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Generalized Method for Sampling Spatially Correlated Heterogeneous Speckled Imagery

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This paper presents a general result for the simulation of correlated heterogeneous targets, which are present in images corrupted by speckle noise. This technique is based on the use of a correlation mask and Gaussian random variables, in order to obtain spatially dependent Gamma deviates. These Gamma random variables, in turn, allow the obtainment of correlated $\mathcal{K}$ deviates with specified correlation structure. The theoretical properties of the procedure are presented, along with the corresponding algorithm.

Keywords and phrases: image modelling, speckle noise, spatial correlation, stochastic simulation.

1. INTRODUCTION AND DEFINITIONS

One of the most challenging activities in data analysis is the assessment of the performance of processing algorithms and analysis procedures. Once the metrics for the evaluation are proposed, there are two main approaches: the one based on their analytical derivation and, when this is unfeasible, the one based on Monte Carlo experiences. A Monte Carlo experience relies on the specification of realistic and controllable models, and on the obtainment of large amounts of outcomes of these models. These outcomes are used to measure the quality of the technique under assessment.

This paper deals with the simulation of images corrupted by speckle noise. Speckle noise appears in images obtained with coherent illumination, for example, B-scan ultrasound, sonar and synthetic aperture radar (SAR) imagery. This noise deviates from the classical model which hypothesizes that the corruption is a Gaussian noise, independent of the signal, that adds to the true value. The speckle noise enters the data in a multiplicative fashion, and in the amplitude and intensity formats it does not obey the Gaussian law. Speckle noise is known to make image analysis difficult, since its "salt-and-pepper effect" tends to corrupt the information or ground truth.

Techniques are continuously being proposed to alleviate the influence of this noise, since it makes both the visual and automatic interpretations hard to perform. The quantitative assessment of these techniques is essential for users and researchers and, since its analytical derivation is, in most of the cases, too cumbersome, stochastic methods are welcome.

In particular, SAR imagery is a very important modality of remote sensing. This technology employs an active sensor, that illuminates the target with microwave radiation in order to form an image. In the SAR community the multiplicative model has been widely adopted for the modelling of these images [1, 2]. This model assumes that the value in every pixel, in the intensity format, is the observation of a stochastic process Z defined as the product of two (mutually independent) stochastic processes: σ and Y, where σ represents the ground truth and Y models the speckle noise

$$Z = \sigma \cdot Y. \tag{1}$$

Amplitude format is the square root of the intensity signal. Only intensity data will be treated here.

It is possible to assume that the speckle noise is a white noise process, that is, formed by independent variables, and that they all obey an exponential distribution with unitary mean. Since this noise is very intense and makes difficult the direct use of the images, it is customary to process the images in order to be able to work with *multilook* data. These data are obtained by taking the mean over n (ideally independent) samples of the same image, where from one observation (look) to the next the only possible variation is due to the noise. These samples are obtained in the processing stage and, thus, there is no time elapsed among them.

Calling Y_r the intensity speckle in each look, and assuming that they all obey a standard exponential distribution, it is well known that the mean $Y = n^{-1} \sum_{r=1}^{n} Y_r$ obeys a Gamma distribution, denoted $Y \sim \Gamma(n, n)$ and characterized by the density

$$g_Y(y) = \frac{n^n}{\Gamma(n)} y^{n-1} \exp(-ny), \quad y, n > 0. \qquad (2)$$

This is a commonly accepted characterization of the multilook speckle noise in intensity format. In order to derive the law that governs the observed data, it is necessary to postulate distributions for the ground truth σ.

A widely used model for the ground truth of heterogeneous and homogeneous targets is the $\Gamma(\alpha, \beta)$ distribution, characterized by the density

$$g_\sigma(\sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)} \sigma^{\alpha-1} \exp(-\beta\sigma), \quad \sigma, \alpha, \beta > 0, \qquad (3)$$

where α is referred to as the shape parameter and β as the scale parameter. This distribution, besides showing good, fits to a wide range of targets, can be derived from the physical modelling of the way matter and radiation interact in the image formation. This interference phenomenon is present in every image that uses coherent illumination.

The model for the observed data Z, that is, for the product of the mutually independent processes σ and Y, has marginal intensity $\mathcal{K}$ distribution. For a detailed discussion of this model and its extension, the reader is referred to [1, 2]. The correlation introduced in the model of σ will induce a certain correlation structure in the process Z. This spatial correlation is needed in order to increase the adequacy of the model to real situations. As can be seen in Figure 1, SAR images often exhibit texture that can be modelled through statistical dependence among observations in neighboring sites.

A weakly stationary model will be used for the σ field, with a nontrivial correlation structure. This departure from the white noise model requires a precise and unique definition of the family of distributions to be simulated, since the joint density is no longer the product of the marginal densities, as is with Gaussian random variables.

In order to be consistent with the multiplicative model, it is imperative to impose that the marginal distributions obey Gamma laws, but there is not a unique definition of what a vector of correlated Gamma random variables is. The definition provided in [3, 4] will be adopted here, since it allows the treatment of uncorrelated Gamma random variables as a particular case of correlated ones.

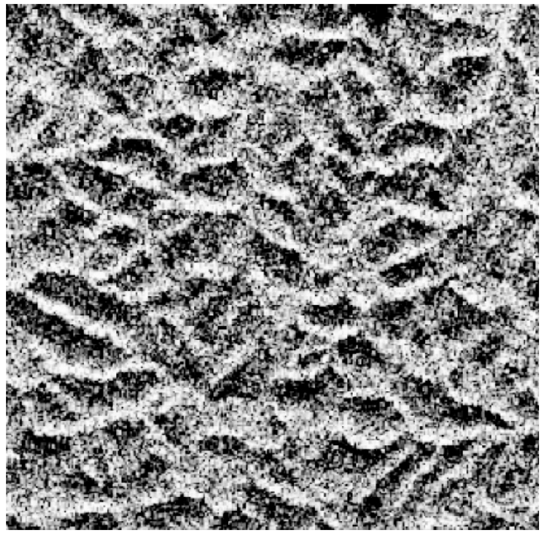

FIGURE 1: Primary Amazon forest over undulated relief, as seen by the JERS sensor.

Definition 1. The random vector $\mathbf{X}$ obeys a correlated Gamma law if each of its components X_i marginally obeys a Gamma law.

Definition 2. The stochastic process $\mathbf{X}$ has a correlated Gamma law if each finite subset of $\mathbf{X}$ has a correlated Gamma law.

This paper presents a review of the main available techniques for the simulation of correlated Gamma random variables, along with some examples and the discussion of their advantages and drawbacks (Section 2). One of the most attractive methods is the one based on the sum of squared Gaussian random variables, and a more general proof than those previously available of the theoretical foundations of this method is presented (Section 2.4). This new proof leads to a result that generalizes previous applications of this technique and, then, a very general technique is proposed along with an algorithm for its implementation (Section 3). Additional proofs are collected in an appendix.

2. GENERATION OF CORRELATED GAMMA DEVIATES

Differently from the Gaussian case, where the correlation matrix and the marginal distributions completely specify the joint distribution, these two ingredients do not induce a unique joint distribution for correlated Gamma random variables.

The applications that we bear in mind, namely, model validation and the evaluation of estimators under the presence of dependence [5], only require the specification of the marginal distributions and the correlation structure. The remaining components, that is, higher order moments, are induced by the way the process is constructed.

Given a set of shape parameters $\alpha_1, \dots, \alpha_n$, a set of scale parameters $\beta_1, \dots, \beta_n$, and correlations $\rho_{i,j}$, with $1 \le i, j \le n$, it is desired to obtain observations from the random vector $\mathbf{X} = (X_1, \dots, X_n)^T$ such that marginal distributions are Gamma (i.e., $X_i \sim \Gamma(\alpha_i, \beta_i)$) and that the correlation structure is as specified (i.e., $\mathrm{Corr}(X_i, X_j) = \rho_{i,j}$).

