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On the Channel Capacity of Multiantenna Systems with Nakagami Fading

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We discuss the channel capacity of multiantenna systems with the Nakagami fading channel. Analytic expressions for the ergodic channel capacity or its lower bound are given for SISO, SIMO, and MISO cases. Formulae for the outage probability of the capacity are presented. It is shown that the channel capacity could be increased logarithmically with the number of receive antennas for SIMO case; while employing 3–5 transmit antennas (irrespective of all other parameters considered herein) can approach the best advantage of the multiple transmit antenna systems as far as channel capacity is concerned for MISO case. We have shown that for a given SNR, the outage probability decreases *considerably* with the number of receive antennas for SIMO case, while for MISO case, the upper bound of the outage probability decreases with the number of transmit antennas when the transmission rate is lower than some value, but increases instead when the transmission rate is higher than another value. A critical transmission rate is identified.

1. INTRODUCTION

Since Foschini and Gans [1] and Telatar [2] established that the capacity of a Rayleigh distributed flat fading channel will increase almost linearly with the minimum of the number of transmit and receive antennas when the receiver has access to perfect channel state information but not the transmitter, multiple transmit and receive antenna (MIMO) systems and spacetime coding have received great attention as a means of providing substantial performance improvement against channel fading in wireless communication systems. In [3], Jayaweera and Poor extended the capacity result to the case of Rician fading channel, considering that Rician fading is a better model for some fading environments. For example, when there is a direct line of sight (LOS) path in addition to the multiple scattering paths, the natural fading model is Rician.

In this paper, we will investigate how the capacity of MIMO systems changes in a Nakagami fading [4] environment. The main reason that motivated our study is that in some communication scenarios such as ultra-wideband (UWB) wireless communications, which has become a very hot topic recently, the Nakagami fading gives a better fitting for the channel model [5]. Another reason is that the Nakagami fading is an extension to the Rayleigh fading, and therefore the results to be presented in this paper will be a generalization of previous MIMO results. On the other hand, there are no reports in the literature on the study of the channel capacity of MIMO systems with the Nakagami fading to the best of the authors' knowledge.

This paper is organized as follows. Section 2 describes the model we are considering. The ergodic channel capacity for the case of single transmit antenna and single receive antenna is discussed in Section 3. Then the MIMO case is studied in Section 4. The outage probability about the capacity is discussed in Section 5. In Section 6, numerical results are provided to demonstrate the dependence of the channel capacity on various kinds of channel parameters. Finally, concluding remarks are given in Section 7.

Notation 1. The notation in this paper is fairly standard. $\mathbf{I}$ is an identity matrix whose dimension is either implied by context or indicated by its subscript when necessary, Pr denotes the probability of an event, $P_A(x)$ represents the cumulative distribution function of a random variable A, that is, $P_A(x) = \Pr\{A \le x\}$, $p_A(x)$ stands for the probability density function of a random variable A, that is, $p_A(x) = \Pr\{x \le A < x + dx\}/dx$, $\varphi_A(\nu)$ represents the characteristic function of a random variable A, $\mathcal{E}_A(f(A))$ stands for the expectation of a function of a random variable A, taking expectation over the statistics of A, and tr represents the trace of a square matrix. Throughout this paper, the function log is understood as the natural logarithm of its argument. Hence the unit of the channel capacity is nat.

2. MODEL DESCRIPTION

Consider a single user communications link in which the transmitter and receiver are equipped with m_X and m_Y antennas, respectively. The received signal in such a system can be written in vector form as

$$\mathbf{Y}(t) = \mathbf{A}(t)\mathbf{X}(t) + \mathbf{N}(t), \tag{1}$$

where $\mathbf{X}(t) \in R^{m_X}$ and $\mathbf{Y}(t) \in R^{m_Y}$ are the transmitted and received signals, respectively, $\mathbf{A}(t) = [a_{nm}(t)]_{m_Y \times m_X}$ is a random matrix characterizing the amplitude fading of the channel, and $\mathbf{N}(t) \in R^{m_Y}$ is the receiver noise. Note that all the signals considered in this paper are in real spaces, in accordance with some communication scenarios such as UWB.

Throughout this paper we will assume that all the random processes are blockwise stationary. Therefore the notation of time will be omitted for briefness.

To make the analysis tractable, the following assumptions are needed.

Assumption 2. It is assumed that all a_{nm}, $n = 1, \dots, m_Y$, $m = 1, \dots, m_X$, are independent and identically distributed.

Assumption 3. The noise $\mathbf{N}$ is zero-mean Gaussian with covariance matrix $\sigma_{\mathbf{N}}^2 \mathbf{I}_{m_Y}$.

Assumption 4. The power of the transmitted signal is bounded by $\overline{S}$, that is, $E(\mathbf{X}^T\mathbf{X}) \le \overline{S}$.

Assumption 5. The receiver possesses complete knowledge of the instantaneous channel parameters, while the transmitter is not aware of the information about the channel parameters.

In the following, we will describe the statistical property of the matrix $\mathbf{A}$. For this purpose, we will generically use a to denote each entry of $\mathbf{A}$. We suppose that the magnitude of a, denoted as $|a|$, takes a Nakagami distribution, whose general form of the probability density function (pdf) is as follows:

$$p_{|a|}(x) = \begin{cases} \dfrac{2m^m x^{2m-1}}{\Gamma(m)\Omega^m} \mathrm{e}^{-mx^2/\Omega} & \text{when } x \ge 0, \\ 0 & \text{when } x < 0, \end{cases} \quad m \ge \frac{1}{2}, \tag{2}$$

where Γ denotes the Gamma function, $\Omega = \mathcal{E}(a^2)$, and $m = [\mathcal{E}(a^2)]^2/\operatorname{Var}[a^2]$. In this paper, we substitute m with another parameter κ by simply defining $\kappa = 2m$. Hence it is clear that $\kappa \ge 1$. By doing so the pdf of $|a|$ in (2) can be rewritten as

$$p_{|a|}(x) = \begin{cases} 2\left(\dfrac{\kappa}{2\Omega}\right)^{\kappa/2} \dfrac{1}{\Gamma(\kappa/2)} x^{\kappa-1} \mathrm{e}^{-\kappa x^2/2\Omega} & \text{when } x \ge 0, \\ 0 & \text{when } x < 0, \end{cases} \quad \kappa \ge 1. \tag{3}$$

Note that we should specify the statistics of the sign of a to describe completely the fading of a. However, for the purpose of this paper, we do not need it.

