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Spatial aliasing for efficient direction-of-arrival estimation based on steering vector reconstruction

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Abstract

A new technique is proposed to reduce the computational complexity of the multiple signal classification (MUSIC) algorithm for direction-of-arrival (DOA) estimate using a uniform linear array (ULA). The steering vector of the ULA is reconstructed as the Kronecker product of two other steering vectors, and a new cost function with spatial aliasing at hand is derived. Thanks to the estimation ambiguity of this spatial aliasing, mirror angles mathematically relating to the true DOAs are generated, based on which the full spectral search involved in the MUSIC algorithm is highly compressed into a limited angular sector accordingly. Further complexity analysis and performance studies are conducted by computer simulations, which demonstrate that the proposed estimator requires an extremely reduced computational burden while it shows a similar accuracy to the standard MUSIC.

Keywords: Multiple signal classification (MUSIC), Direction-of-arrival (DOA) estimation, Spatial aliasing, Steering vector reconstruction

1 Introduction

Direction-of-arrival (DOA) estimation using a sensor array has been an important topic that arises in many fields such as radar, sonar, passive localization, and wireless communication [1, 2]. Over several decades, numerous outstanding algorithms including maximum-likelihood (ML) [3], multiple signal classification (MUSIC) [4], estimation of signal parameters via rotational invariance technique (ESPRIT) [5], and subspace fitting [6] have been developed to estimate the DOA of narrow-band sources. Among state-of-the-art parameter estimation techniques, the MUSIC algorithm which can offer a so-called super-resolution probability for two sufficiently closely spaced sources is one of the most popular one. The primary advantage of the MUSIC algorithm over the other subspace-based methods is distinguished by its easy implementation with arbitrary geometries [7, 8]. However, since the standard MUSIC involves a subspace decomposition step and a tremendous spectral search step, the computational complexity of MUSIC is usually expensive, and a variety of low-complexity approaches have been proposed to reduce this computational burden.

The research in [9] and [10] suggest that the spectral search step dominates the complexity of the MUSIC algorithm, therefore, avoiding the spectral search step or limiting the range for this spectral search becomes the key to reducing the complexity of MUSIC, and numerous modifications of the standard MUSIC have been presented from this point of view. For example, when the array geometry satisfies the rotational invariant property, DOAs can be computed without spectral search by the well-known ESPRIT method. Although ESPRIT saves a much lower complexity, it sacrifices a significant estimation accuracy as compared to MUSIC on the other hand. Another premising search-free method is the root-MUSIC algorithm [11], which can be taken as a special extension of MUSIC with uniform linear arrays (ULAs). Using the Vandermonde sturcture of a ULA, the root-MUSIC algorithm transforms the MUSIC function into a polynomial and finds signal DOAs by rooting instead of spectral search. Although root-MUSIC has been extended to nonuniform linear arrays (NULAs)

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[12–17], and has an improved computational load and threshold performances as compared to the standard MUSIC [18], the former usually requires to find all the roots of a polynomial whose order is about twice of the number of sensors. Therefore, the computational burden of root-MUSIC is in fact much higher than expected, especially when large numbers of sensors are used (Fig. 1).

An outstanding structural (instead of algorithmic) method is recently proposed to overcome the complexity problem that arises in the famous ML DOA estimator [19]. The algorithm is based on a complex doubly scaled aperture nested array (DSANA) which can generate *spatial aliasing* (also known as ambiguous DOAs) in the ML spectral. The spatial aliasing is used to highly compress the full grid search into small angular sectors to reduce significant computational burden, and unique determination of DOA is finally obtained by implementing the ML algorithm with different sub-arrays selected from DSANA. Following this idea, we propose in this paper a new method with a uniform linear array (ULA) to reduce the computational complexity of the MUSIC algorithm. Unlike DSANA, the array geometry in this work is much simpler and more popular in practice. Furthermore, the proposed technique can be used to reduce the complexity of both MUSIC and ML while that in [19] cannot be used for MUSIC directly (since implementing MUSIC on DSANA requires the number of any sub-arrays selected from DSANA being larger than that of sources).

The proposed method starts at reconstructing the steering vector of a ULA as the Kronecker product of two other steering vectors, which has been successfully used in two-dimensional (2D) polarization estimate [20], MIMO radar [21], and nested array [22]. Using the Kronecker product, the high-cost MUSIC function is further reformulated as an eigenvector-based optimization problem, and a new low-complexity cost function with spatial aliasing is derived. Due to the ambiguous DOAs caused by spatial aliasing, the full spectral search is finally compressed into a limited angular sector, and hence a significant computational burden as compared to the standard MUSIC is saved.

The outline of this paper is as follows. The narrow-band signal model and the conventional MUSIC algorithm are introduced in Section 2. In Section 3, the steering vector of a ULA that used in the standard MUSIC is firstly reconstructed as two other steering vectors, based on which the proposed low-complexity cost function with spatial aliasing is addressed. The characteristics of this spatial aliasing are analyzed, and the compressed spectral range is discussed in detail finally in this section. The complexity of our method is analyzed in Section 4, and simulation results are conducted to validate the effectiveness of new method in Section 5.