An immediate difficulty that arises with this requirement is that not every set of correlations $\{\rho_{i,j}\}_{1\le i,j\le n}$ is consistent with an arbitrary set of scale parameters $\{\alpha_i\}_{1\le i\le n}$, since the latter set imposes restrictions on the former. Another limitation is that, even with consistent scale parameters and correlations at hand, there might not be a suitable algorithm for the obtainment of the deviates. This is the reason why all the available procedures for the generation of correlated Gamma variables are effective in restricted domains. There are simple algorithms that allow the simulation of both positively- and negatively-correlated low-dimensional Gamma random vectors. When more than two or three random variables are sought, the restrictions are severe. In Section 2.1, the main simulation procedures available for the generation of correlated Gamma random deviates will be presented.

2.1. Low-dimensional vectors

A number of methods for the simulation of pairs and triplets of correlated Gamma variates has been proposed in the literature. They all require a generator of outcomes of independent Gamma random variables (many alternatives are available in [6]). Some of these methods are

- the one outlined in [7], which allows the generation of a two-dimensional random vector (X_1, X_2) with marginal Gamma distributions with any shape and scale parameters (namely α_1, α_2 and β_1, β_2), but imposes the following restriction on the correlation between the components: $0 \le \rho < \min\{\alpha_1, \alpha_2\}/\sqrt{\alpha_1\alpha_2}$;
- methods based on the Laplace-Stieltjes transform [8, 9], that offer more control over the higher-order moments of the distribution, but are harder to use in nontrivial situations;
- the Beta-Gamma transformation, that allows the simulation of negatively-correlated Gamma random variables [10, 11] and can be generalized to higher-order vectors, but that becomes easily intractable.

2.2. Multidimensional vectors and generalized moving averages

Bivariate or trivariate random vectors cannot represent images well, where hundreds or even millions of observations have to be modelled. This section reviews the main available methods for the generation of high-dimensional correlated Gamma vectors, and the generalized method is presented.

It is already known that if a moving average filter of size L is applied to a vector of uncorrelated Gamma random variables, then the result is a vector of correlated Gamma random variables with triangular shaped autocorrelation function, where the shape parameter is multiplied by L. More generally, every filter with finite impulse response and binary coefficients will preserve the Gamma marginal distribution and will introduce some correlation structure.

Using this property, Ronning [4] and Blacknell [12] proposed methods for the simulation of correlated Gamma deviates. The drawback of these methods is that, it is hard to find the filters that have to be applied, and that they only allow the generation of very simple correlation structures. They are presented in the following sections.

2.2.1 Incidence matrix method

This technique was introduced by Ronning [4] as a generalization of methods for bivariate generation, where only nonnegative correlation is obtained.

Consider $\gamma^{(1)}$ and $\gamma^{(2)}$ vectors of positive constants and dimensions N^2 and M, respectively, with $M \ge N^2$. Assume that $\xi^{(1)} = (\xi_1^{(1)} \cdots \xi_{N^2}^{(1)})$ and $\xi^{(2)} = (\xi_1^{(2)} \cdots \xi_M^{(2)})$ are independent random vectors such that $\xi_i^{(1)} \sim \Gamma(\gamma_i^{(1)}, 1)$ and $\xi_j^{(2)} \sim \Gamma(\gamma_j^{(2)}, 1)$ for every $1 \le j \le M$, and $1 \le i \le N^2$. Then the covariance matrices of $\xi^{(1)}$ and $\xi^{(2)}$ are, respectively,

$$\Gamma_1 = \mathrm{Diag}\left(\gamma_1^{(1)}, \dots, \gamma_{N^2}^{(1)}\right), \qquad \Gamma_2 = \mathrm{Diag}\left(\gamma_1^{(2)}, \dots, \gamma_M^{(2)}\right). \tag{4}$$

Consider $\mathbf{T}$ an *incidence matrix*, that is, $\mathbf{T}$ is an $N^2 \times M$ matrix such that $\mathbf{T}_{i,j} \in \{0, 1\}$. Defining the vector $\eta = \xi^{(1)} + \mathbf{T}\xi^{(2)}$ it is possible to prove that

(1) the covariance matrix of η is $\Sigma = \Gamma_1 + \mathbf{T}\Gamma_2\mathbf{T}^T$;
(2) if $\alpha = (\alpha_1, \dots, \alpha_{N^2})$ is the diagonal of the covariance matrix of η, then $\alpha = \gamma^{(1)} + \mathbf{T}\gamma^{(2)}$;
(3) denoting every element of Σ by $\sigma_{i,j}$, then

$$\sigma_{i,i} = \gamma_i^{(1)} + \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{i,k}, \qquad \sigma_{i,j} = \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{j,k}; \tag{5}$$

(4) every component η_i has $\Gamma(\alpha_i, 1)$ distribution for every $1 \le i \le N^2$.

In this manner, the vector η has a correlated Gamma distribution with means $\alpha = (\alpha_1, \dots, \alpha_{N^2})$ and covariance matrix Σ.

In order to introduce different scale parameters, consider the positive numbers $\beta_1, \dots, \beta_{N^2}$ and the matrix $\mathbf{B} = \mathrm{Diag}(1/\beta_1, \dots, 1/\beta_{N^2})$. If $\Psi = \mathbf{B}\eta$, it is immediate that the marginal distributions are $\psi_k \sim \Gamma(\alpha_k, \beta_k)$. It is also possible to see that the correlation between ψ_k and ψ_j is the same as the correlation between η_k and η_j.

With these results, in order to generate Gamma correlated deviates with a certain correlation structure, it is necessary to derive the incidence matrix $\mathbf{T}$ as well as $\Gamma^{(1)}$ and $\Gamma^{(2)}$ in order to have η with the desired correlation matrix Σ. An algorithm for this is as follows:

(1) Define $M = N^2(N^2 - 1)/2$, Σ is the correlation matrix and B is the diagonal matrix with the desired scale parameters.

(2) Choose $\gamma^{(2)}$ a vector of constants and $\mathbf{T}$ an incidence matrix such that $\sigma_{i,j} = \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{j,k}$ for $i \neq j$.

(3) Generate random deviates from $\xi_1^{(2)}, \ldots, \xi_M^{(2)}$, that is, independent samples from $\Gamma(\gamma_i^{(2)}, 1)$ distributions.

(4) Define $\gamma_i^{(1)} = \sigma_{i,i} - \sum_{k=1}^{M} \mathbf{T}_{i,k}\gamma_k^{(2)}\mathbf{T}_{i,k}$, for every $1 \leq i \leq N^2$.

(5) Obtain samples of $\xi_1^{(1)}, \ldots, \xi_{N^2}^{(1)}$, that is, independent deviates from $\Gamma(\gamma_i^{(1)}, 1)$ random variables.

(6) Return $\Psi = \mathbf{B}(\xi^{(1)} + \mathbf{T}\xi^{(2)})$.

This method cannot yield negatively-correlated Gamma random variables, and the shape parameters are imposed by the desired correlations. Moreover, it is often numerically instable.

2.2.2 Moving average filter method

This technique, due to Blacknell [12], is based on the use of moving average filters over independent Gamma random variables. The analysis of the filter is performed using the moment generating function of the result.

If $X \sim \Gamma(\alpha, \beta)$, then its moment generating function is $M_X(s) = E(\exp(Xs)) = (1 - s/\beta)^{\alpha}$.