Define $\eta = a^2$. It is easy to get the pdf of η as follows:

$$p_\eta(x) = \begin{cases} \left(\dfrac{\kappa}{2\Omega}\right)^{\kappa/2} \dfrac{1}{\Gamma(\kappa/2)} x^{\kappa/2-1} \mathrm{e}^{-\kappa x/2\Omega} & \text{when } x \ge 0, \\ 0 & \text{when } x < 0. \end{cases} \tag{4}$$

In the sequel development, we need the characteristic function of the random variable η. First, we calculate the moment generating function of η through which the characteristic function of η can be easily obtained. The moment generating function of η is given by

$$\begin{aligned} \psi_\eta(s) &= \int_{-\infty}^{+\infty} \mathrm{e}^{sx} p_\eta(x) dx \\ &= \int_0^{+\infty} \mathrm{e}^{sx} \frac{1}{(2\Omega/\kappa)^{\kappa/2}\Gamma(\kappa/2)} x^{\kappa/2-1} \mathrm{e}^{-\kappa x/2\Omega} dx. \end{aligned} \tag{5}$$

Substituting the integral variable x with $y = (\kappa/2\Omega - s)x$ and using the definition of the Gamma function, we obtain

$$\begin{aligned} \psi_\eta(s) &= \frac{1}{(2\Omega/\kappa)^{\kappa/2}\Gamma(\kappa/2)(\kappa/2\Omega - s)^{\kappa/2}} \int_0^{+\infty} y^{\kappa/2-1} \mathrm{e}^{-y} dy \\ &= \frac{1}{(2\Omega/\kappa)^{\kappa/2}\Gamma(\kappa/2)(\kappa/2\Omega - s)^{\kappa/2}} \Gamma\left(\frac{\kappa}{2}\right) \\ &= \frac{1}{(1 - (2\Omega/\kappa)s)^{\kappa/2}}. \end{aligned} \tag{6}$$

Thus according to the relationship between moment generating function and characteristic function [6], the latter is given by

$$\varphi_\eta(\nu) = \psi_\eta(\mathrm{j}\nu) = \frac{1}{(1 - \mathrm{j}(2\Omega/\kappa)\nu)^{\kappa/2}}. \tag{7}$$

Notice that the distribution defined by pdf (4) can be regarded as a modified χ^2 distribution with κ degree of freedom (we can write $\eta = (\Omega/\kappa)\chi^2$).

Now we are ready to discuss the channel capacity.

3. THE CASE OF SINGLE TRANSMIT AND RECEIVE ANTENNA (SISO)

First we study the SISO case. For this case, the input-output relation is simplified to

$$Y(t) = a(t)X(t) + N(t), \tag{8}$$

where X, Y, and N become scalars, a assumes the distribution described by (3), and the noise N is zero-mean Gaussian and white with variance σ_N^2.

In this and next sections we will discuss ergodic channel capacity. So in the following we will assume that the fading process is ergodic, so that averaging the classical channel capacity over the amplitude fading is of operational significance.

The channel capacity for an AWGN channel with a given fading amplitude a is given by

$$C|_a = W_X \log\left(1 + \frac{a^2\overline{S}}{\sigma_N^2}\right), \tag{9}$$

where W_X is the bandwidth of the channel. So the ergodic channel capacity, denoted as C_e, turns out to be

$$\begin{aligned} C_e &= \mathcal{E}_a(C|_a) = W_X \int_{-\infty}^{\infty} \log\left(1 + \frac{a^2\overline{S}}{\sigma_N^2}\right) p_a(a)da \\ &= W_X \int_{-\infty}^{\infty} \log\left(1 + \frac{\eta\overline{S}}{\sigma_N^2}\right) p_\eta(\eta)d\eta \\ &= W_X \int_0^{\infty} \log\left(1 + \frac{x\overline{S}}{\sigma_N^2}\right)\left(\frac{\kappa}{2\Omega}\right)^{\kappa/2} \\ &\quad \times \frac{1}{\Gamma(\kappa/2)} x^{\kappa/2-1} \mathrm{e}^{-\kappa x/2\Omega} dx, \end{aligned} \tag{10}$$

where $\eta = a^2$ and the distribution of η is given by (4). Substituting the variable x with $x = (2\Omega/\kappa)u$ in the above integral yields

$$C_e = \frac{W_X}{\Gamma(\kappa/2)} \int_0^{\infty} \log\left(1 + \frac{u}{\beta}\right) u^{\kappa/2-1} \mathrm{e}^{-u} du, \tag{11}$$

where

$$\beta := \frac{\kappa\sigma_N^2}{2\Omega\overline{S}} = \frac{\kappa}{2\,\mathrm{SNR}}, \tag{12}$$

$$\mathrm{SNR} := \frac{\Omega\overline{S}}{\sigma_N^2}, \tag{13}$$

can be considered as the ratio of signal power (at the receiver side) to the noise power. Let us define

$$J(\kappa;\beta) := \int_0^{\infty} \log\left(1 + \frac{u}{\beta}\right) u^{\kappa/2-1} \mathrm{e}^{-u} du. \tag{14}$$

Integrating the above integral by parts, we obtain

$$\begin{aligned} J(\kappa;\beta) &= \int_0^{\infty} \frac{1}{u+\beta} u^{\kappa/2-1} \mathrm{e}^{-u} du \\ &\quad + \left(\frac{\kappa}{2} - 1\right) \int_0^{\infty} \log\left(1 + \frac{u}{\beta}\right) u^{(\kappa-2)/2-1} \mathrm{e}^{-u} du \\ &= \int_0^{\infty} \frac{1}{u+\beta} u^{\kappa/2-1} \mathrm{e}^{-u} du + \left(\frac{\kappa}{2} - 1\right) J(\kappa-2;\beta). \end{aligned} \tag{15}$$

From [7, page 319], one sees that

$$\int_0^{\infty} \frac{1}{u+\beta} u^{\kappa/2-1} \mathrm{e}^{-u} du = \mathrm{e}^{\beta} \beta^{(\kappa-2)/2} \Gamma\left(\frac{\kappa}{2}\right) \Gamma\left(1 - \frac{\kappa}{2}, \beta\right), \tag{16}$$

where it is required that $\kappa \geq 1$ to guarantee the integral to converge and $\Gamma(\alpha, z)$ denotes the incomplete Gamma function, defined by (see [7, page 940])

$$\Gamma(\alpha, z) = \int_z^{\infty} \mathrm{e}^{-u} u^{\alpha-1} du. \tag{17}$$

Thus we have

$$J(\kappa;\beta) = \mathrm{e}^{\beta} \beta^{(\kappa-2)/2} \Gamma\left(\frac{\kappa}{2}\right) \Gamma\left(1 - \frac{\kappa}{2}, \beta\right) + \left(\frac{\kappa}{2} - 1\right) J(\kappa-2;\beta). \tag{18}$$

To use the recursive formula (18) to calculate $J(\kappa;\beta)$, we need to know $J(1;\beta)$ and $J(2;\beta)$, respectively. By definition, we have

$$\begin{aligned} J(1;\beta) &= \int_0^{\infty} \frac{\log(1+u/\beta)\mathrm{e}^{-u}}{\sqrt{u}} du \\ &= \pi^{3/2} \operatorname{erfi}\left(\sqrt{\beta}\right) - (\gamma + 2\log 2 + \log\beta)\sqrt{\pi} - 2\sqrt{\pi}\beta \\ &\quad \cdot {}_2F_2\left([1,1], \left[2, \frac{3}{2}\right], \beta\right) \\ &= \sqrt{\pi}\Big[\pi \operatorname{erfi}\left(\sqrt{\beta}\right) - \gamma - 2\log 2 - \log\beta - 2\beta \\ &\quad \cdot {}_2F_2\left([1,1], \left[2, \frac{3}{2}\right], \beta\right)\Big], \end{aligned} \tag{19}$$