2 Signal model and standard MUSIC

Assume that L narrowband signals with unknown DOAs simultaneously impinge from far-field on a ULA with omnidirectional antenna elements indexed by $\{0, 1, 2, \ldots, M-1\}$, as shown in Fig. 1, the received signal can be written in matrix form as [1–19]

$$\mathbf{x}(t) = \mathbf{A}(\theta)\,\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{A}(\theta) = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \ldots, \mathbf{a}(\theta_L)] \triangleq \mathbf{A}$ is the $M \times L$ matrix of the signal direction vectors and

$$\mathbf{a}(\theta) \triangleq \Big[1, e^{-j\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta}, e^{-j\cdot 2\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta}, \ldots, e^{-j\cdot(M-1)\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta}\Big]^T \tag{2}$$

is the $M \times 1$ steering vector, $\mathbf{s}(t)$ is the $L \times 1$ source waveforms, and $\mathbf{n}(t)$ is the $M \times 1$ sensor noise. In addition, $j \triangleq \sqrt{-1}$, λ is center wavelength, d is array interval, and $(\cdot)^T$ is transpose.

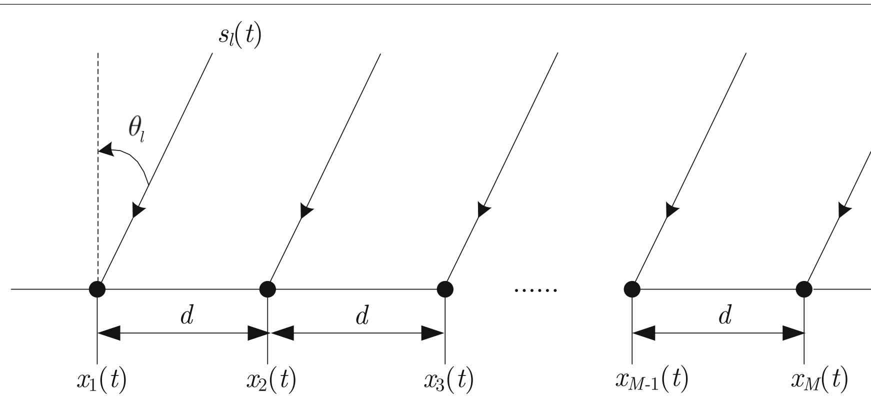

Fig. 1 Uniform linear array of *M* sensors with interval *d*

Given N snapshots of the received signals $\{\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(N)\}$, the main task of DOA estimation is to determine the incident signal angles $\{\theta_1, \theta_2, \ldots, \theta_L\}$, provided that the total number of signals L is detected in advance. This task can be equivalently regarded as the estimation of a set of steering vectors as a set of directions and that of steering vectors are related in a one-to-one correspondence [19]. The study follows will focus on the angle estimation of the incident signals to simplify the DOA estimation problem with the assumption that the number of signals has been detected in advance.

Almost all of the existent super-resolution DOA estimators need to compute the covariance of $\mathbf{x}(t)$, which is given by

$$\mathbf{R} = E\left[\mathbf{x}(t)\mathbf{x}^H(t)\right] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M. \tag{3}$$

In practical applications, the theoretical $\mathbf{R}$ is unavailable, and it is usually estimated by $\{\mathbf{x}(1), \mathbf{x}(2), \ldots, \mathbf{x}(N)\}$ as

$$\widehat{\mathbf{R}} = \frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t) \tag{4}$$

where $\mathbf{R}_s = E[\,\mathbf{s}(t)\mathbf{s}^H(t)]$ is the $L \times L$ signal covariance matrix, $\mathbf{I}_M$ is $M \times M$ identity matrix, σ_n^2 is noise variance, $E[\,\cdot]$ is mathematical expectation, and $(\cdot)^H$ is Hermitian transpose. The eigenvalue decompositions (EVDs) of the theoretical $\mathbf{R}$ given in (3) and the practical estimated $\widehat{\mathbf{R}}$ given in (4) can be defined, respectively, in a standard way as

$$\begin{aligned} \mathbf{R} &= \mathbf{V}_s\Pi_s\mathbf{V}_s^H + \mathbf{V}_n\Pi_n\mathbf{V}_n^H \\ \widehat{\mathbf{R}} &= \widehat{\mathbf{V}}_s\widehat{\Pi}_s\widehat{\mathbf{V}}_s^H + \widehat{\mathbf{V}}_n\widehat{\Pi}_n\widehat{\mathbf{V}}_n^H \end{aligned} \tag{5}$$

where

$$\begin{aligned} \mathbf{V}_s &= [\mathbf{v}_1, \ldots, \mathbf{v}_K] \\ \mathbf{V}_n &= [\mathbf{v}_{K+1}, \ldots, \mathbf{v}_M] \\ \Pi_s &= \mathrm{diag}\{\pi_1, \ldots, \pi_K\} \\ \Pi_n &= \mathrm{diag}\{\pi_{K+1}, \ldots, \pi_M\} \\ \widehat{\mathbf{V}}_s &= [\widehat{\mathbf{v}}_1, \ldots, \widehat{\mathbf{v}}_K] \\ \widehat{\mathbf{V}}_n &= [\widehat{\mathbf{v}}_{K+1}, \ldots, \widehat{\mathbf{v}}_M] \\ \widehat{\Pi}_s &= \mathrm{diag}\{\widehat{\pi}_1, \ldots, \widehat{\pi}_K\} \\ \widehat{\Pi}_n &= \mathrm{diag}\{\widehat{\pi}_{K+1}, \ldots, \widehat{\pi}_M\} \end{aligned} \tag{6}$$

with the subscripts s and n standing for the signal and noise subspace, respectively.