Consider $\mathbf{X} = (X_1, \ldots, X_N)^T$ part of a weakly stationary process; if for every $\mathbf{s} = (s_1, \ldots, s_N)$, with $|\mathbf{s}| < \delta$, it holds that $M_{\mathbf{X}}(\mathbf{s}) = E(\prod_{j=1}^{N} \exp(X_j s_j)) < \infty$, then $M_{\mathbf{X}}$ is called moment generating function of $\mathbf{X}$. Note that $M_{X_i}(s_i) = M_{\mathbf{X}}((0, \ldots, s_i, \ldots, 0)^T)$ for every s_i, therefore if marginal Gamma distributions are sought for each X_i with shape and scale parameters α and β, the following conditions must be verified:

$$\begin{aligned} M_{\mathbf{X}}\left((s, 0, \ldots, 0)^T\right) &= M_{\mathbf{X}}\left((0, s, \ldots, 0)^T\right) \\ &= M_{\mathbf{X}}((0, \ldots, s)^T) = \left(1 - \frac{s}{\beta}\right)^{\alpha}. \end{aligned} \tag{6}$$

We also have that $E(X_i X_j) = (\partial/\partial j)[(\partial/\partial i)M_{\mathbf{X}}(\mathbf{0})]$ and, therefore, if the correlation ρ_j is desired at lag j, then it must be imposed that,

$$\rho_j(\mathbf{X}) = \frac{(\partial/\partial i)[(\partial/(\partial i + j))M_{\mathbf{X}}(\mathbf{0})] - E(X_1)^2}{\operatorname{Var}(X_1)}. \tag{7}$$

The method proposed by Blacknell consists of obtaining $\mathbf{X}$ as $\sum_{r=1}^{R} H_r^T \mathbf{Y}_r$, with $R \geq 1$ finite, H_r being $N \times N$ matrices and $\mathbf{Y}_1, \ldots, \mathbf{Y}_R$ independent random vectors, each one formed by independent identically distributed random variables obeying Gamma distributions such that $M_{\mathbf{X}}$ has the required properties.

Now notice that if $\mathbf{Y}$ is an N-dimensional random vector, H is an invertible matrix and $\mathbf{X} = H^T\mathbf{Y}$, then $\mathbf{X}$ has its moment generating function given by $M_{\mathbf{X}}(\mathbf{s}) = M_{\mathbf{Y}}(H\mathbf{s})$. Also if $\mathbf{Y}_1, \ldots, \mathbf{Y}_R$ are independent random vectors, $H_1, \ldots, H_R$ are $N \times N$ matrices, and $\mathbf{X} = \sum_{r=1}^{R} H_r^T Y_r$, then $M_{\mathbf{X}}(\mathbf{s}) = \prod_{r=1}^{R} M_{\mathbf{Y}_r}(H_r \mathbf{s})$.

Given L such that $1 \leq L \leq N$, define $\mathcal{V}_L = \{\ell = (\ell_1, \ldots, \ell_N) : \ell_1 = 1,\ \ell_i \in \{0, 1\},\ \sum_{i=1}^{N} \ell_i = L\}$; then for each $\ell \in \mathcal{V}_L$, the circulant $N \times N$ matrix $H_{\ell,L}$ is defined as

$$H_{\ell,L} = \frac{1}{L}\begin{bmatrix} \ell_1 & \ell_2 & \cdots & \ell_N \\ \ell_N & \ell_1 & \cdots & \ell_{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ \ell_2 & \ell_3 & \cdots & \ell_1 \end{bmatrix}. \tag{8}$$

These matrices have the property that rows and columns have L nonzero values.

Consider $\mathbf{Y} = (Y_1, \ldots, Y_N)^T$ a vector of uncorrelated $Y_i \sim \Gamma(a\alpha/L, \beta/L)$ distributed random variables, then for every $\mathbf{t}$ such that $|\mathbf{t}| < \beta/L$, $M_{\mathbf{Y}}(\mathbf{t}) = \prod_{i=1}^{N}(1 - Lt_i/\beta)^{-a\alpha/L}$. Therefore, if $\mathbf{X} = H_{\ell,L}\mathbf{Y}$, one has that

$$\begin{aligned} M_{\mathbf{X}}\left((s_1, \ldots, s_N)^T\right) &= M_{\mathbf{Y}}(H_{\ell,L}\mathbf{s}) \\ &= \prod_{i=1}^{N}\left(1 - \frac{1}{\beta}L\left(\sum_{j=1}^{N} h_{i,j}s_j\right)\right)^{-a\alpha/L}. \end{aligned} \tag{9}$$

Therefore, for each $1 \leq k \leq N$, it holds that $M_{\mathbf{X}}((0, \ldots, s_k, \ldots, 0)^T) = (1 - s_k/\beta)^{-\alpha a}$ because there are only L rows where $h_{i,k} \neq 0$ and, thus, $\mathbf{X}$ obeys the correlated Gamma distribution with $X_i \sim \Gamma(a\alpha, \beta)$. The coefficients of correlation can be evaluated using (7), or from the moment generating function at the desired lag j_0:

$$M_{\mathbf{X}}(s_{k_0}, s_{k_0+j_0}) = \prod_{i=1}^{N}\left(1 - \frac{L(h_{ik_0}s_{k_0} + h_{i(k_0+j_0)}s_{k_0+j_0})}{\beta}\right)^{-a\alpha/L}, \tag{10}$$

and comparing this function with the bivariate case, since

$$M_{X_1,X_2}(s_1, s_2) = \left[\left(1 - \frac{s_1}{\beta}\right)\left(1 - \frac{s_2}{\beta}\right)\right]^{-\alpha(1-\rho)}\left(1 - \frac{(s_1 + s_2)}{\beta}\right)^{-\alpha\rho}. \tag{11}$$

From this, it is immediate the identification of the coefficient ρ.

In both cases the obtainment of ρ_j as a function of a and L is complicated, and only available in particular cases. It is also noteworthy that it is not possible to establish a specific correlation with a single free parameter, so additional parameters are required.

Finally, the algorithm for the generation of the vector $\mathbf{X}$ with correlated Gamma distribution can be posed as

(1) Define $\rho_1, \ldots, \rho_R$ the desired correlation coefficients for the first R lags, α and β are the shape and scale parameters for the final marginal distributions, and N is the dimension of the final vector.

(2) Define $L_1, \ldots, L_m$ integers with $1 \leq L_i \leq N$ and for each of them let $\ell_i \in \mathcal{V}_{L_i}$ be such that they generate filters $H_{\ell_1}, \ldots, H_{\ell_m}$. These filters induce non-null autocorrelation functions only in the first R lags.

(3) Calculate $a_1, \ldots, a_m$ such that

$$\begin{bmatrix} 1 \\ \rho_1 \\ \vdots \\ \rho_R \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ \rho_1(H_{\ell_1}Y_1) \\ \vdots \\ \rho_R(H_{\ell_1}Y_1) \end{bmatrix} + \cdots + a_m \begin{bmatrix} 1 \\ \rho_1(H_{\ell_m}Y_m) \\ \vdots \\ \rho_R(H_{\ell_m}Y_m) \end{bmatrix}, \tag{12}$$

where $\mathbf{Y}_1, \ldots, \mathbf{Y}_m$ are N-dimensional independent vectors, with marginal distributions $Y_{i,j} \sim \Gamma(a_i\alpha/L, \beta/L)$, for every $1 \le i \le R$ and every $1 \le j \le N$.

(4) Return $\mathbf{X} = \sum_{i=1}^{m} H_{\ell_i,L_i}\mathbf{Y}_i$.

Note that, since $M_{\mathbf{X}}(\mathbf{s}) = \prod_{r=1}^{m}\prod_{i=1}^{N}(1 - (L/\beta) \times \sum_{j=1}^{N} h_{r,i,j}s_j)^{-a_r\alpha/L}$, then it holds that $M_{\mathbf{X}}((0,\ldots, s_k,\ldots,0)^T) = (1 - s_k/\beta)^{-\alpha}$ and, therefore, $\mathbf{X}$ has marginal Gamma distributions with the desired parameters α and β.