where $\gamma \approx 0.5772$ is the Euler's constant, $\operatorname{erfi}(z)$ and ${}_2F_2([\alpha_1, \alpha_2], [\alpha_3, \alpha_4], z)$ are the imaginary error function and generalized hypergeometric function, respectively, which are defined by (cf. [7, page 1045])

$$\begin{aligned} \operatorname{erfi}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z \mathrm{e}^{u^2} du, \\ {}_2F_2([\alpha_1, \alpha_2], [\alpha_3, \alpha_4], z) &= \sum_{k=0}^{\infty} \frac{(\alpha_1)_k (\alpha_2)_k}{(\alpha_3)_k (\alpha_4)_k} \frac{z^k}{k!}, \end{aligned} \tag{20}$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1) = \Gamma(\alpha+k)/\Gamma(\alpha)$. While using the definition of the incomplete Gamma function, we obtain

$$\begin{aligned} J(2;\beta) &= \int_0^{\infty} \log\left(1 + \frac{u}{\beta}\right) \mathrm{e}^{-u} du = \int_0^{\infty} \frac{\mathrm{e}^{-u}}{u+\beta} du \\ &= \mathrm{e}^{\beta} \int_{\beta}^{\infty} \frac{\mathrm{e}^{-v}}{v} \mathrm{d}v = \mathrm{e}^{\beta} \Gamma(0, \beta). \end{aligned} \tag{21}$$

Finally, the ergodic channel capacity can be calculated according to

$$C_e = \frac{W_X}{\Gamma(\kappa/2)} J(\kappa;\beta). \tag{22}$$

From (22), (12), and (13), it is interesting to observe that the channel capacity depends only on parameters W_X, κ, and β, and that parameter Ω plays the same role as $\overline{S}$. This is an expected result since Ω is proportional to the power of a, which can be seen from the fact that $\mathcal{E}(a^2) = \Omega$.

4. THE CASE OF MULTIPLE TRANSMIT AND RECEIVE ANTENNAS

In this case, the input-output relation (channel model) is described by (1). The mutual information between $\mathbf{X}$ and $\mathbf{Y}$ for a given $\mathbf{A}$ is

$$\begin{aligned} \mathcal{I}(\mathbf{X};\mathbf{Y} \mid \mathbf{A}) &= \mathcal{H}(\mathbf{Y} \mid \mathbf{A}) - \mathcal{H}(\mathbf{Y} \mid \mathbf{X},\mathbf{A}) \\ &= \mathcal{H}(\mathbf{Y} \mid \mathbf{A}) - \mathcal{H}(\mathbf{N}), \end{aligned} \tag{23}$$

where $\mathcal{H}$ denotes the entropy of a random variable, whose definition can be found in [8] for the case of continuous random variables. It is well known that if $\mathbf{X}$ is constrained to have covariance $\mathbf{Q}$, the choice of $\mathbf{X}$ that maximizes $\mathcal{I}(\mathbf{X};\mathbf{Y} \mid \mathbf{A})$ is a Gaussian random variable with covariance $\mathbf{Q}$. Thus the channel capacity for a given fading matrix $\mathbf{A}$ turns out to be [2, 8]

$$\begin{aligned} C|_{\mathbf{A}} &= \max_{p_{\mathbf{X}}(\mathbf{x})} \mathcal{I}(\mathbf{X};\mathbf{Y} \mid \mathbf{A}) \\ &= W_X \log\det\left(\sigma_{\mathbf{N}}^2 \mathbf{I}_{m_Y} + \mathbf{A}\mathbf{Q}\mathbf{A}^T\right) - W_X \log\det\left(\sigma_{\mathbf{N}}^2 \mathbf{I}_{m_Y}\right) \\ &= W_X \log\det\left(\mathbf{I}_{m_Y} + \frac{1}{\sigma_{\mathbf{N}}^2}\mathbf{A}\mathbf{Q}\mathbf{A}^T\right), \end{aligned} \tag{24}$$

where $\mathbf{A}^T$ represents the transpose of matrix $\mathbf{A}$. Let us define

$$\Psi(\mathbf{Q}) = \mathcal{E}_{\mathbf{A}}\left[\log\det\left(\mathbf{I}_{m_Y} + \frac{1}{\sigma_{\mathbf{N}}^2}\mathbf{A}\mathbf{Q}\mathbf{A}^T\right)\right]. \tag{25}$$

Then the ergodic channel capacity is given by

$$C_e = W_X \max_{\operatorname{tr}(\mathbf{Q}) \le \overline{S}} \Psi(\mathbf{Q}). \tag{26}$$

The optimization problem described by (25) and (26) is difficult to solve. In the following we will solve the suboptimal problem described by (25) and

$$\underline{C}_e = W_X \max_{\substack{\operatorname{tr}(\mathbf{Q}) \le \overline{S} \\ \mathbf{Q} \text{ is diagonal}}} \Psi(\mathbf{Q}). \tag{27}$$

The constraint in (27) says that the transmitted signals among all antennas are uncorrelated. As is well known, a nice property for the case of the (complex) Gaussian fading channel is that the optimal solution of $\mathbf{Q}$ for problem (25)-(26) is a diagonal matrix, but for our problem, whether or not $\mathbf{Q}$ is diagonal is still an open problem. In principle, a nondiagonal $\mathbf{Q}$ may yield a greater maximum mutual information than a diagonal Q for general fading matrix $\mathbf{A}$. Therefore, we will generally have $C_e \ge \underline{C}_e$. In some cases, we will see $C_e = \underline{C}_e$.

Now following the same argument as that in [2], we show that the optimal solution of $\mathbf{Q}$ for problem (25) and (27) is

$$\mathbf{Q}_{\text{opt}} = \frac{\overline{S}}{m_X}\mathbf{I}. \tag{28}$$

Suppose that $\mathbf{Q}$ is any given nonnegative diagonal matrix satisfying $\operatorname{tr}(\mathbf{Q}) \le \overline{S}$ and $\mathbf{\Pi}$ is any permutation matrix. Consider $\mathbf{Q}^{\mathbf{\Pi}} := \mathbf{\Pi}\mathbf{Q}\mathbf{\Pi}^T$ and $\mathbf{A}^{\mathbf{\Pi}} := \mathbf{A}\mathbf{\Pi}^T$. Since $\mathbf{A}^{\mathbf{\Pi}}$ is obtained by interchanging two corresponding columns, it can be inferred from the independence of the elements in $\mathbf{A}$ that $p_{\mathbf{A}}(\mathbf{Z}) = p_{\mathbf{A}^{\Pi}}(\mathbf{Z})$, where $\mathbf{Z}$ is a matrix with the same dimension as $\mathbf{A}$. Therefore, we have

$$\begin{aligned} \Psi(\mathbf{Q}) &= \mathcal{E}_{\mathbf{A}}\left[\log\det\left(\mathbf{I}_{m_Y} + \frac{1}{\sigma_{\mathbf{N}}^2}\mathbf{A}\mathbf{\Pi}^T\left(\mathbf{\Pi}\mathbf{Q}\mathbf{\Pi}^T\right)\mathbf{\Pi}\mathbf{A}^T\right)\right] \\ &= \mathcal{E}_{\mathbf{A}^{\Pi}}\left[\log\det\left(\mathbf{I}_{m_Y} + \frac{1}{\sigma_{\mathbf{N}}^2}\mathbf{A}^{\mathbf{\Pi}}Q^{\Pi}\left(A^{\Pi}\right)^T\right)\right] = \Psi\left(\mathbf{Q}^{\mathbf{\Pi}}\right). \end{aligned} \tag{29}$$