Using the facts $\mathrm{span}(\mathbf{V}_s) \perp \mathrm{span}(\mathbf{V}_n)$ and $\mathrm{span}(\mathbf{A}) = \mathrm{span}(\mathbf{V}_s)$, the MUSIC algorithm suggests to estimate DOAs by spectral search

$$\begin{aligned} \min_\theta\ & P_{\mathrm{MUSIC}}(\theta) \triangleq \mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\widehat{\mathbf{V}}_n^H\mathbf{a}(\theta) \\ \text{s.t.}\ & \theta \in [-\pi/2, \pi/2] \end{aligned} \tag{7}$$

to find the L peaks of $P_{\mathrm{MUSIC}}(\theta)$ which indicate source DOAs. The advantage of the MUSIC algorithm over the other subspace-based methods is its easy implementation and high resolution [7, 8]. However, the complexity of the spectral search step is typically substantially high since for each point, the product $\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\widehat{\mathbf{V}}_n^H\mathbf{a}(\theta)$ has to be computed [9].

3 The proposed algorithm

3.1 Steering vector reconstruction

Let us consider the number of sensors M. For any $M > L$, if M is not a prime number and $M \geqslant 4$, we can find two integers M_1 and M_2 that satisfy

$$M = M_1 \cdot M_2, \qquad M_1 \geqslant 2, M_2 \geqslant 2. \tag{8}$$

Define $z \triangleq e^{-j\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta}$, the steering vector $\mathbf{a}(\theta)$ given in (2) can be rewritten as

$$\mathbf{a}(\theta) = \begin{bmatrix} 1 \\ z \\ z^2 \\ \cdots \\ z^{M_1\cdot M_2-1} \end{bmatrix} = \begin{bmatrix} 1 \times 1 \\ 1 \times z \\ \cdots \\ 1 \times z^{M_1-1} \\ \cdots\cdots\cdots \\ z^{M_1} \times 1 \\ z^{M_1} \times z \\ \cdots \\ z^{M_1} \times z^{M_1-1} \\ \cdots\cdots\cdots \\ \cdots \\ \cdots\cdots\cdots \\ z^{(M_2-1)\cdot M_1} \times 1 \\ z^{(M_2-1)\cdot M_1} \times z \\ \cdots \\ z^{(M_2-1)\cdot M_1} \times z^{M_1-1} \end{bmatrix} \triangleq \mathbf{b}(\theta) \otimes \mathbf{c}(\theta) \tag{9}$$

where $\otimes$ denotes the Kronecker product and

$$\mathbf{b}(\theta) = \begin{bmatrix} 1 \\ z^{M_1} \\ z^{2\cdot M_1} \\ \cdots \\ z^{(M_2-1)\cdot M_1} \end{bmatrix}, \quad \mathbf{c}(\theta) = \begin{bmatrix} 1 \\ z \\ z^2 \\ \cdots \\ z^{M_1-1} \end{bmatrix}. \tag{10}$$

It can be seen from (9) and (10) that $\mathbf{a}(\theta)$ can be reconstructed as the Kronecker product of $\mathbf{b}(\theta)$ and $\mathbf{c}(\theta)$, where $\mathbf{b}(\theta)$ and $\mathbf{c}(\theta)$ can be regarded as the steering vectors of two ULAs with M_2 and M_1 sensors, respectively. The differences between the two steering vectors $\mathbf{b}(\theta)$ and $\mathbf{c}(\theta)$ is that the array interval for $\mathbf{c}(\theta)$ is d while that for $\mathbf{b}(\theta)$ is $M_1 d$.

3.2 Proposed cost function

Now, substituting (9) into (7), the MUSIC algorithm can be rewritten as

$$\begin{aligned}\min_{\theta} P_{\mathrm{MUSIC}}(\theta) &= [\mathbf{b}(\theta)\otimes\mathbf{c}(\theta)]^H\,\widehat{\mathbf{V}}_n\widehat{\mathbf{V}}_n^H\,[\mathbf{b}(\theta)\otimes\mathbf{c}(\theta)]\\ &= \mathbf{c}^H(\theta)\widehat{\mathbf{Q}}_\theta\mathbf{c}(\theta)\\ \text{s.t.}\quad &\theta\in[-\pi/2,\pi/2]\end{aligned}\tag{11}$$

where

$$\mathbf{Q}_\theta \triangleq \left[\mathbf{b}(\theta)\otimes\mathbf{I}_{M_1}\right]^H\mathbf{V}_n\mathbf{V}_n^H\left[\mathbf{b}(\theta)\otimes\mathbf{I}_{M_1}\right].\tag{12}$$

It can be concluded from (11) that searching the minima of $P_{\mathrm{MUSIC}}(\theta)$ over $\theta\in[-\pi/2,\pi/2]$ is equivalent to finding the optimal solution of $\mathbf{c}(\theta)$ that minimizes the product $\mathbf{c}^H(\theta)\widehat{\mathbf{Q}}_\theta\mathbf{c}(\theta)$.