This algorithm is relatively simple, though more and more expensive from the computational point of view as the number of non-null correlated random variables increases. This limits its usefulness to "small" cases, where the biggest non-null correlation lags are of order 2 or 3 at the most.

2.3. *Transformation method*

An alternative approach to the problem of generating outcomes from correlated Gamma vectors is a method based in three steps

(1) generating independent outcomes from a convenient distribution;
(2) introducing correlation in these data;
(3) transforming the correlated observations into the desired marginal properties [2].

The transformation that guarantees the validity of this procedure is obtained from the cumulative distribution functions of the data obtained in step (2) and from the desired set of distributions. Recall that if U is a continuous random variable with cumulative distribution function F_U, then $F_U(U)$ obeys a $\mathcal{U}(0,1)$ law and, reciprocally, if V obeys a $\mathcal{U}(0,1)$ distribution, then $F_U^{-1}(V)$ is F_U distributed. In order to use this method, it is necessary to know the correlation that the random variables will have after the transformation.

In principle, there are no restrictions on the possible order parameters values that can be obtained by this method, but numerical issues must be taken into account. Another important point is that not every desired final correlation structure is mapped onto a feasible intermediate correlation structure.

Consider any $\alpha > 0$ and let G be the cumulative distribution function of a $\Gamma(\alpha,\alpha)$ distributed random variable

$$G(y) = \frac{(\alpha)^\alpha}{\Gamma(\alpha)}\int_0^y x^{\alpha-1}e^{-\alpha x}\,dx. \tag{13}$$

Now let Φ be the cumulative distribution function of a standard Gaussian random variable (denote this distribution $\mathcal{N}(0,1)$). Since $U \sim \mathcal{N}(0,1)$, then the variable $G^{-1}(\Phi(U)) = X \sim \Gamma(\alpha,\alpha)$.

Consider now the N^2-dimensional random vector $(U_1,\ldots,U_{N^2})$ with $\mathcal{N}(\mathbf{0},\Sigma)$ distribution, where

$$\Sigma = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,N^2} \\ \rho_{1,2} & 1 & \cdots & \rho_{2,N^2} \\ \vdots & \ddots & \cdots & \vdots \\ \rho_{1,N^2} & \rho_{2,N^2} & \cdots & 1 \end{bmatrix} \tag{14}$$

with $0 \le |\rho_{i,j}| < 1$, for every $1 \le i \le N^2-1$ and every $i+1 \le j \le N^2$. Define for every $1 \le k \le N^2$ the random variable $X_k = G^{-1}(\Phi(U_k))$; then $\mathbf{X} = (X_1,\ldots,X_{N^2})^T$ has a correlated Gamma distribution with $\rho_{k,\ell} = \rho(X_k,X_\ell) = \alpha(E(X_kX_\ell) - 1)$. Now

$$E(X_kX_\ell) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G^{-1}(\Phi(u_k))G^{-1}(\Phi(u_\ell)) \times \phi_2(u_k,u_\ell)\,du_k\,du_\ell, \tag{15}$$

where

$$\phi_2(u_k,u_\ell) = \frac{1}{2\pi\sqrt{\left(1-\rho_{k,\ell}^2\right)}}\exp\left(-\frac{u_k^2 - 2\rho_{k,\ell}u_ku_\ell + u_\ell^2}{2\left(1-\rho_{k,\ell}^2\right)}\right). \tag{16}$$

The problem now consists of specifying the correlation matrix Σ that yields the desired correlation structure $E(X_kX_\ell)$. Since the function G^{-1} is only available using numerical methods, it is an approximation that may impose restrictions to the use of this simulation method.

2.4. *The sum of squared normals*

It is known that the sum of the squares of n independent identically standard Gaussian random variables obeys a Gamma distribution with shape parameter $n/2$. This is the basis for the method presented in this section. This procedure, described for the bivariate case in [13], is easily generalized to any finite number of Gamma random variables. It has the disadvantage of only allowing shape parameters taking values $n/2$ with n integer, and of restricting the correlation between components to being the square root of the final desired correlation. In Section 2.5 this scheme will be extended to allow the use of convolution of Gaussian vectors in a more general manner than the one presented in [2], where it is used for the simulation of SAR images.

Proposition 3. *Consider the independent random vectors $\underline{\xi}_1,\ldots,\underline{\xi}_\alpha$, each of dimension N^2, obeying the $\mathcal{N}(\mathbf{0},\Sigma)$ distribution, such that Σ is of the form*

$$\Sigma = \frac{1}{2}\Sigma_1, \tag{17}$$

with Σ_1 given in (14), $0 \le \rho_{i,j} < 1$ for every $1 \le i \le N^2-1$ and $i+1 \le j \le N^2$. Consider $\underline{\eta} = \sum_{j=1}^{\alpha}\underline{\xi}_j^2$, then $\underline{\eta} = (\eta_1,\ldots,\eta_{N^2})^T$ has correlated Gamma distribution with $\eta_i \sim \Gamma(\alpha/2,1)$ and with $\rho_{i,j}^2$ as the correlation between η_i and η_j, for every $1 \le i \le N^2-1$ and $i+1 \le j \le N^2$. Also if $\beta_1^{-1},\ldots,\beta_{N^2}^{-1}$ are positive integers and if B is the diagonal matrix formed by these constants, then $X' = B\underline{\eta}$ has correlated Gamma distribution with marginals $\Gamma(\alpha/2,\beta_k)$ with correlation between X_i' and X_j' given by $\rho_{i,j}^2$, for every $1 \le i \le N^2-1$ and $i+1 \le j \le N^2$.

The proof of this proposition is given in Appendix A.

2.5. Proposal: multivariate reduction

The last method has a restriction on the possible values for the shape parameter, but it has the advantage of being easy to implement. The aforementioned restriction may be of no practical importance for the applications that we bear in mind. That method relies on the obtainment of correlated Gamma random variables, with correlations that are the square root of the desired value.

The use of convolution filters for the generation of such correlated Gamma deviates is proposed in this work, using independent normal random variables as input. A particular case of this method was presented in [14]. The procedure can be outlined as

(1) Generate independent normal observations.

(2) Choose the correlation as the square of a suitable function E, defined on $\mathbb{Z}^2$.

(3) Calculate the mask θ that the convolution filter will use, such that $\theta * \theta = E$.

(4) Apply the convolution filter to the independent normal deviates, obtaining outcomes from the processes with correlation E in each component.

(5) Return the sum of the squares of each normal deviate.

This procedure is valid for the family of functions E such that

(1) Periodicity: $E : \mathbb{Z}^2 \to \mathbb{R}$ is a periodic function with fundamental period $R_N = \{(s_1, s_2) : 0 \le s_1,\ s_2 \le N-1\}$.

(2) Separability: there is a unidimensional periodic function E_1 such that $E(s_1, s_2) = E_1(s_1)E_1(s_2)$.

(3) There is a real characteristic function c such that

$$E_1(s) = \begin{cases} c(s), & 0 \le s \le \dfrac{N}{2}, \\ c(N-s), & \dfrac{N}{2} + 1 \le s \le N-1. \end{cases} \tag{18}$$

In this manner, this proposition extends previous results since it allows the use of characteristic functions for the correlation structure of the process. In order to have a real characteristic function $c(t) = \int e^{itx}\, dF(x)$, the corresponding distribution F has to be symmetric around the origin. Distributions with complex characteristic functions can be translated or reflected over the origin in order to obtain real valued characteristic functions. Another interesting property is that if c is a characteristic function, then so is $|c|^2$. For a comprehensive account of characteristic functions and their properties, the reader is referred to [15, 16].