Let $\widetilde{\mathbf{Q}} = (1/m_X!)\sum_{\mathbf{\Pi}} \mathbf{Q}^{\mathbf{\Pi}}$. It is well known [8] that the mapping $\mathbf{Q} \mapsto \Psi(\mathbf{Q})$ is convex $\cap$ (in the convention of [8]) over the set of positive definite matrices. Thus it follows that $\Psi(\widetilde{\mathbf{Q}}) \ge \Psi(\mathbf{Q})$. Notice that $\widetilde{\mathbf{Q}}$ is simply a multiple of the identity matrix and $\operatorname{tr}(\widetilde{\mathbf{Q}}) = \operatorname{tr}(\mathbf{Q})$. Thus $\underline{C}_e$ is achieved by letting $\mathbf{Q} = \alpha\mathbf{I}$. Applying the trace constraint to $\mathbf{Q}$ yields that $\alpha = \overline{S}/m_X$. Therefore, we arrive at

$$C_e \ge \underline{C}_e = W_X \mathcal{E}_{\mathbf{A}}\left[\log\det\left(\mathbf{I}_{m_Y} + \frac{\overline{S}}{m_X \sigma_{\mathbf{N}}^2}\mathbf{A}\mathbf{A}^T\right)\right]. \tag{30}$$

Equation (30) provides a lower bound for the channel capacity of the Nakagami fading channels. The conservativeness of the lower bound comes from the diagonal assumption on $\mathbf{Q}$. If, on the other hand, $\mathbf{Q}$ is nondiagonal, some kind of knowledge, either statistical property on or the exact value of the fading matrix should be provided to the transmitter. Considering Assumption 5, we can conclude that the lower bound described by (30) is a useful performance measure for the wireless systems with the Nakagami fading.

To use (30), we need to know the distribution of $\det(\mathbf{I}_{m_Y} + (\overline{S}/m_X\sigma_{\mathbf{N}}^2)\mathbf{A}\mathbf{A}^T)$ or that of the eigenvalues of matrix $\mathbf{A}\mathbf{A}^T$. Unfortunately, these distributions are known only when $\mathbf{A}$ possesses some special distribution (typically normal distribution) if both $m_X > 1$ and $m_Y > 1$, see, for example, [9]. Therefore, we will consider some special cases in the following.

Here we would like to point out that, in the above derivation, we have used the property that the distribution of $\mathbf{A}$ is invariant under permutation transformations, but this property does not hold for $\mathbf{A}$ under general unitary transformations, such as the case of normal distribution discussed in [2].

4.1. *Single transmit and multiple receive antennas*

In this case, $m_X = 1$ and $m_Y > 1$. We denote $\mathbf{A} = [a_1, \ldots, a_{m_Y}]^T$. First notice the fact that for any two matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ with compatible dimensions, we have

$$\det\left(\mathbf{I} + \mathbf{M}_1\mathbf{M}_2\right) = \det\left(\mathbf{I} + \mathbf{M}_2\mathbf{M}_1\right). \tag{31}$$

Note also that in this case the matrix $\mathbf{Q}$ reduces to a scalar. Applying these two facts to (30), one sees that

$$C_e = \underline{C}_e = W_X \mathcal{E}_{\mathbf{A}}\left[\log\left(1+\frac{\overline{S}}{\sigma_{\mathbf{N}}^2}\mathbf{A}^T\mathbf{A}\right)\right]. \tag{32}$$

Let

$$\Upsilon := \mathbf{A}^T\mathbf{A} = \sum_{l=1}^{m_Y} a_l^2 := \sum_{l=1}^{m_Y}\Upsilon_l, \tag{33}$$

where $\Upsilon_l := a_l^2$, $l = 1,\dots,m_Y$. According to (7) and noticing the fact that $\{\Upsilon_l, l = 1,\dots,m_Y\}$ are independent, we can see that the characteristic function of Υ is given by

$$\varphi_\Upsilon(\nu) = \left[\frac{1}{(1-\mathrm{j}(2\Omega/\kappa)\nu)^{\kappa/2}}\right]^{m_Y} = \frac{1}{(1-\mathrm{j}(2\Omega/\kappa)\nu)^{(\kappa/2)m_Y}}. \tag{34}$$

From (34) we can see (cf. [6, page 148]) that $\varphi_\Upsilon(\nu)$ is the characteristic function of the Gamma distribution. Thus Υ has the following pdf:

$$p_\Upsilon(x) = \begin{cases}\dfrac{1}{(2\Omega/\kappa)^{(\kappa/2)m_Y}\Gamma((\kappa/2)m_Y)}x^{(\kappa/2)m_Y-1}\mathrm{e}^{-\kappa x/2\Omega} & \text{when } x \ge 0,\\ 0 & \text{when } x < 0.\end{cases} \tag{35}$$

Therefore, the ergodic channel capacity is given by

$$\begin{aligned}C_e &= W_X\mathcal{E}_\Upsilon\left(\log\left(1+\frac{1}{\sigma_{\mathbf{N}}^2}\overline{S}\Upsilon\right)\right)\\ &= W_X\int_0^\infty \log\left(1+\frac{\overline{S}}{\sigma_{\mathbf{N}}^2}x\right)\\ &\qquad\times\frac{1}{(2\Omega/\kappa)^{(\kappa/2)m_Y}\Gamma((\kappa/2)m_Y)}x^{(\kappa/2)m_Y-1}\mathrm{e}^{-\kappa x/2\Omega}dx\\ &= W_X\int_0^\infty \log\left(1+\frac{2\Omega\overline{S}}{\kappa\sigma_{\mathbf{N}}^2}y\right)\frac{1}{\Gamma((\kappa/2)m_Y)}y^{(\kappa/2)m_Y-1}\mathrm{e}^{-y}dy\\ &= \frac{W_X}{\Gamma((\kappa/2)m_Y)}J(m_Y\kappa;\beta).\end{aligned} \tag{36}$$

4.2. *Multiple transmit and single receive antennas*

In this case, $m_X > 1$ and $m_Y = 1$. Thus $\mathbf{AA}^T$ is a scalar. Define $\overline{\Upsilon} = \mathbf{AA}^T$. It is clear that $\overline{\Upsilon}$ has the following distribution:

$$p_{\overline{\Upsilon}}(x) = \begin{cases}\dfrac{1}{(2\Omega/\kappa)^{(\kappa/2)m_X}\Gamma((\kappa/2)m_X)}x^{(\kappa/2)m_X-1}\mathrm{e}^{-\kappa x/2\Omega} & \text{when } x \ge 0,\\ 0 & \text{when } x < 0.\end{cases} \tag{37}$$

Therefore, from (30), we have

$$\begin{aligned}C_e \ge \underline{C}_e &= W_X\mathcal{E}_{\overline{\Upsilon}}\left[\log\left(1+\frac{\overline{S}}{m_X\sigma_{\mathbf{N}}^2}\overline{\Upsilon}\right)\right]\\ &= W_X\int_0^\infty \log\left(1+\frac{\overline{S}}{m_X\sigma_{\mathbf{N}}^2}x\right)\\ &\qquad\times\frac{1}{(2\Omega/\kappa)^{(\kappa/2)m_X}\Gamma((\kappa/2)m_X)}x^{(\kappa/2)m_X-1}\mathrm{e}^{-\kappa x/2\Omega}dx\\ &= \frac{W_X}{\Gamma((\kappa/2)m_X)}J(m_X\kappa;m_X\beta).\end{aligned} \tag{38}$$

Remark 6. Notice that when $\kappa = 2$, the fading model for each element of $\mathbf{A}$ reduces to Rayleigh distribution, which corresponds to the classic narrowband wireless communication channel. So we expect that the results obtained for this specific κ also recover the results obtained in [2]. Substituting $\kappa = 2$ into (36) and (38), respectively, readily reveals that (36) and (38) indeed reduce to (12) and (13) in [2], respectively.