Noting that $\|\mathbf{c}(\theta)\|^2=\mathbf{c}^H(\theta)\mathbf{c}(\theta)=M_1$, we now define two vector sets $\mathbf{C}_1$ and $\mathbf{C}_2$ as follows

$$\mathbf{C}_1 \triangleq \left\{\mathbf{c}\in\mathbb{C}^{M_1\times 1}|\mathbf{c}^H\mathbf{c}=M_1\right\}\tag{13}$$

$$\mathbf{C}_2 \triangleq \left\{\mathbf{c}\in\mathbb{C}^{M_1\times 1}|\mathbf{c}=\begin{bmatrix}1\\ z\\ z^2\\ \cdots\\ z^{M_1-1}\end{bmatrix},\; z=e^{-j\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta},\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right\}.\tag{14}$$

Clearly, for any vector $\mathbf{c}_1\in\mathbf{C}_1$, we must have $\mathbf{c}_1^H\mathbf{c}_1=M_1$. On the other hand, for each $\theta\in[-\pi/2,\pi/2]$, we can compute $\mathbf{c}_2=[1,z,z^2,\ldots,z^{M_1-1}]^T$ using $z=e^{-j\cdot 2\pi\cdot\frac{d}{\lambda}\cdot\sin\theta}$, and vector $\mathbf{c}_2$ must also satisfy $\mathbf{c}_2^H\mathbf{c}_2=M_1$. This indicates that any vector in $\mathbf{C}_2$ must be in the form of a steering vector while that in $\mathbf{C}_1$ may not be. Therefore, we have

$$\mathbf{C}_2\subset\mathbf{C}_1.\tag{15}$$

To minimize $\mathbf{c}^H(\theta)\widehat{\mathbf{Q}}_\theta\mathbf{c}(\theta)$, we can search over the entire angle set $[-\pi/2,\pi/2]$ to find an appropriate angle θ_0 which satisfies $\mathbf{c}^H(\theta_0)\widehat{\mathbf{Q}}_{\theta_0}\mathbf{c}(\theta_0)=\min$. In other words, the MUSIC algorithm can be directly reformulated as the following optimization problem

$$\begin{aligned}\min\quad &\mathbf{c}^H(\theta)\widehat{\mathbf{Q}}_\theta\mathbf{c}(\theta)\\ \text{s.t.}\quad &\mathbf{c}(\theta)\in\mathbf{C}_2.\end{aligned}\tag{16}$$

However, the above optimization problem is in fact identical to the standard MUSIC and there is no reduction on the computational complexity.

To reduce the complexity, we note that for a given $\mathbf{Q}_\theta$ computed by a given angle θ, the minimizing task of $\mathbf{c}^H(\theta)\widehat{\mathbf{Q}}_\theta\mathbf{c}(\theta)$ can be performed over $\mathbf{C}_1$ instead of $\mathbf{C}_2$ since we have $\mathbf{C}_2\subset\mathbf{C}_1$. This means that we can omit the parameter θ and let $\mathbf{c}=\mathbf{c}(\theta)$. Hence, the MUSIC algorithm can be equivalently reformulated as the following optimization problem

$$\begin{aligned}\min\quad &\mathbf{c}^H\widehat{\mathbf{Q}}_\theta\mathbf{c}\\ \text{s.t.}\quad &\mathbf{c}\in\mathbf{C}_1.\end{aligned}\tag{17}$$

The differences between (16) and (17) are that both $\widehat{\mathbf{Q}}_\theta$ and $\mathbf{c}(\theta)$ must be considered as functions of θ in the former while in the latter, we only need to consider $\widehat{\mathbf{Q}}_\theta$ for each $\theta\in[-\pi/2,\pi/2]$. The derivation follows indicate that this can lead to a significant complexity reduction as compared to the standard MUSIC.

Using the Lagrange multiplier technique with respect to the restriction $\mathbf{c}^H\mathbf{c}=M_1$, we obtain the following Lagrangian

$$f[\mathbf{c}]=\mathbf{c}^H\mathbf{Q}_\theta\mathbf{c}-\xi\left[\mathbf{c}^H\mathbf{c}-M_1\right]\tag{18}$$

where ξ is the Lagrange multiplier.

Setting the gradient of (18) with respect to $\mathbf{c}$ to zero yields

$$\mathbf{Q}_\theta\mathbf{c}=\xi\mathbf{c}.\tag{19}$$

Equation (19) can be identified as the characteristic one for matrix $\mathbf{Q}_\theta$. Thus, ξ is an eigenvalue of $\mathbf{Q}_\theta$ and $\mathbf{c}$ is the eigenvector that associated with ξ. Since $\mathbf{c}^H\mathbf{c}=M_1$, it follows from (19) that

$$\mathbf{c}^H\mathbf{Q}_\theta\mathbf{c}=\xi\mathbf{c}^H\mathbf{c}=M_1\xi.\tag{20}$$

Therefore, minimizing $\mathbf{c}^H\widehat{\mathbf{Q}}_\theta\mathbf{c}$ is in fact equivalent to finding $\widehat{\xi}_{\min}$, i.e, finding the minimum eigenvalue of $\widehat{\mathbf{Q}}_\theta$. Additively, the optimized solution for $\mathbf{c}$, i.e., $\mathbf{c}_{\mathrm{opt}}$, is equivalent to the associated eigenvector respect to $\widehat{\xi}_{\min}$. Hence, $\mathbf{c}_{\mathrm{opt}}$ can be given as follows

$$\mathbf{c}_{\mathrm{opt}}=\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}\tag{21}$$

where $\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}$ is the eigenvector associated with $\widehat{\xi}_{\min}$.

Substituting (21) into (11), DOAs can finally estimated by

$$\begin{aligned}\min_{\theta}\widehat{P}(\theta) &= \left[\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}\right]^H\widehat{\mathbf{Q}}_\theta\left[\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}\right]\\ \text{s.t.}\quad &\theta\in\Theta\end{aligned}\tag{22}$$

where $\Theta\subset[-\pi/2,\pi/2]$ is a limited angular sector, which is to be discussed in the subsection that follows.