Proposition 4. *There is a periodic function $\theta : \mathbb{Z}^2 \to \mathbb{R}$ with fundamental period R_N that satisfies*

$$\begin{aligned} \theta * \theta(s_1, s_2) &= \sum_{t_1=0}^{N-1} \sum_{t_2=0}^{N-1} \theta(t_1, t_2)\theta(s_1 - t_1, s_2 - t_2) \\ &= E(s_1, s_2), \end{aligned} \tag{19}$$

and such that

$$\theta(s_1, s_2) = \begin{cases} \theta(N - s_1, s_2) & \text{if } (s_1, s_2) \in R_2, \\ \theta(s_1, N - s_2) & \text{if } (s_1, s_2) \in R_3, \\ \theta(N - s_1, N - s_2) & \text{if } (s_1, s_2) \in R_4, \end{cases} \tag{20}$$

where $R_1 = \{\mathbf{s} : 0 \le s_1,\ s_2 \le N/2\}$, $R_2 = \{\mathbf{s} : N/2 + 1 \le s_1 \le N-1,\ 0 \le s_2 \le N/2\}$, $R_3 = \{\mathbf{s} : 0 \le s_1 \le N/2,\ N/2 + 1 \le s_2 \le N-1\}$, and $R_4 = \{\mathbf{s} : N/2 + 1 \le s_1,\ s_2 \le N-1\}$.

The proof of this proposition can be found in Appendix B.

Definition 5. Consider ζ_k, $1 \le k \le 2\alpha$, are independent Gaussian white noise periodic stochastic processes with fundamental period R_N. Define ξ_k, $1 \le k \le 2\alpha$, as periodic processes with fundamental period R_N, as $\xi_k(s_1, s_2) = (\theta * \zeta_k)(s_1, s_2)$.

Proposition 6. *The processes ξ_k as previously defined satisfy the following properties:*

(1) *$\xi_k(s_1, s_2) \sim N(0, (\theta * \theta)(0,0)/2)$, that is, ξ_k are stochastic processes with Gaussian marginals with zero mean and variances* $1/2$.

(2) $E(\xi_k(0,0)\xi_k(s_1, s_2)) = (\theta * \theta)(s_1, s_2)/2 = E(s_1, s_2)/2$.

(3) $\rho(\xi_k(0,0), \xi_k(s_1, s_2)) = E(s_1, s_2)$.

The proof of this proposition can be seen in Appendix C.

Definition 7. Define the periodic stochastic process η with fundamental period R_N as $\eta(s_1, s_2) = \sum_{k=1}^{2\alpha} \xi_k^2(s_1, s_2)$ for every $(s_1, s_2) \in R_N$, and assume $\beta > 0$. The periodic stochastic process σ is defined as $\sigma(s_1, s_2) = (1/\beta)\eta(s_1, s_2)$ for every $(s_1, s_2) \in R_N$.

Proposition 8. *The following properties hold:*

(1) *The process η is a weakly stationary stochastic process with correlated Gamma distribution such that $\eta(s_1, s_2) \sim \Gamma(\alpha, 1)$.*

(2) *The process σ is a weakly stationary stochastic process with correlated Gamma distribution such that*

(a) *$\sigma(s_1, s_2) \sim \Gamma(\alpha, \beta)$, then $E(\sigma(s_1, s_2)) = \alpha/\beta$ and* $\mathrm{Var}(\sigma(s_1, s_2)) = \alpha/\beta^2$.

(b) *The coefficient of correlation at* $\mathrm{lag}(s_1, s_2)$ *is* $\rho(\sigma_{(s_1, s_2)}, \sigma_{(0,0)}) = E^2(s_1, s_2)$.

The proof of this proposition is presented in Appendix D.

2.6. Summary

Table 1 presents the main properties of the presented methods for the generation of correlated multidimensional Gamma random variables. It is important to note here that no computational comparison amongst methods was performed, since the ones already available in the literature are less general or much more difficult to implement than our proposal, or both.

3. SIMULATING HETEROGENEOUS IMAGES

The general method presented in previous sections was implemented using the following algorithm:

(1) Generate the Gaussian white noises ζ_k, with variance $1/2$ for every $1 \le k \le 2\alpha$.

(2) Define a convenient function E_1, obeying the aforementioned conditions.

(3) Compute $\psi_2(s_1, s_2) = \sqrt{\mathcal{F}(E_1)(s_1) \cdot \mathcal{F}(E_1)(s_2)}$, the frequency domain mask.

Table 1: Main features and limitations of multivariate simulation of correlated Gamma random variables.

Method	Features	Limitations
Incidence matrix	Generalises bivariate generation	Nonnegative correlation, generation shape parameters induced by correlation structure, numerical instabilities
Moving average	Simple	Long length correlation structures are hard to implement
Transformation	Very general	Numerical instabilities
Sum of normals	Simple	Restricted to simple situations, nonnegative correlation, restricted scale parameters
Multivariate reduction	Very general	Nonnegative correlation

(4) Calculate $\xi_k = \mathcal{F}^{-1}(\psi_2 \cdot \mathcal{F}(\zeta_k))$, for every $1 \le k \le 2\alpha$.

(5) Obtain $\sigma = \beta^{-1} \sum_{k=1}^{2\alpha} \xi_k^2$.

(6) Generate independent random variables identically distributed as $\Gamma(n, n)$, where n is the desired equivalent number of looks, Y.

(7) Return $Z = \sigma \cdot Y$.

In this algorithm $\mathcal{F}(U)$ and $\mathcal{F}^{-1}(U)$ represent the direct and inverse Fourier transforms, respectively, of the input U, usually computed using routines based on the fast Fourier transform algorithm. It is noteworthy that the bigger the parameter α, the slower will be the execution of this procedure.

The return simulated with this procedure obeys an intensity $\mathcal{K}$ distribution, characterized by the density

$$f_Z(z) = \frac{2\beta n}{\Gamma(\alpha)\Gamma(n)} (\beta n x)^{(\alpha+n)/2-1} K_{\alpha-n}\left(2\sqrt{\beta n x}\right), \quad (21)$$

where $z, \alpha, \beta, n > 0$ and K_ν is the modified Bessel function of the third kind and order ν. This is the distribution of a random variable obtained as the product of two independent random variables that obey $\Gamma(n, n)$ and $\Gamma(\alpha, \beta)$ distributions. This distribution has been consecrated in the SAR literature as an excellent model for heterogeneous and homogeneous targets. More details about this and other distributions arising from the multiplicative model can be seen in [1]. Examples of this density are shown in Figures 4 and 7 for the $n = 3$ and $n = 1$ cases, respectively.

The use of this technique will be illustrated with three particular (useful and widely employed) characteristic functions: normal, sinc, and exponential functions. Though all the examples shown belong to the family of parametric correlation structures, the method here presented allows simulating nonparametric situations, provided the restrictions on the function E are respected.

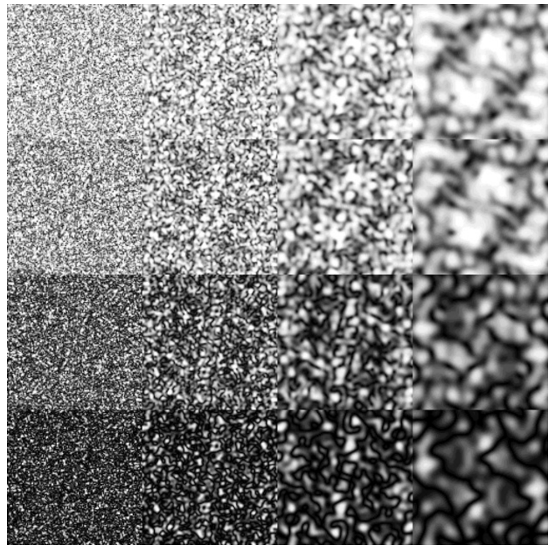

Figure 2: Simulated σ fields obeying $\Gamma(\alpha, 1)$ distributions with normal correlation, α varying in the rows and correlation lag varying in the columns.