5. CAPACITY VERSUS OUTAGE PROBABILITY

The results we have obtained in the previous sections apply to the case where the fading matrix is ergodic and there are no constraints on the decoding delay on the receiver. In practical communication systems, we often run into the case where the fading matrix is generated or chosen randomly at the beginning of the transmission, while no significant channel variability occurs during the whole transmission. In this case, the fading matrix is clearly not ergodic. We suppose that the fading matrix still has the distribution defined in the previous sections. In this case it is more important to investigate the channel capacity in the sense of outage probability. An outage is defined as the event where the communication channel does not support a target data rate. Thus, according to [10], outage probability, denoted by $P_{\text{out}}(R)$, is defined as follows. With a given rate R, we associate a set Θ_R in the space of the fading matrix $\mathbf{A}$. The set is the largest possible set for which C_Θ, the capacity of the compound channel with parameter $\mathbf{A} \in \Theta_R$, satisfies $C_\Theta \ge R$. The outage probability is then defined as $P_{\text{out}}(R) = \Pr\{\mathbf{A} \notin \Theta_R\}$. Thus it is clear that

$$P_{\text{out}}(R) = \Pr\{\mathbf{A} \notin \Theta_R\} = \Pr\{C|_{\mathbf{A}} < R\} = \Pr\{C|_{\mathbf{A}} \le R\}, \tag{39}$$

that is, the outage probability can be actually viewed as the cumulative distribution function (cdf) of the conditional Shannon capacity. Notice that the last equality of the above equation follows from the fact that $C(\mathbf{X};\mathbf{Y}|\mathbf{A})$ is a continuous function of the continuous random matrix $\mathbf{A}$.

Based on the above discussion, we can evaluate the outage probability for the following three cases.

(i) SISO case

Let us define $\eta = a^2$. Recall that η has the pdf defined by (4). Thus its cdf is as follows:

$$\begin{aligned} P_\eta(x) &= \Pr\{\eta \le x\} \\ &= \int_0^x \left(\frac{\kappa}{2\Omega}\right)^{\kappa/2} \frac{1}{\Gamma(\kappa/2)} y^{\kappa/2-1} e^{-\kappa y/2\Omega} dy \\ &= \frac{1}{\Gamma(\kappa/2)} \gamma\left(\frac{\kappa}{2}, \frac{\kappa}{2\Omega}x\right), \end{aligned} \tag{40}$$

where $\gamma(\alpha, z)$ is the incomplete Gamma function, defined by (cf. [7, page 940])

$$\gamma(\alpha, z) = \int_0^z e^{-x} x^{\alpha-1} dx. \tag{41}$$

Therefore, from (9) it follows that

$$\begin{aligned} P_{\text{out}}(R) &= \Pr\left\{ W_X \log\left(1 + \frac{a^2\overline{S}}{\sigma_N^2}\right) \le R \right\} \\ &= \Pr\left\{ \eta \le \frac{\sigma_N^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right\} = P_\eta\left(\frac{\sigma_N^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right) \\ &= \frac{1}{\Gamma(\kappa/2)} \gamma\left(\frac{\kappa}{2}, \beta\left(e^{R/W_X} - 1\right)\right). \end{aligned} \tag{42}$$

(ii) SIMO case

Recalling the definition of Υ and its pdf (35), we can obtain its cdf as follows:

$$\begin{aligned} P_\Upsilon(x) &= \Pr\{\Upsilon \le x\} \\ &= \int_0^x \frac{1}{(2\Omega/\kappa)^{(\kappa/2)m_Y}\Gamma((\kappa/2)m_Y)} y^{(\kappa/2)m_Y-1} e^{-\kappa y/2\Omega} dy \\ &= \frac{1}{\Gamma((\kappa/2)m_Y)} \gamma\left(\frac{\kappa}{2}m_Y, \frac{\kappa}{2\Omega}x\right). \end{aligned} \tag{43}$$

Following the same argument as (32), the conditional capacity can be derived as

$$C|_{\mathbf{A}} = W_X \log\left(1 + \frac{1}{\sigma_{\mathbf{N}}^2}\overline{S}\mathbf{A}^T\mathbf{A}\right). \tag{44}$$

Thus the outage probability turns out to be

$$\begin{aligned} P_{\text{out}}(R) &= \Pr\left\{ W_X \log\left(1 + \frac{1}{\sigma_{\mathbf{N}}^2}\overline{S}\mathbf{A}^T\mathbf{A}\right) \le R \right\} \\ &= \Pr\left\{ \Upsilon \le \frac{\sigma_{\mathbf{N}}^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right\} = P_\Upsilon\left(\frac{\sigma_{\mathbf{N}}^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right) \\ &= \frac{1}{\Gamma((\kappa/2)m_Y)} \gamma\left(\frac{\kappa}{2}m_Y, \beta\left(e^{R/W_X} - 1\right)\right). \end{aligned} \tag{45}$$

(iii) MISO case

First, we have

$$P_{\overline{\Upsilon}}(x) = \frac{1}{\Gamma((\kappa/2)m_X)} \gamma\left(\frac{\kappa}{2}m_X, \frac{\kappa}{2\Omega}x\right). \tag{46}$$

Then according to (38), we have

$$\begin{aligned} P_{\text{out}}(R) &= \Pr\{C|_{\mathbf{A}} \le R\} \\ &\le \Pr\left\{ W_X \log\left(1 + \frac{\overline{S}}{m_X\sigma_{\mathbf{N}}^2}\mathbf{A}\mathbf{A}^T\right) \le R \right\} \\ &= \Pr\left\{ \overline{\Upsilon} \le \frac{m_X\sigma_{\mathbf{N}}^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right\} \\ &= P_{\overline{\Upsilon}}\left(\frac{m_X\sigma_{\mathbf{N}}^2}{\overline{S}}\left(e^{R/W_X} - 1\right)\right) \\ &= \frac{1}{\Gamma((\kappa/2)m_X)} \gamma\left(\frac{\kappa}{2}m_X, m_X\beta\left(e^{R/W_X} - 1\right)\right) \\ &= \overline{P}_{\text{out}}(R). \end{aligned} \tag{47}$$

$\overline{P}_{\text{out}}$, as defined in (47), provides an upper bound for the concerned outage probability.