3.3 Search range compression

In this subsection, we show source DOAs can be efficiently estimated by spectral search over only a limited angular sector Θ instead of $[-\pi/2,\pi/2]$. Hence, a significant complexity is reduced as compared to MUSIC.

Recall the maximum inter-element spacing criterion of an array, which is defined as [19]

$$d_{\max} = \frac{\lambda}{2\,|\sin\theta_{\max}|} \tag{23}$$

where $\theta_{\max}$ is the maximum searching angle. Because a ULA scans at most $\pm 90°$, the element spacing of such an array must be less than half of the wavelength; otherwise, the impinging signals will result in estimation ambiguous. This phenomenon happens when there are replica DOAs θ_{re} producing the same optimized solution for $\mathbf{c}(\theta)$ as the true DOAs θ_{ge} such that

$$\mathbf{c}_{\mathrm{opt}}\left(\theta_{\mathrm{ge}}\right) = \mathbf{c}_{\mathrm{opt}}\left(\theta_{\mathrm{re}}\right). \tag{24}$$

Since the array interval of $\mathbf{b}(\theta)$ is M_1 times that of $\mathbf{c}(\theta)$, the relationship between θ_{re} and θ_{ge} is

$$\frac{M_1}{2}\sin\theta_{\mathrm{ge}} + k = \frac{M_1}{2}\sin\theta_{\mathrm{re}} \tag{25}$$

or equivalently given by

$$\theta_{\mathrm{re}} = \sin^{-1}\left[\sin\theta_{\mathrm{ge}} + \frac{2k}{M_1}\right] \tag{26}$$

where k is an integer. As $|\sin\theta| \le 1$, different integer values of k will be given according to θ_{ge} as follows

$$\frac{M_1}{2}\left[-1+\sin\theta_{\mathrm{ge}}\right] < k < \frac{M_1}{2}\left[1-\sin\theta_{\mathrm{ge}}\right]. \tag{27}$$

The maximum searching angle without estimation ambiguity can be determined by inserting $d_{\max} = M_1\lambda/2$ into (23) as

$$\theta_{\max,\mathrm{re}} = \sin^{-1}\left(\frac{1}{M_1}\right). \tag{28}$$

Therefore, the compressed searching range Θ is given by

$$\Theta = \left[-\sin^{-1}\left(\frac{1}{M_1}\right), \sin^{-1}\left(\frac{1}{M_1}\right)\right]. \tag{29}$$

The above analysis implies that searching $P(\theta)$ over Θ will reduce L candidate angles $\{\widehat{\theta}_1^{\mathrm{ge}}, \widehat{\theta}_2^{\mathrm{ge}}, \ldots, \widehat{\theta}_L^{\mathrm{ge}}\}$, with which the other M_1L ones $\left\{\widehat{\theta}_1^{\mathrm{re}}, \widehat{\theta}_2^{\mathrm{re}}, \ldots, \widehat{\theta}_{M_1L}^{\mathrm{re}}\right\}$ can be computed immediately by (26). Finally, the L source DOAs can be obtained efficiently by selecting L angles among $\left\{\widehat{\theta}_1^{\mathrm{re}}, \widehat{\theta}_2^{\mathrm{re}}, \ldots, \widehat{\theta}_{M_1L}^{\mathrm{re}}\right\}$ that minimize $\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\widehat{\mathbf{V}}_n^H\mathbf{a}(\theta)$.

3.4 Summary of the new algorithm

Detailed steps for implementing the proposed algorithm are summarized in Algorithm 1.

Figure 2 is conducted to show more clearly the implementations of the new method. First, it is computed by (29) that

$$\Theta = \left[-\sin^{-1}(1/3), \sin^{-1}(1/3)\right] \approx \left[-19.47°, 19.47°\right].$$

Next, two candidate angles $\theta_{11}^{\mathrm{re}} = -9.6°$ and $\theta_{21}^{\mathrm{re}} = -1.4°$ are found by searching $\widehat{P}(\theta)$ over only Θ, and the other four ones are computed by using (26) as follows

$$k = 1,\quad \theta_{12}^{\mathrm{re}} = \sin^{-1}\left[\sin(-9.6°) + 2/3\right] \approx 29.99°$$
$$k = -1,\quad \theta_{13}^{\mathrm{re}} = \sin^{-1}\left[\sin(-9.6°) - 2/3\right] \approx -56.45°$$
$$k = 1,\quad \theta_{22}^{\mathrm{re}} = \sin^{-1}\left[\sin(-1.4°) + 2/3\right] \approx 39.95°$$
$$k = -1,\quad \theta_{23}^{\mathrm{re}} = \sin^{-1}\left[\sin(-1.4°) - 2/3\right] \approx -43.71°.$$

Finally, we calculate the numerical values of the cost function at all candidate angles in Table 1, from which we can conclude that the product $\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$ get its $L = 2$ minimum values at angles $\theta_{12}^{\mathrm{re}} = 29.99°$ and $\theta_{22}^{\mathrm{re}} = 39.95°$. Since the total number of signals L is supposed to be detected in advance, the above two angles are selected as the estimated DOAs, which match the true angles.