The adequacy of the simulation procedures was checked comparing desired correlation structures with the observed ones, for a variety of parameters, and the results are compatible with the theory. Detailed results on estimation procedures for the spatial dependence of SAR data will be reported elsewhere; it is an active research area where the simulation procedure here presented will be used. It is noteworthy that the simulated fields show a striking resemblance with those from real targets (cf. [1, 2, 17]).

3.1. Normal case

Consider the generation of random fields with a correlation structure given by the Gaussian function

$$E_1(s) = \exp\left(-\frac{s^2}{(2\ell^2)}\right), \quad 0 \le s \le \frac{N}{2}. \quad (22)$$

Figure 2 shows sixteen simulated $\Gamma(\alpha, 1)$ fields of size 128×128 each with varying shape parameter α (rows with $\alpha \in \{0.5, 1, 1.5, 2\}$) and correlation length ℓ (columns with $\ell \in \{1, 2, 4, 8\}$). Figure 3 shows the images that should be returned by a three-looks system, corresponding to the truth images shown in Figure 2. The marginal densities corresponding to these return images ((21) with $\beta = 1$ and $n = 3$) are shown in Figure 4.

3.2. Sinc case

Consider the generation of random fields with a correlation structure given by the sinc function $E_1(s) = \sin(\ell s/2)/(\ell s/2)$.

Figure 5 shows sixteen simulated $\Gamma(\alpha, 1)$ fields of size 128×128 each, with varying shape parameter α (rows with $\alpha \in \{0.5, 1, 1.5, 2\}$) and correlation length ℓ (columns with $\ell \in \{8, 4, 2, 1\}$). Figure 6 shows the images that should be

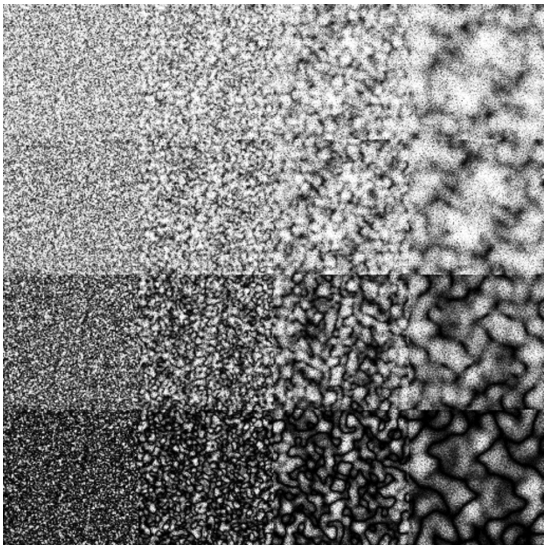

FIGURE 3: The observed data, with three looks and the normally correlated ground truth.

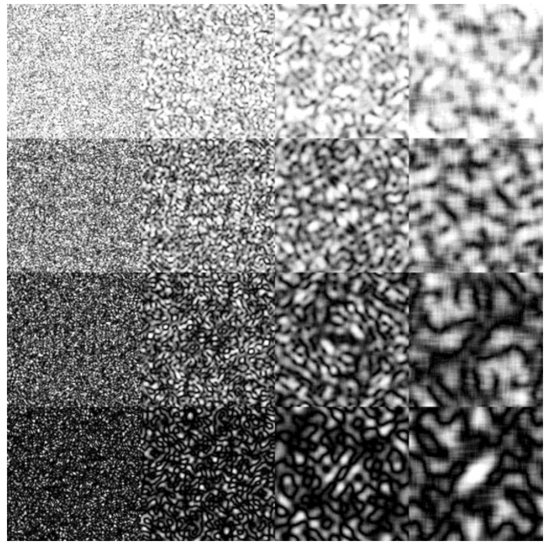

FIGURE 5: Simulated σ fields obeying $\Gamma(\alpha, 1)$ distributions and sinc correlation, α varying in the rows and correlation lag varying in the columns.

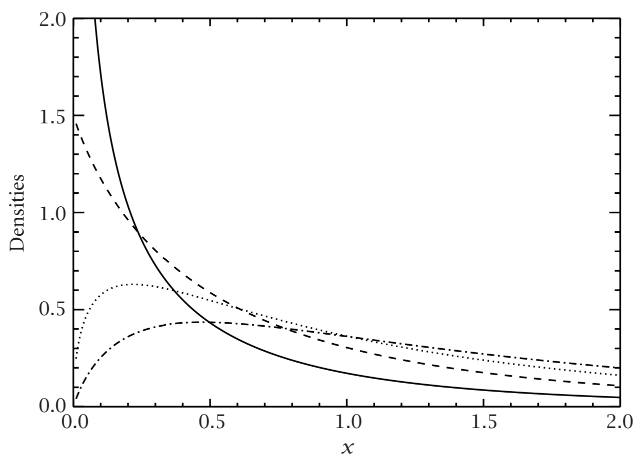

FIGURE 4: Marginal densities of the simulated three-looks Z processes: $\mathcal{K}(\alpha, 1, 3)$, with $\alpha \in \{0.5, 1, 1.5, 2\}$ (solid, dash, dots, dot-dash).

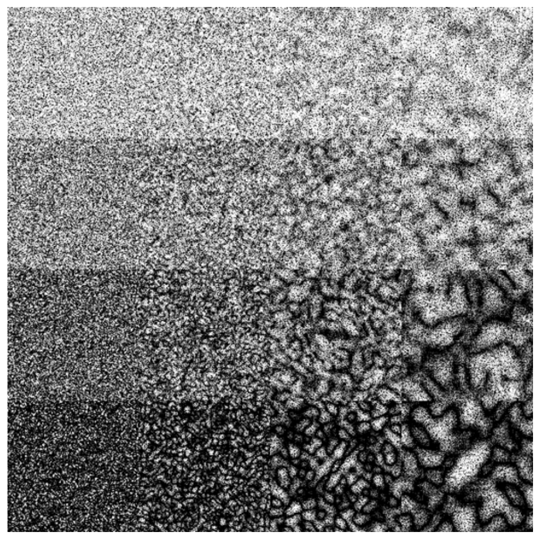

FIGURE 6: The observed data, with one look and sinc correlated ground truth.

returned by a one-look system, corresponding to the truth images shown in Figure 5. The marginal densities corresponding to these return images ((21) with $\beta = n = 1$) are shown in Figure 7.

3.3. *Exponential case*

Consider the generation of random fields with a correlation structure given by the exponential function $E_1(s) = \exp(-|s|/\ell), \ell > 0$.

Figure 8 shows sixteen simulated $\Gamma(\alpha, 1)$ fields of size 128×128 each, with varying shape parameter α (rows with $\alpha \in \{0.5, 1, 1.5, 2\}$) and correlation length ℓ (columns with $\ell \in \{1, 2, 4, 8\}$). Figure 9 shows the images that should be returned by a one-look system, corresponding to the truth images shown in Figure 8. The marginal densities of the return are those presented in Figure 7.