6. NUMERICAL RESULTS

In this section, we will investigate the variation of channel capacity with respect to various kinds of parameters. It is found from (12), (22), (36), and (38) that the ergodic channel capacity depends only on channel bandwidth W_X, the number κ, m_Y, m_X, and the signal-to-noise power ratio SNR in the sense defined by (13), respectively, and it depends on W_X linearly, so we let $W_X = 1$ and only focus our attention on the variation of C_e with respect to κ, SNR, m_Y, and m_X, respectively.

Figure 1 depicts the variation of channel capacity C_e with respect to the number κ for SISO case. We can see from this figure that even though the channel capacity increases with the number κ, the quantities increased are not large compared to the base case ($\kappa = 1$), especially when $\kappa \ge 10$. For example, for the case of SNR $= 0$ dB, when κ is increased from 1 to 10, C_e increases $(0.6695 - 0.5335)/0.5335 = 25.5\%$, while when κ is increased from 10 to 30, C_e increases only by 2.31%.

Figure 2 demonstrates the relationship between channel capacity and SNR for SISO case, which shows that when SNR becomes large, C_e is approximately a logarithmic function of SNR. This is a result coinciding with our expectation.

Figure 3 shows the relationship between the channel capacity and the number of receive antennas for SIMO case. It can be seen from this figure that $\underline{C}_e$ increases with m_Y almost logarithmically. This phenomenon is similar to the corresponding one in the case of the Rayleigh fading channels (cf. [2, Example 3]).

Figure 4 shows the relationship between the lower bound of the channel capacity and the number of transmit antennas for MISO case. It is interesting to observe from this figure

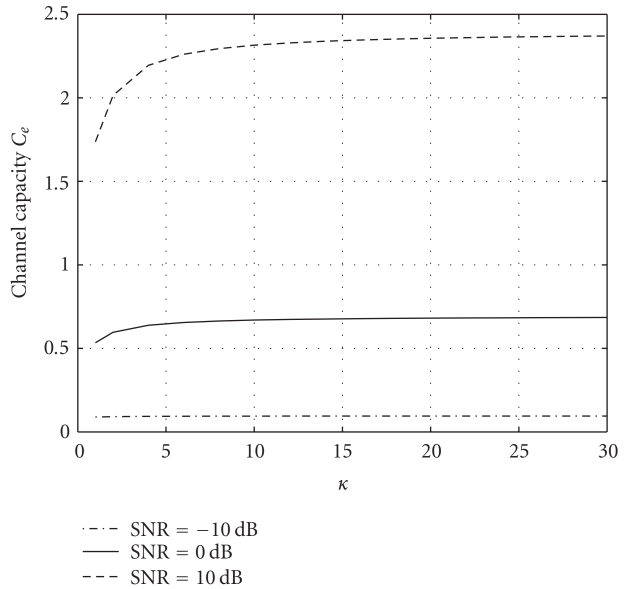

FIGURE 1: Variation of ergodic channel capacity C_e (in nats/s/Hz) with the number κ for SISO case.

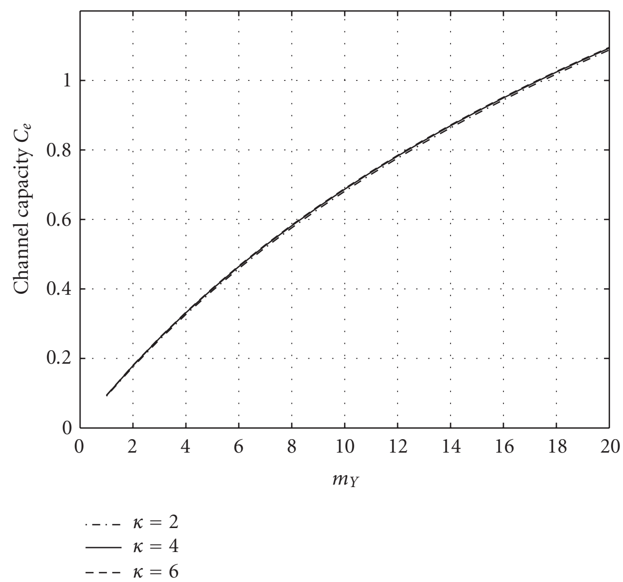

FIGURE 3: Variation of ergodic channel capacity C_e (in nats/s/Hz) with the number of receive antennas m_Y for SIMO case (SNR = −10 dB).

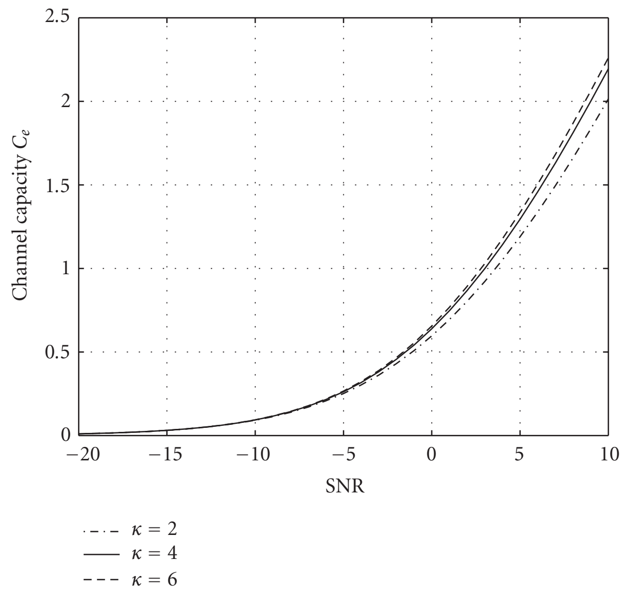

FIGURE 2: Variation of ergodic channel capacity C_e (in nats/s/Hz) with the ratio SNR (in dB) for SISO case.

that the capacity increases with m_X rapidly when m_X is small ($m_X \leq 6$), however, the increase is very slow when m_X becomes large ($m_X > 6$). This phenomenon is different from the one in the case of the Rayleigh fading channel, see [2, Example 4], where it is found that C_e does not change with m_X when $m_X \geq 2$. An important phenomenon can also be observed by comparing Figures 4(a) and 4(b), that is, when the signal-to-noise ratio is low, the benefit obtained by distributing the available power to different transmit antennas is very limited as far as the average capacity is concerned.

From Figures 3 and 4, we can see that increasing the number of receiver antennas can obtain more benefit in channel capacity than increasing the number of transmit antennas. Principally, the channel capacity could be increased infinitely by employing a large number of receive antennas, but it appears to increase only logarithmically in this number; while employing 3—5 receive antennas can approach the best advantage of the multiple transmit antenna systems (for the case of single receive antenna). The reason for this phenomenon is two fold. First, the power is constrained to be a constant, for different m_X, among all the transmit antennas, while no such constraint is applied to receive antennas. Second, it is assumed that the receiver possesses the full knowledge about the channel state.