Remarks:

(1) Note that the eigenvector $\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}$ may not be necessarily in the form of a steering vector. Consequently, the

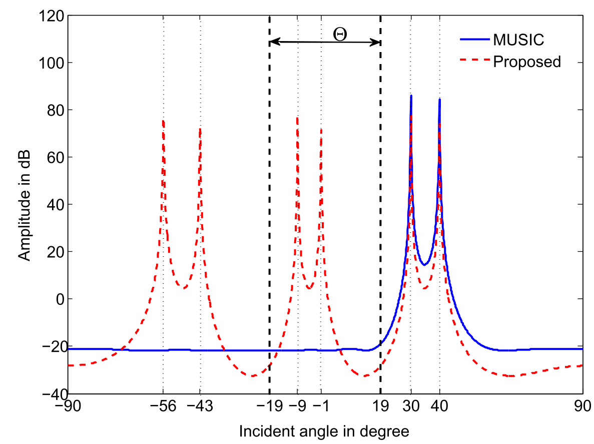

Fig. 2 Direction finding by the proposed method, SNR = 20 dB, $N = 200$, $M_1 = M_2 = 3$. Two sources at $\theta_1 = 30°$ and $\theta_2 = 40°$, half-wavelength ULA with nine sensors

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Algorithm 1 The proposed algorithm

Require: $\{\widehat{\mathbf{x}}(t)\}_{t=1}^N$: N snapshots of array output vector.
1: Use $\{\widehat{\mathbf{x}}(t)\}_{t=1}^N$ to get $\widehat{\mathbf{R}}$ by (4) and compute $\widehat{\mathbf{V}}_n$ by (5) ;
2: Choose appropriate integers M_1 and M_2 according to (8) and compute Θ by (29);
3: **for** each $\theta \in \Theta$ **do**
4: Calculate $\widehat{\mathbf{Q}}_\theta$ and find smallest eigenvector $\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}$;
5: Compute $\widehat{P}(\theta)$ by (22);
6: **end for**
7: Find the L angles $\left\{\widehat{\theta}_1^{\mathrm{ge}}, \widehat{\theta}_2^{\mathrm{ge}}, \ldots, \widehat{\theta}_L^{\mathrm{ge}}\right\}$ minimizing $\widehat{P}(\theta)$;
8: Calculate the M_1L angles $\left\{\widehat{\theta}_1^{\mathrm{re}}, \widehat{\theta}_2^{\mathrm{re}}, \ldots, \widehat{\theta}_{M_1L}^{\mathrm{re}}\right\}$ by (26);
9: Select the L DOAs among $\left\{\widehat{\theta}_1^{\mathrm{re}}, \widehat{\theta}_2^{\mathrm{re}}, \ldots, \widehat{\theta}_{M_1L}^{\mathrm{re}}\right\}$ by minimizing the product $\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$;
10: **return** the L estimated source DOAs.

Table 1 Numerical values of product $\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$ at all candidate angles

Candidate angles	−56.45°	−43.71°	−9.6°	−1.4°	29.99°	39.95°
$\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$	8.7873	8.8731	8.8810	8.7873	2.89e-05	1.63e-05
True or virtual	Virtual	Virtual	Virtual	Virtual	True	True

optimized solution $\mathbf{c}_{\text{opt}}$ may not be in the form of a steering vector either. This means that we may have $\mathbf{c}_{\text{opt}} \in \mathbf{C}_1$ but $\mathbf{c}_{\text{opt}} \notin \mathbf{C}_2$. However, since $\mathbf{Q}_\theta$ is computed by a given angle θ using (12), DOA information are contained is in both $\mathbf{Q}_\theta$ and $\mathbf{c}_{\text{opt}}$. Therefore, signal DOA can be successfully estimated by (19). □

(2) The proposed method uses $\mathbf{V}_n$ to construct its spectrum, which is similar to MUSIC. Hence, the maximum number of sources which can be identified by the new algorithm is $L < M$, which shows an outstanding advantage over the ADSANS-ML method [19]. An example is given in Fig. 3 to illustrate this more clearly. □

(3) It is the last step in Algorithm 1 that helps the proposed method to select the true angles from the computed candidate angles in various scenarios. An example for selecting true sources in the case where all the three true angles map to the same candidate angles is given in Fig. 4. Note that each of the three true angles has six candidate DOAs, which can be computed by using (26). Also note that the three true angles share the same ambiguous angles. To select the true angles, the values of $\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$ are computed in Table 2. Because the number of signals $L = 3$ is supposed to be detected in advance, the true angles can be easily selected by selecting the $L = 3$ angels that minimize $\|\mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n\|^2$. □

(4) The compressed spectral search by the proposed method is in fact resulted from the ambiguous DOAs, which is similar to the RV-MUSIC [8], C-MUSIC [10], and DSANA-ML [19] algorithms. Therefore, techniques suggested by C-MUSIC and DSANA-ML can be directly exploited for the proposed method to solve closely spaced candidate DOAs. □

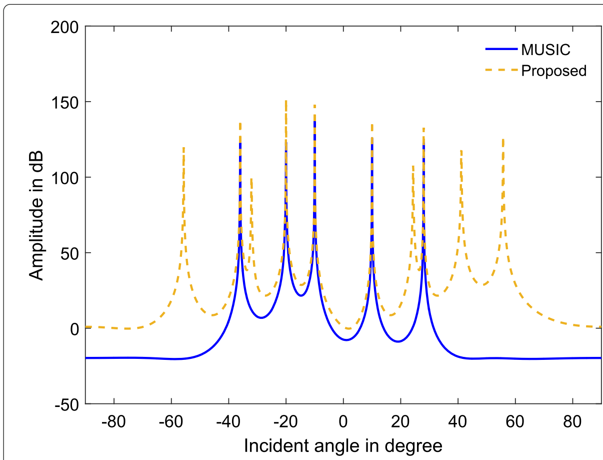

Fig. 3 The maximum number of resolvable sources by the proposed method. SNR = 30 dB, $N = 500$, $M_1 = 2$, $M_2 = 3$. Five sources at $\theta_1 = -36°, \theta_2 = -20°, \theta_3 = -10°, \theta_4 = 10°$, and $\theta_5 = 28°$, half-wavelength ULA with six sensors