4. CONCLUSIONS

Methods for the generation of correlated Gamma fields have been presented and discussed, aiming at the simulation of correlated $\mathcal{K}$ fields for the simulation of images corrupted by speckle noise. Moving averages has the advantage of allowing any shape parameter and a wide variety of autocorrelation functions but, in practice, it is too cumbersome to be implemented but in very simple situations. Methods based on random variables transformations are also very general and have the least restrictions of all the techniques considered, but they rely on numerical approximations which are seldom effective. The sum of squares of Gaussian random variables limits the

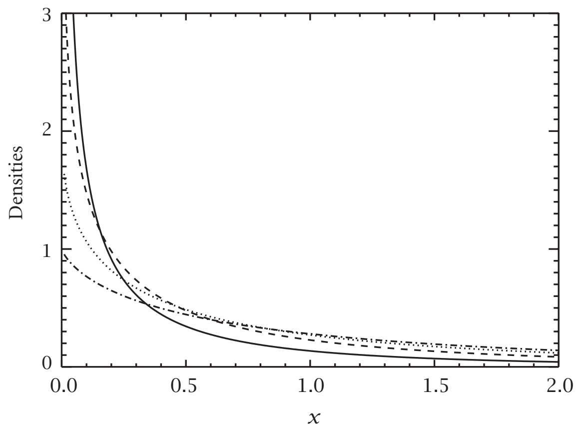

FIGURE 7: Marginal densities of the simulated one-look Z processes: $\mathcal{K}(\alpha, 1, 1)$, with $\alpha \in \{0.5, 1, 1.5, 2\}$ (solid, dash, dots, dot-dash).

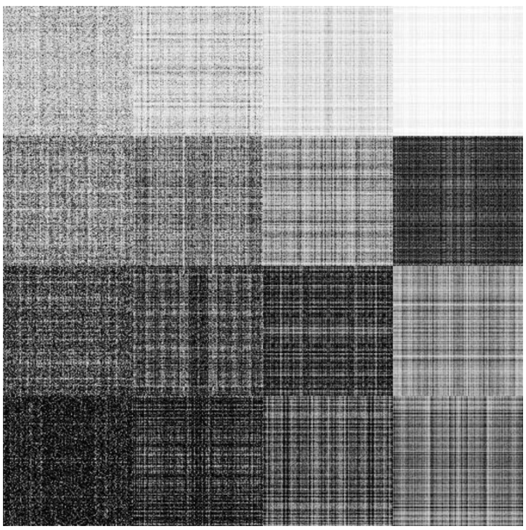

FIGURE 8: Simulated σ fields obeying $\Gamma(\alpha, 1)$ distributions and exponential correlation, α varying in the rows and correlation lag varying in columns.

shape parameters to halves of integers, but if this restriction is of little or no significance, it is the recommended method.

A family of autocorrelation functions was proposed in this article, and a simulation methodology was presented for it. Members of this family have been previously used for the modelling of forest data [18], assuming that the spatial correlation decays exponentially at distance ℓ. Simulations were presented for several parameter values and three particular cases.

APPENDIX A: PROOF OF PROPOSITION 3

Proof. Some useful and well-known results are

(1) If $\xi \sim \mathcal{N}(0, 1/2)$, then $\xi^2 \sim \Gamma(1/2, 1)$.

(2) Consider the independent identically distributed random variables $\xi_1, \dots, \xi_\alpha$ obeying the $N(0, 1/2)$ law, then $\xi_1^2 + \cdots + \xi_\alpha^2 \sim \Gamma(\alpha/2, 1)$.

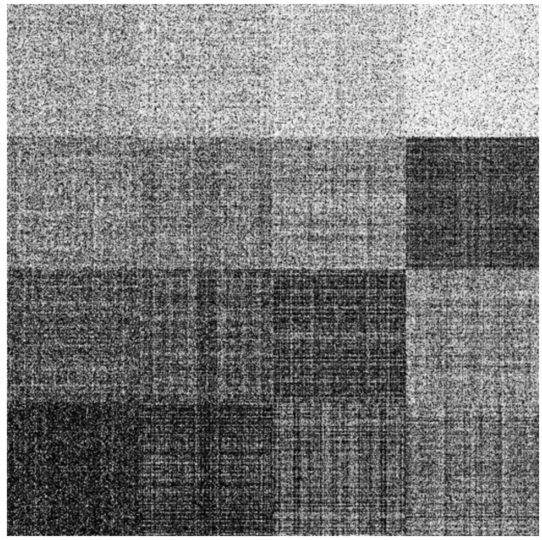

FIGURE 9: The observed data, with one look and exponentially correlated ground truth.

(3) Consider $\underline{\xi} = (\xi_1, \dots, \xi_{N^2})^t$ an N^2-dimensional vector with $\mathcal{N}(\mathbf{0}, \Sigma)$ distribution, where Σ is as given in (17). Let $\underline{\xi}_j = (\xi_{1,j}, \dots, \xi_{N^2,j})^t$, with $1 \le j \le \alpha$, be N^2-dimensional independent random vectors, each having an $\mathcal{N}(\mathbf{0}, \Sigma)$ distribution, with Σ of the form given in (17). Define $\underline{\eta} = \sum_{j=1}^{\alpha} \underline{\xi}_j^2 = (\sum_{j=1}^{\alpha} \xi_{1,j}^2, \dots, \sum_{j=1}^{\alpha} \xi_{N^2,j}^2)^t$ and let $\eta_i = \sum_{j=1}^{\alpha} \xi_{i,j}^2$, with $1 \le i \le N^2$. Then $\eta_i \sim \Gamma(\alpha/2, 1)$, with $E(\eta_j) = \alpha/2$ and $\mathrm{Var}(\eta_j) = \alpha/2$. In other words, $\underline{\eta}$ has a correlated Gamma distribution.

In order to compute the correlation between η_i and η_j, we verify first that if (U, V) is an $\mathcal{N}((0,0), \Sigma)$ distributed vector with covariance matrix of the form

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \tag{A.1}$$

then $E(U^2V^2) = \sigma_1^2\sigma_2^2(1 + 2\rho^2)$. Using this, since $\mathrm{Cov}(\eta_i, \eta_j) = E(\eta_i\eta_j) - E(\eta_i)E(\eta_j)$, and $E(\eta_j) = E(\eta_i) = \alpha/2$, we must compute $E(\eta_i\eta_j)$. Using the fact that the vectors $\underline{\xi}_j$ are independent, the previous result and the fact that $E(\xi_{i,k}^2) = 1/2$, we have

$$\begin{aligned} E(\eta_i\eta_j) &= E\left(\sum_{h=1}^{\alpha} \xi_{i,h}^2 \sum_{k=1}^{\alpha} \xi_{j,k}^2\right) = \sum_{h=1}^{\alpha}\sum_{k=1}^{\alpha} E\left(\xi_{i,h}^2\xi_{j,k}^2\right) \\ &= \sum_{h=1}^{\alpha} E\left(\xi_{i,h}^2\xi_{j,h}^2\right) + \sum_{h=1}^{\alpha}\sum_{k\neq h} E\left(\xi_{i,k}^2\right)E\left(\xi_{j,k}^2\right) \\ &= \alpha E\left(\xi_{i,i}^2\xi_{j,i}^2\right) + \alpha(\alpha-1)E\left(\xi_{1,1}^2\right)^2 \\ &= \frac{\alpha}{4}\left(1 + 2\rho_{i,j}^2\right) + \alpha(\alpha-1)\frac{1}{4} \\ &= \frac{\alpha}{4}\left(\alpha + 2\rho_{i,j}^2\right). \end{aligned} \tag{A.2}$$

Then $\mathrm{Cov}(\eta_i, \eta_j) = (1/4)\alpha^2 + (\alpha/2)\rho_{i,j}^2 - (\alpha^2/4) =$

$(\alpha/2)\rho_{i,j}^2$, so

$$\rho(\eta_i,\eta_j) = \frac{\mathrm{Cov}(\eta_i,\eta_j)}{\sqrt{\mathrm{Var}(\eta_i)\,\mathrm{Var}(\eta_j)}} = \frac{(\alpha/2)\rho_{i,j}^2}{\alpha/2} = \rho_{i,j}^2. \quad (\text{A.3})$$

□

Summarizing, consider $\underline{\eta} = (\eta_1,\dots,\eta_{N^2})^t$ is an N^2-dimensional vector with $\eta_j \sim \Gamma(\alpha/2,1)$ and with $\rho(\eta_i,\eta_j) = \rho_{i,j}^2$. If $B = \mathrm{Diag}(\beta_1^{-1},\dots,\beta_{N^2}^{-1})$ with $\beta_i > 0$ and $X' = B\underline{\eta}$, then $\eta_i \sim \Gamma(\alpha/2,\beta_i)$ and $\mathrm{Cov}(X_i',X_j') = \mathrm{Cov}(\eta_i,\eta_j) = \rho_{i,j}^2$.