The variations of outage probability P_{out} or $\overline{P}_{\text{out}}$ with respect to the transmission rate (in nats/s/Hz), R/W_X, are shown in Figures 5, 6, and 7 for SISO, SIMO, and MISO cases, respectively. From these figures, it can be observed that for a given SNR, the outage probability decreases considerably with the number of receive antennas in the range of whole transmission rate, while $\overline{P}_{\text{out}}(R)$ decreases with the number of transmit antennas when R/W_X is lower than some value (denoted as R_1), but increases instead when R/W_X is larger than another value (denoted as R_2). Notice that the outage probability is so large when R/W_X is larger than R_2 that to transfer information at this rate is of little practical interest. Therefore, we can conclude that increasing the number of transmit antennas is of some significance at a transmission rate of practical communications with tolerable outage probability.

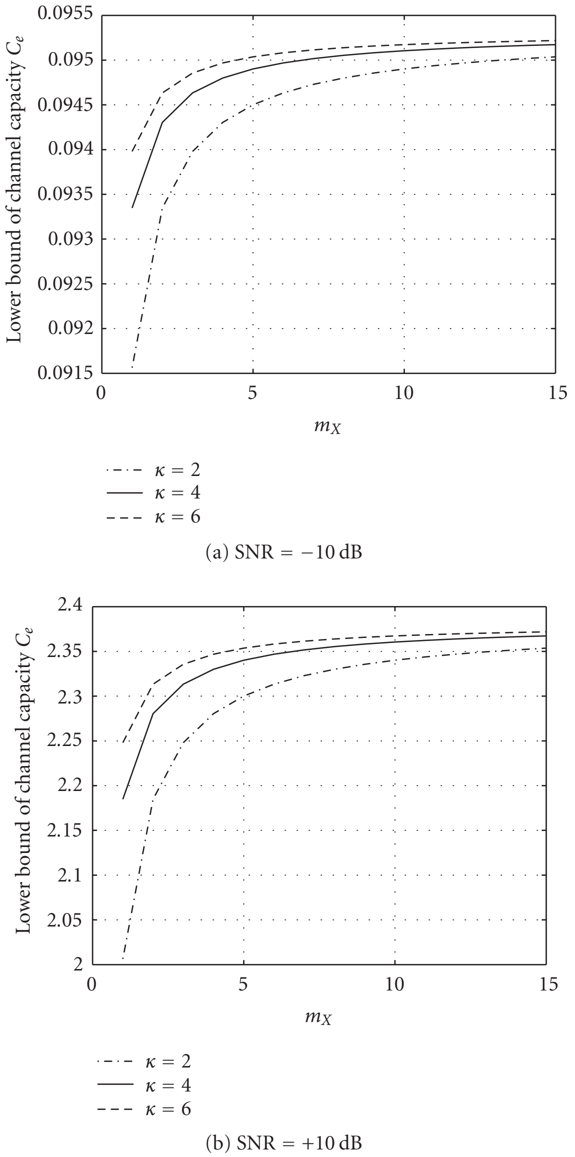

Figure 4: Variation of the lower bound of the ergodic channel capacity C_e (in nats/s/Hz) with the number of transmit antennas m_X for MISO case.

It would be difficult to calculate R_2 exactly. However, R_1 can be calculated in the following way. Notice the fact that the total signal power $\overline{S}$ is equally distributed among transmit antennas for MISO case with m_X transmit antennas. Thus the received power from the useful signals should be

$$\overline{S}_Y = \sum_{k=1}^{m_X} a_k^2 \cdot \frac{\overline{S}}{m_X} = \frac{\sum_{k=1}^{m_X} a_k^2}{m_X} \cdot \overline{S}. \tag{48}$$

Therefore it is clear that the larger the number m_X, the smaller the variance of the received power from the useful signals. In the extreme case, when m_X approaches infinity, we have

$$\overline{S}_Y \longrightarrow \Omega\overline{S} \text{ with probability 1} \quad \text{as } m_X \longrightarrow \infty, \tag{49}$$

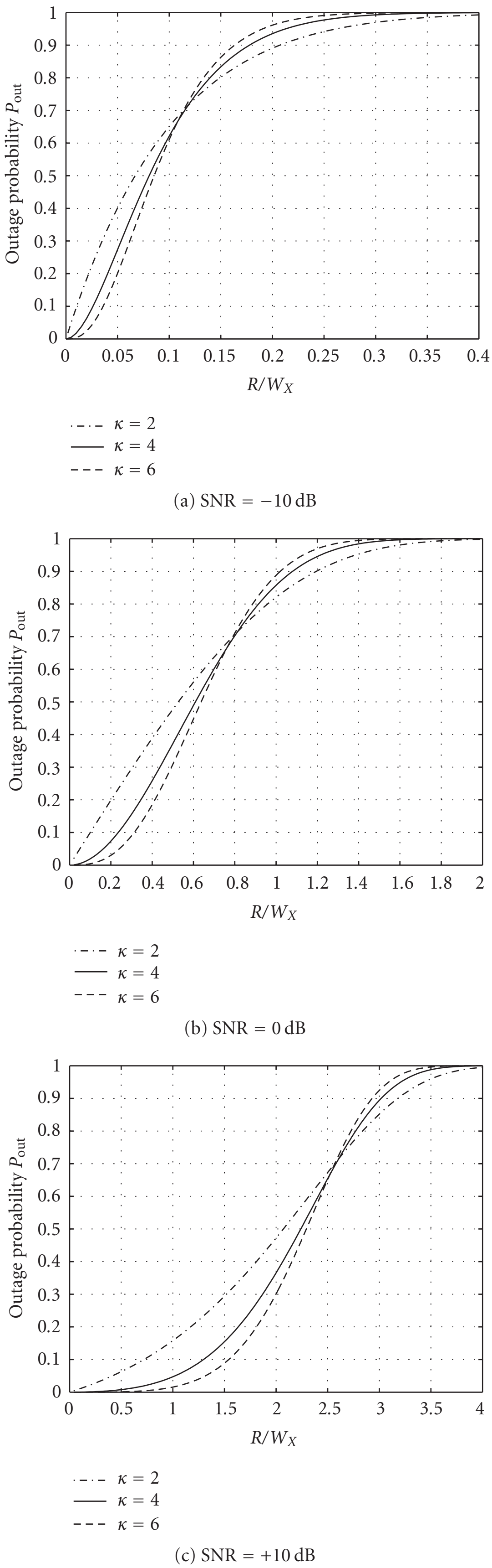

Figure 5: Outage probability P_{out} versus transmission rate R/W_X (in nats/s/Hz) for various κ and SNR for SISO case.