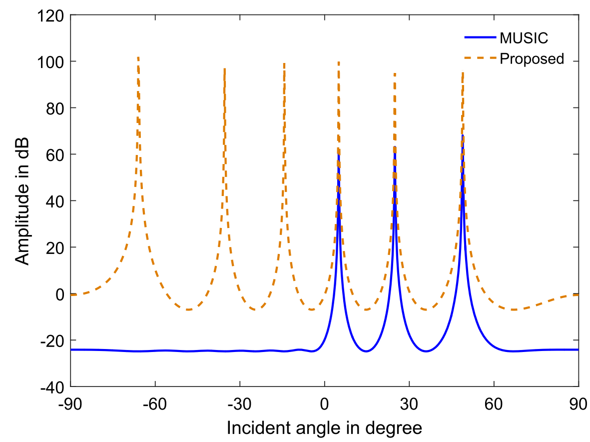

Fig. 4 Scenario in which some of the true sources map to the same candidate angles. SNR = 10 dB, $N = 200$, $M_1 = 6$, $M_2 = 2$. Three sources at $\theta_1 = 5°, \theta_2 = 24.87°$ and $\theta_3 = 48.92°$, half-wavelength ULA with 12 sensors

4 Complexity analysis

Using the fast subspace decomposition (FSD) technique [23], the complexity of MUSIC is given by [8, 10]

$$C_{\text{MUSIC}} = M^2 (L+2) + J(M+1)(M-L) \text{ flops} \quad (30)$$

where J is the number of sample points in $[-\pi/2, \pi/2]$.

For each point in Θ, the new method has to compute three items including $\widehat{\mathbf{Q}}_\theta$, the singular valued decomposition (SVD) of $\widehat{\mathbf{Q}}_\theta$ and $\widehat{P}(\theta)$. Since there are at most M_2 non-zeros in each column of $\mathbf{b}(\theta) \otimes \mathbf{I}_{M_1}$, the computation of $\widehat{\mathbf{Q}}_\theta$ requires $M_2 (M-L)(M_1+M_2)$ flops. As $\widehat{\mathbf{Q}}_\theta$ is of dimensions $M_1 \times M_1$ and we only need to find $\widehat{\gamma}_{\min}^{\widehat{\mathbf{Q}}_\theta}$, the SVD of $\widehat{\mathbf{Q}}_\theta$ costs M_1^2 flops [24]. Note that the width of Θ is $2\sin^{-1}(1/M_1)$ and for each point in Θ and computing $\widehat{P}(\theta)$ needs $M_1^2 + M_1$ flops.

On the other hand, each true source generate M_1 ambiguous DOAs. Therefore, the total number of candidate angles generated by the proposed method is $M_1 L$. For each candidate angle, we need to compute the product, which costs $(M+1)(M-L)$ flops. Therefore, the complexity for solving the ambiguous DOAs is given by $M_1 L(M+1)(M-L)$.

Table 2 Angle selecting for Fig. 4

Simulated angles	$\theta_1 = 5°$, $\theta_2 = 24.87°$, $\theta_3 = 48.92°$					
Candidate angles	−65.90°	−35.42°	−14.25°	5°	24.86°	48.92°
$\Vert \mathbf{a}^H(\theta)\widehat{\mathbf{V}}_n \Vert^2$	11.9990	11.9990	11.9990	1.02e-04	1.15e-04	1.17e-04
True or virtual	Virtual	Virtual	Virtual	True	True	True

Based on the above analysis, the total complexity of the proposed method is given by

$$C_{\text{Proposed}} = M^2(L+2) + \frac{\sin^{-1}(1/M_1)}{90} J \times \left[2M_1^2 + M_1 + M_2(M-L)(M_1+M_2)\right] + M_1 L(M+1) \times (M-L) \text{ flops}. \tag{31}$$

Figure 5 plots the complexity as functions of the number of sensors. It is seen from the figure that the proposed method costs a heavier complexity as compared to MUSIC for $M_1 = 2$, with which Θ reaches its maximum width. As M_1 increases, Θ gets smaller, and the new method costs significantly reduced complexities as compared to MUSIC. This implies that larger M_1 yields lower complexity. Since the number of ambiguous DOAs equals to M_1, larger M_1 also leads to more ambiguities. Hence, M_1 cannot be chosen too large in practice.

5 Performance study

Simulations with 500 independent Monte Carlo trials are conducted to assess the mean square error (MSE) performance of the proposed method, where the MSE is defined as

$$\text{MSE} \triangleq \frac{1}{500}\sum_{i=1}^{500}\left(\widehat{\theta}_i - \theta\right)^2 \tag{32}$$

with θ and $\widehat{\theta}_i$ presenting the true and the estimated DOA of the ith trial, respectively. Five algorithms including MUSIC [4], ESPRIT [5], C-MUSIC with $\beta = 2$ angular sectors [10], U-MUSIC [25], RV-MUSIC [8], and the unconditional Cramér-Rao Lower Bound (CRLB) [26] are also applied for references.