APPENDIX B: PROOF OF PROPOSITION 4

Proof. We prove that there exists $\theta_1 : \mathbb{Z} \to \mathbb{R}$ with period $R = \{0,\dots,N-1\}$ such that

(1) $\theta_1 * \theta_1(s_1) = \sum_{j=0}^{N-1}\theta_1(j)\theta_1(s_1-j) = E_1(s_1)$ for every $s_1 \in \mathbb{Z}$,
(2) $\theta_1(s_1) = \theta_1(N-s_1)$, if $(N/2)+1 \le s_1 \le N-1$.

If such θ_1 exists, it suffices to define the function θ in separable form, that is, $\theta(s_1,s_2) = \theta_1(s_1)\theta_1(s_2)$ in order to hold the proposition.

Using Lemma 9, the Fourier transform of E_1

$$\hat{E}_1(s_1) = \frac{1}{N}\sum_{k=0}^{N-1} E_1(k)\omega_{s_1k,N}^* \quad (\text{B.1})$$

is a real positive function, then we can define the periodic function ψ as $\psi = \sqrt{\hat{E}_1}$. This function satisfies that $\psi(k) = \psi(N-k)$, since

$$\begin{aligned}\hat{E}_1(N-k) &= \frac{1}{N}\sum_{\ell=0}^{N-1} E_1(\ell)\omega_{(N-k)\ell,N}^* \\ &= \frac{1}{N}\sum_{\ell=0}^{N-1} E_1(N-\ell)\omega_{(N-k)\ell,N}^* \\ &= \hat{E}_1(k).\end{aligned} \quad (\text{B.2})$$

In order to obtain this result, the properties of the unit roots and the definition of E_1 are used. Consider now θ_1, the inverse Fourier transform of ψ is given by

$$\theta_1(s_1) = \tilde{\psi}(s_1) = \sum_{k=0}^{N-1}\psi(k)\omega_{ks_1,N}. \quad (\text{B.3})$$

Then, by the properties of the periodic Fourier transform and the definition of θ_1, $\widehat{\theta_1 * \theta_1} = \hat{\theta}_1\hat{\theta}_1 = \hat{\tilde{\psi}}\cdot\hat{\tilde{\psi}} = \psi\cdot\psi = \hat{E}_1$, and by the unicity of the transform one has that $\theta_1 * \theta_1 = E_1$ verifying, thus, the first condition.

The second condition stems from the fact that the inverse Fourier transform always satisfies that $\tilde{\psi}(N-k) = \tilde{\psi}(k)^*$ for every $0 \le k \le N-1$, and that

$$\begin{aligned}\tilde{\psi}(k)^* &= \sum_{\ell=0}^{N-1}\psi(l)\omega_{k\ell,N}^* \\ &= \sum_{\ell=0}^{N-1}\psi(N-l)\omega_{(N-\ell)k,N} = \tilde{\psi}(k).\end{aligned} \quad (\text{B.4})$$

From these, one has that $\theta_1(s_1) = \theta_1(N-s_1)$ if $N/2+1 \le s_1 \le N-1$. □

Lemma 9. *The Fourier transform of E_1, given by $\hat{E}_1(s_1) = (1/N)\sum_{k=0}^{N-1}E_1(k)\omega_{s_1k,N}^*$, is a real positive function.*

Proof. We will check that $\hat{E}_1$ is a real positive function. Remember that $E_1(j) = c(j)$, for every $0 \le j \le N/2$ and $E_1(N-j) = c(N-j)$ in every $1 \le j \le N/2-1$, with c a real characteristic function. Since c is a positive definite function, then

$$M = \begin{bmatrix} E_1(0) & E_1(1) & \cdots & E_1(N-1) \\ E_1(N-1) & E_1(0) & \cdots & E_1(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ E_1(1) & E_1(2) & \cdots & E_1(0) \end{bmatrix} \quad (\text{B.5})$$

is a circulant positive definite matrix. Therefore (see [19]), its eigenvalues are positive real numbers. These eigenvalues are, for every $0 \le j \le N-1$, given by $\lambda_j = \sum_{k=0}^{N-1}E_1(k)\omega_{jk,N}^* = N\hat{E}_1(j) > 0$ and, therefore, $\hat{E}_1$ is a real positive function. □

APPENDIX C: PROOF OF PROPOSITION 6

Proof. Since the periodic convolution is a finite linear combination, the processes ξ_k obey Gaussian distributions since the processes ζ_k are independent white noise Gaussian processes. In order to verify the second item, the same reason is used along with the definition of θ,

$$\begin{aligned}&E(\xi_k(0,0)\xi_k(s_1,s_2)) \\ &= E\left(\sum_{t,n}\zeta_k(t_1,t_2)\theta(s_1-t_1,s_2-t_2)\zeta_k(n_1,n_2)\theta(-n_1,-n_2)\right) \\ &= \sum_{t,n}\theta(-n_1,-n_2)\theta(s_1-t_1,s_2-t_2)E(\zeta_k(t_1,t_2)\zeta_k(n_1,n_2)) \\ &= \sum_{n}\theta(-n_1,-n_2)\theta(s_1-n_1,s_2-n_2)E(\zeta_k(n_1,n_2)\zeta_k(n_1,n_2)) \\ &= \frac{1}{2}\sum_{n}\theta(n_1,n_2)\theta(s_1-n_1,s_2-n_2) \\ &= \frac{1}{2}(\theta*\theta)(s_1,s_2) = \frac{1}{2}E(s_1,s_2).\end{aligned} \quad (\text{C.1})$$

Also note that $\rho(\xi(0,0),\xi(s_1,s_2)) = E(s_1,s_2)$. □

APPENDIX D: PROOF OF PROPOSITION 8

Proof. Note that the processes $\xi_1,\dots,\xi_{2\alpha}$ are independent weakly stationary Gaussian processes, each with

(1) $\xi_k(s_1,s_2) \sim \mathcal{N}(0,1/2)$, $\forall\,(s_1,s_2) \in R_N$,
(2) $E(\xi_k(s_1,s_2)\xi_k(t_1,t_2)) = (1/2)E(s_1-t_1,s_2-t_2)$.

Applying Proposition 3 to ξ_k in R_N, we obtain the correlated Gaussian process η with $\eta(s_1,s_2) \sim \Gamma(\alpha,1)$. Analogously, σ is a periodic process with correlated Gamma distribution and $\sigma \sim \Gamma(\alpha, \beta)$, with coefficients of correlation given by

$$\begin{aligned}\rho(\sigma_{s_1,s_2}, \sigma_{(0,0)}) &= \rho(\eta(s_1,s_2), \eta(0,0)) \\ &= \rho^2(\xi(s_1,s_2), \xi(0,0)) \qquad \text{(D.1)} \\ &= E^2(s_1,s_2). \qquad \square\end{aligned}$$

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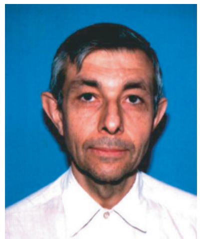

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