of the triangle, $N = (n+1)(n+2)(n+3)/6$; that N_t (the total number of first-order optical systems) is equal to the highest value that appears in the nth row of the triangle, $N_t = (n+2)^2/4$ for $n =$ even, and $N_t = (n+3)(n+1)/4$ for $n =$ odd; that the number of isotropic systems is $n + 1$; and that N_a (the number of anamorphic systems) follows from $N_a = N_t - (n+1)$.

5. CONCLUSIONS

We have shown how all global WD moments of arbitrary order can be measured as intensity moments in the output planes of an appropriate number of first-order optical systems (separable, but generally anamorphic ones), and we have derived the minimum number of such systems that are needed for the determination of these moments. The results followed directly from the general relationship (11) that expresses the intensity moments in the output plane of a separable first-order optical system in terms of the moments in the input plane and the system parameters a_x, b_x, c_x, d_x and a_y, b_y, c_y, d_y.

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Martin J. Bastiaans was born in Helmond, The Netherlands, in 1947. He received the M.S. degree in electrical engineering (with honours) and the Ph.D. degree in technical sciences from the Technische Universiteit Eindhoven (Eindhoven University of Technology), Eindhoven, The Netherlands, in 1969 and 1983, respectively. In 1969 he became an Assistant Professor and since 1985 he has been an Associate Professor with the Department of Electrical Engineering, Technische Universiteit Eindhoven, in the Signal Processing Systems Group, where he teaches electrical circuit analysis, signal theory, digital signal processing, and Fourier optics and holography. His main current research interest is in describing signals by means of a local frequency spectrum (for instance, the Wigner distribution function, the windowed Fourier transform, Gabor's signal expansion, etc.). He is the author and coauthor of more than 140 papers in international scientific journals and proceedings of scientific conferences. Dr. Bastiaans is a Fellow of the Optical Society of America and a Senior Member of the Institute of Electrical and Electronics Engineers.

Tatiana Alieva was born in Moscow, Russia. In 1983 and 1996 she received the M.S. and Ph.D. degrees (with honours) in physics from M. V. Lomonosov Moscow State University and University Autonoma of Madrid, respectively. From 1983 till 1999, she worked for A. L. Mintz Radiotechnical Institute, Academy of Science, Moscow. She spent several years as a Postdoctoral Fellow at the Departments of Electrical Engineering of Catholic University of Leuven, Belgium, and Eindhoven University of Technology, The Netherlands. She is currently a Researcher at the Physical Faculty of Complutense University of Madrid. She has worked in diverse fields of physics, such as theory of high energetic beams channelling in crystals, and radio waves scattering on the anisotropic formations in the ionosphere. Her current research interests are in optical information processing, fractal signal analysis, theory of fractional integral transforms, phase retrieval, and characterization of optical fields by means of phase-space distributions. She is an author of more than eighty scientific publications.