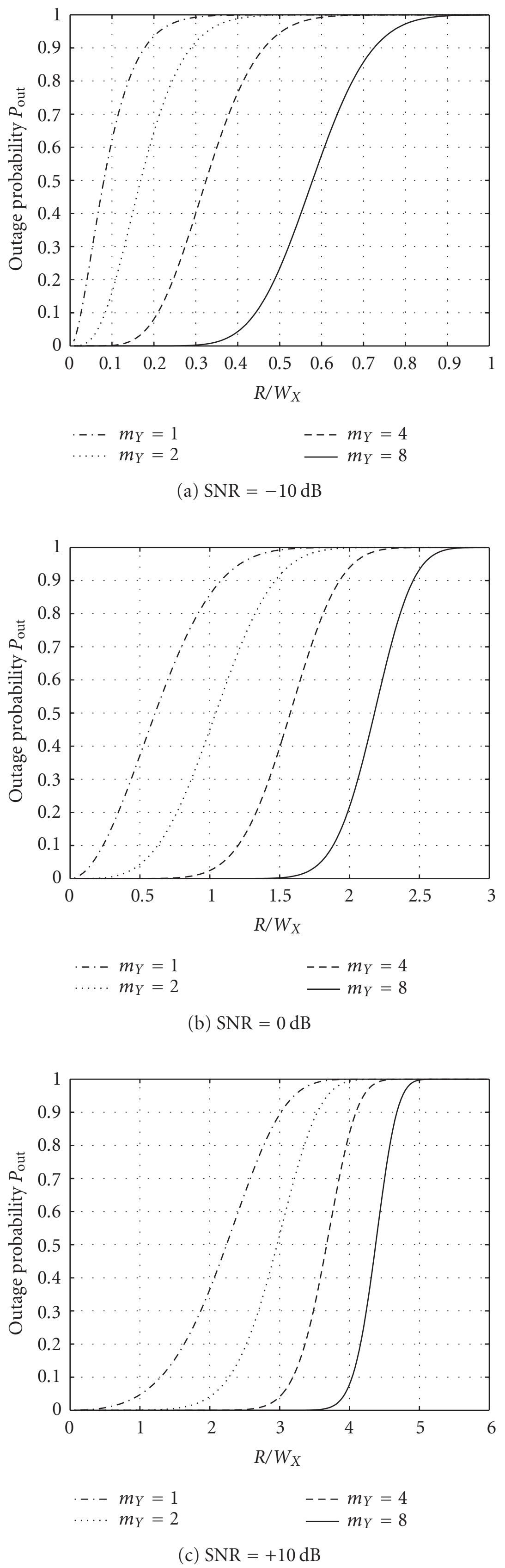

Figure 6: Outage probability P_{out} versus transmission rate R/W_X (in nats/s/Hz) for various m_Y and SNR for SIMO case ($\kappa = 4$).

according to strong law of large numbers [6]. Therefore in this extreme case, we obtain

$$\frac{R}{W_X} = \log\left(1 + \frac{\Omega\overline{S}}{\sigma_N^2}\right) = \log(1 + \text{SNR}) := R_c. \tag{50}$$

The above says that the channel capacity will approach a constant R_c when the number of transmit antennas approaches infinity. We call R_c the critical transmission rate. From Figure 7, we can see that R_1 and R_2 satisfy the relationships

$$R_1 = R_c, \qquad R_2 > R_c. \tag{51}$$

The above analysis yields that $R_c = 0.0953$, 0.6931, and 2.3979 for SNR being −10 dB, 0 dB, and +10 dB, respectively in the case of Figure 7. It is seen that R_2 almost coincides with R_c.

7. CONCLUDING REMARKS

In this paper, the analytic expression for the ergodic channel capacity or its lower bound of wireless communication systems with the Nakagami fading is presented for three special cases: (i) single transmit antenna and single receive antenna, (ii) single transmit and multiple receive antennas, and (iii) multiple transmit and single receive antennas, respectively. Formulae on the outage probability about the channel capacity are also presented. Numerical results are provided to demonstrate the dependence of the channel capacity on various kinds of channel parameters. It is shown that increasing the number of receive antennas can obtain more benefit in channel capacity than increasing the number of transmit antennas. Principally, the channel capacity could be increased infinitely by employing a large number of receive antennas, but it appears to increase only logarithmically in this number for SIMO case; while employing 3—5 transmit antennas can approach the best advantage of the multiple transmit antenna systems (irrespective of all other parameters considered herein) as far as channel capacity is concerned for MISO case. We have also observed that when the signal-to-noise ratio is low, the benefit in average capacity obtained by distributing the available power to different transmit antennas is very limited. We have shown numerically that for a given signal-to-noise ratio, the outage probability decreases *considerably* with the number of receive antennas for SIMO case, while for MISO case, the upper bound of the outage probability decreases with the number of transmit antennas when the communication rate is lower than the critical transmission rate (R_c), but increases when the rate is higher than another value (R_2). The gap between R_2 and R_c is not big for the cases considered here. R_c is determined by the fading power and the signal-to-noise ratio of the system at the transmitter side. We can roughly say that it is not beneficial to use multiple transmit antennas if the required transmission rate (normalized by system bandwidth) is higher than the critical transmission rate.

Due to the fact that the probability density function of the eigenvalues of nonnormal distributed random matrices is

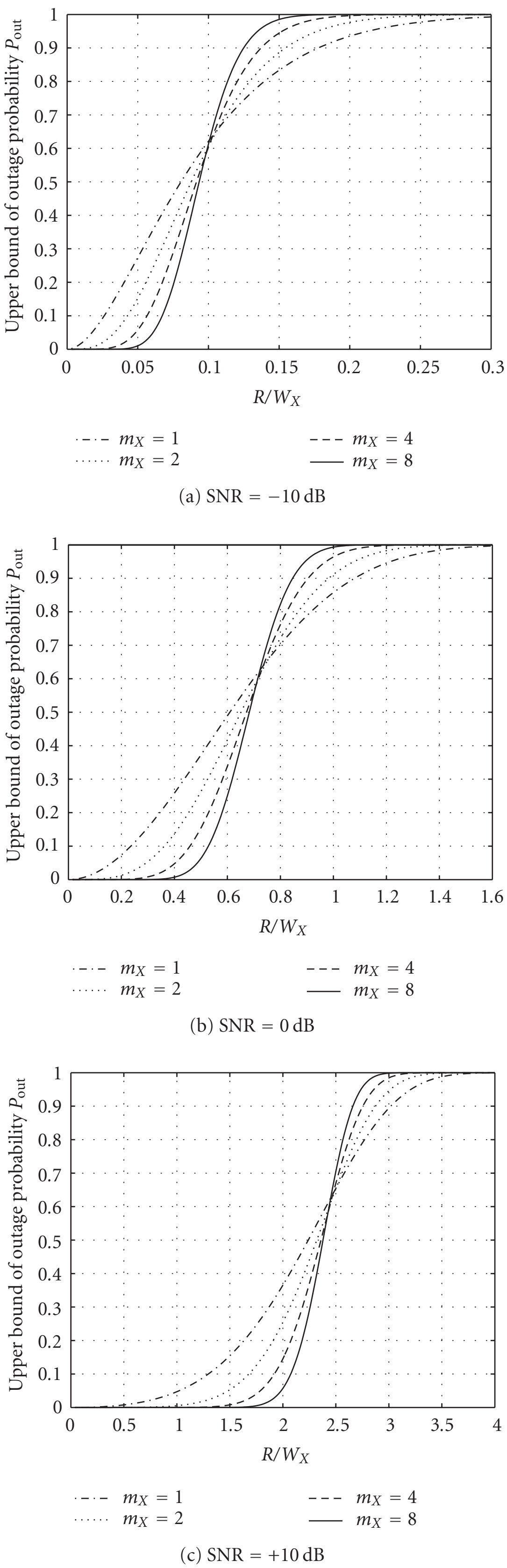

Figure 7: The upper bound of the outage probability $\overline{P}_{out}$ versus transmission rate R/W_X (in nats/s/Hz) for various m_X and SNR for MISO case ($\kappa = 4$).

unknown yet, the problem about the calculation of the channel capacity for the general MIMO case is still open.

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