To see clearly the performance of the proposed algorithm, Fig. 6 plots MSE against the SNR, in which a half-wavelength ULA with $M = 12$ sensors is used to locate $L = 2$ sources at $\theta_1 = 30°$ and $\theta_2 = 40°$. Since the two parameters M_1 and M_2 for the proposed method satisfy $M = M_1 \cdot M_2$, different combinations for the two parameters are considered to provide further insights into the new method. In addition, the number of snapshots is fixed as $N = 500$ and the SNR varies from a wide range from SNR = −20 dB to SNR = 20 dB in the simulation.

It is seen clearly from Fig. 6 that the proposed method shows a similar accuracy to the standard MUSIC and RV-MUSIC, which is much better than the ESPRIT and C-MUSIC techniques. It is also seen from the figure that the MSEs of the new method decrease slightly as M_1 increases. However, the new method still performs closely to MUSIC with $M_1 = 4$. Since $C_{\text{MUSIC}}/C_{\text{Proposed}}\big|_{M_1=4} \approx 23976/7699 \approx 3.11$ ($J = 180$), we can conclude that the proposed method trades off MSE by complexity efficiently.

To verify the efficiency of the developed method and the computational complexity analysis in Section 4, we compare the simulation times costed by the standard MUSIC and the proposed method with different choices for the parameter M_1. In the simulation, we fix

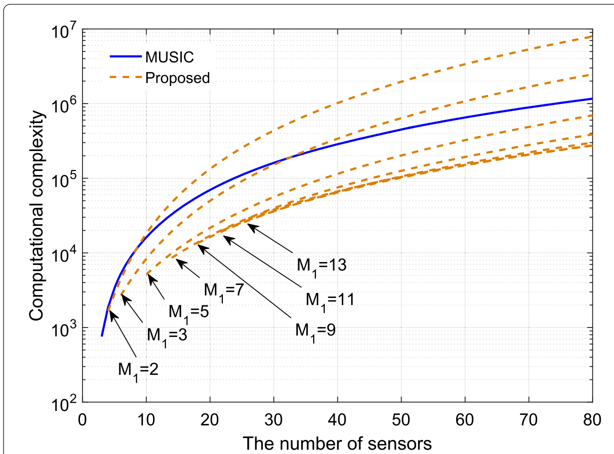

Fig. 5 Complexity versus the number of sensors, $J = 180$, $L = 2$

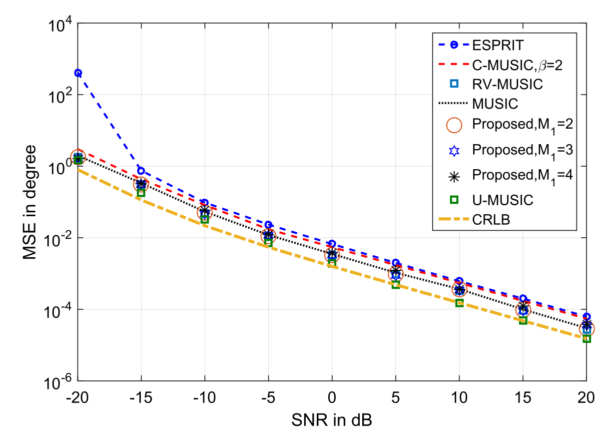

Fig. 6 MSE versus the SNR, $N = 500$, two sources at 30° and 40°, half-wavelength ULA with 12 sensors

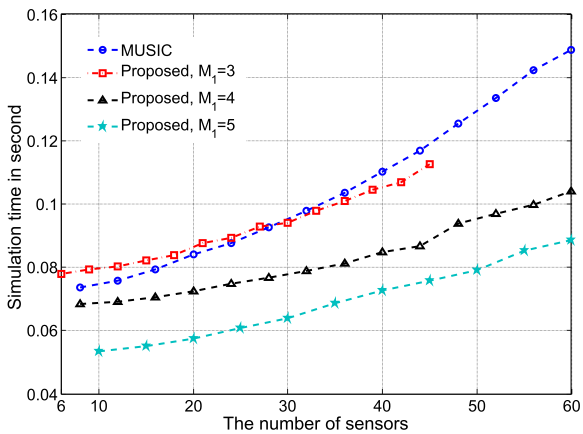

Fig. 7 Simulation time against the number of sensors, SNR = 20 dB, $N = 200$, a fine search grid 0.11° for MUSIC and the proposed method, two sources at 30° and 40°, half-wavelength ULA

SNR = 20 dB, $N = 200$ and a fine search grid 0.11° is applied for spectral search in both MUSIC and the proposed method. The simulations presented here are performed by running the MATLAB codes in the same environment on a personal computer whose CPU configurations and RAM are given by Intel(R) Core(TM) Duo T5870 2.0 GHz and 1 GB, respectively.

It is seen clearly from Fig. 7 that the proposed method costs a much smaller simulation time than the standard MUSIC. It is also seen from the figure that as M_1 increases, the simulation time of the new method decreases dramatically, which matches the complexity comparison results of Fig. 5. It can be concluded from these observations that the proposed method shows an obvious computational efficiency advantage over the standard MUSIC.

6 Conclusions

We have proposed a new computationally efficient algorithm for DOA estimate. The key idea behind is to reconstruct the steering vector of a ULA as the Kronecker product of the two other steering vectors, leading to a limited spectral search with a significantly reduced complexity as compared to MUSIC. Simulations demonstrate that the new method has a very close MSE performance to MUSIC. Future research should be focused on extending the proposed method to arbitrary arrays.

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Competing interests

The authors declare that they have no competing interests.

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